MDOF System Forced Response – Direct Method

- Not in text - text only discusses "modal analysis" approach

$$\bar{M}\ddot{\bar{x}} + \bar{K}\bar{x} = \bar{F}\cos\omega t$$

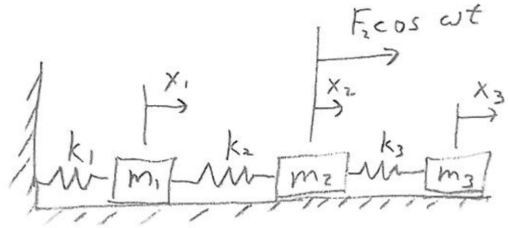
Assume a solution $\bar{x}(t) = \bar{X}\cos\omega t$

$$-\omega^2\bar{M}\bar{X}\cancel{\cos\omega t} + \bar{K}\bar{X}\cancel{\cos\omega t} = \bar{F}\cancel{\cos\omega t}$$

$$(\bar{K} - \omega^2\bar{M})\bar{X} = \bar{F}$$

Example:

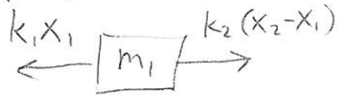
Given:



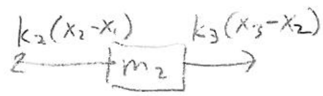
$k_1 = k_2 = k_3 = 10$ N/m
$m_1 = m_2 = m_3 = 0.1$ kg
$\omega = 10$ rad/s
$F_2 = 0.1$ N

Find: (i) $\bar{x}(t)$
(ii) $\bar{X}(\omega)$ from $\omega = 0$ to $\omega = 30$ rad/s

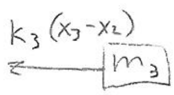
Sol'n (i)



$\sum F_x = m_1\ddot{x}_1 \quad \therefore \quad m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$



$\sum F_x = m_2\ddot{x}_2 \quad \therefore \quad m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 - k_3x_3 = 0$



$\sum F_x = m_3\ddot{x}_3 \quad \therefore \quad m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ 0 \end{Bmatrix}\cos\omega t$$

$$\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}\ddot{\bar{x}}(t) + \begin{bmatrix} 10+10 & -10 & 0 \\ -10 & 10+10 & -10 \\ 0 & -10 & 10 \end{bmatrix}\bar{x}(t) = \begin{Bmatrix} 0 \\ 0.1 \\ 0 \end{Bmatrix}\cos 10t$$

$$(\bar{K} - \omega^2 M)\bar{X} = \bar{F}$$

$$\left(\begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} - 10^2 \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 10 & -10 & 0 \\ -10 & 10 & -10 \\ 0 & -10 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.1 \\ 0 \end{Bmatrix} \qquad \text{Solving} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.01 \end{Bmatrix}$$

$$\bar{X}(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -0.01 \end{Bmatrix} \cos 10t$$

(ii) Follow same procedure with different values of ω to create graph

- For case with viscous damping

$$\bar{M}\ddot{\bar{x}} + \bar{C}\dot{\bar{x}} + \bar{K}\bar{x} = \bar{F}\cos\omega t$$

- Assume a solution $\bar{x}(t) = \bar{X}_1 \cos\omega t + \bar{X}_2 \sin\omega t$

$$-\omega^2 \bar{M}(\bar{X}_1 \cos\omega t + \bar{X}_2 \sin\omega t) + \omega \bar{C}(-\bar{X}_1 \sin\omega t + \bar{X}_2 \cos\omega t) + \bar{K}(\bar{X}_1 \cos\omega t + \bar{X}_2 \sin\omega t) = \bar{F}\cos\omega t$$

- Considering $\sin\omega t$ & $\cos\omega t$ terms separately:

$\cos\omega t$: $(\bar{K} - \omega^2 \bar{M})\bar{X}_1 + \omega \bar{C} \bar{X}_2 = \bar{F}$

$\sin\omega t$: $(K - \omega^2 M)\bar{X}_2 - \omega \bar{C} \bar{X}_1 = \bar{0}$

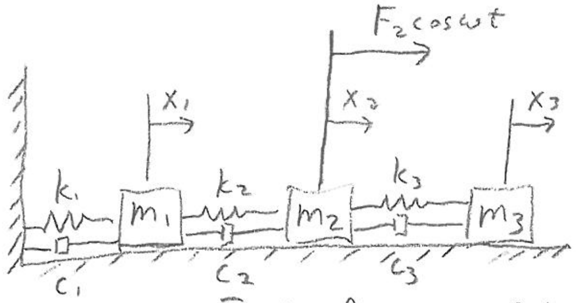
writing this as a matrix equation:

$$\begin{bmatrix} \bar{K} - \omega^2 \bar{M} & \omega \bar{C} \\ -\omega \bar{C} & \bar{K} - \omega^2 \bar{M} \end{bmatrix} \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{F} \\ \bar{0} \end{Bmatrix}$$

- notice that matrix is not symmetric!

Viscous Damping Example

Given:



$m_1 = m_2 = m_3 = 0.1$ kg

$k_1 = k_2 = k_3 = 10$ N/m

$c_1 = c_2 = c_3 = 1$ Ns/m

$\omega = 10$ rad/s

$F_2 = 0.1$ N

Find (i) $\bar{x}(t)$ ii) $\bar{X}(\omega)$ from $\omega = 0$ to $\omega = 30$ rad/s

$$\bar{M}\ddot{\bar{x}} + \bar{C}\dot{\bar{x}} + \bar{K}\bar{x} = \bar{F}\cos\omega t$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix}\cos\omega t$$

$$\bar{x}(t) = \bar{X}_1 \cos\omega t + \bar{X}_2 \sin\omega t$$

$$\begin{bmatrix} \bar{K}-\omega^2\bar{M} & \omega\bar{C} \\ -\omega\bar{C} & \bar{K}-\omega^2\bar{M} \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} \bar{F} \\ \bar{0} \end{bmatrix}$$

Homework: Problem 4.~~70~~ 77

Solve for "response" using "direct method" from class (not using modal analysis).

$f(t) = 10\sin t$ N

i) Use $c_1 = c_2 = c_3 = 0$

ii) Use $c_1 = c_3 = 0$; $c_2 = 4000 \frac{Ns}{m}$

MDOF Forced Vibration Response (undamped)

ω (rad) 10

$$K = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \quad \text{(N/m)}$$

$$M = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \quad \text{(kg)} \qquad \omega\text{^}2\, M = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \quad \text{(N/m)}$$

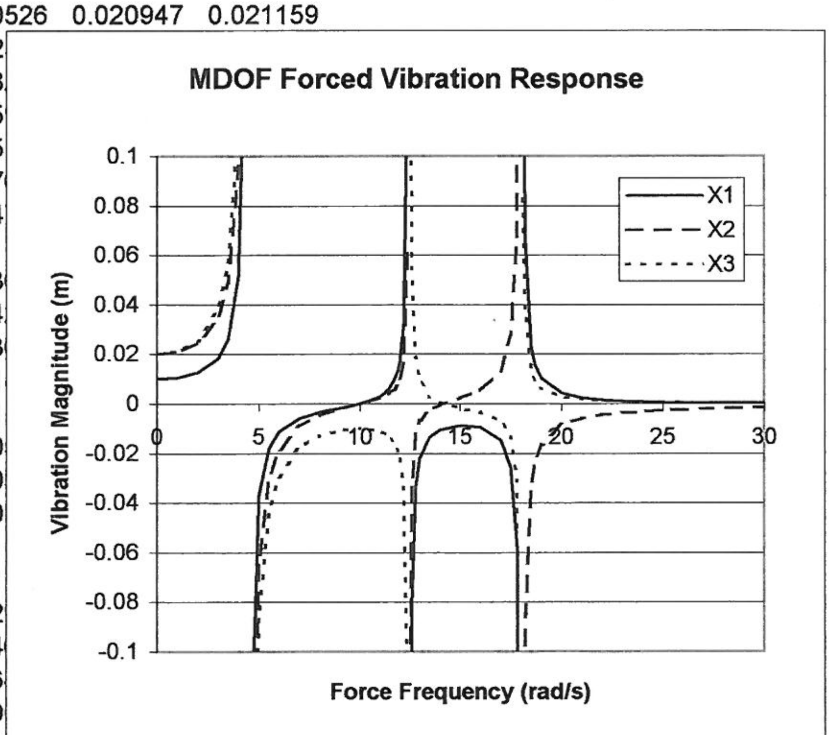
$$K-\omega\text{^}2\, M = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10 & -10 \\ 0 & -10 & 0 \end{bmatrix}$$

$$\text{inv } K-\omega\text{^}2\, M = \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0 & 0 & -0.1 \\ -0.1 & -0.1 & 0 \end{bmatrix} \quad \text{(m/N)} \qquad F = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} \quad \text{(N)} \qquad \underline{\text{Solution}}\ X = \begin{bmatrix} 0 \\ 0 \\ -0.01 \end{bmatrix} \quad \text{(m)}$$

Solution

	X magnitude (m)		
ω (rad)	X_1	X_2	X_3
10	0	0	-0.01

	X magnitude (m)		
ω (rad)	X_1	X_2	X_3
0	0.01	0.02	0.02
1	0.010526	0.020947	0.021159
2	0.012		
3	0.018		
3.5	0.025		
4	0.05		
4.3	0.147		
4.4	0.434		
4.5	-0.43		
4.6	-0.14		
5	-0.03		
5.5	-0.01		
6	-0.01		
7	-0.00		
8	-0.00		
9	-0.00		
10			
10.5	0.001		
11	0.002		
11.3	0.004		
11.5	0.006		
11.7	0.009		
11.9	0.013		



MDOF Forced Vibration Response with ~~Coulomb~~ Viscous Damping

ω (rad) = 10

$$\mathbf{K} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \text{ (N/m)}$$

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \text{ (N s/m)} \qquad \omega\,\mathbf{C} = \begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \text{ (N/m)}$$

$$\mathbf{M} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \text{ (kg)} \qquad \omega\text{^}2\,\mathbf{M} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{ (N/m)}$$

$$\mathbf{K}\text{-}\omega\text{^}2\,\mathbf{M} = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10 & -10 \\ 0 & -10 & 0 \end{bmatrix} \text{ (N/m)} \qquad \mathbf{F} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} \text{ (N)}$$

$$\begin{bmatrix} \mathbf{K}\text{-}\omega\text{^}2\,\mathbf{M} & \omega\,\mathbf{C} \\ -\omega\,\mathbf{C} & \mathbf{K}\text{-}\omega\text{^}2\,\mathbf{M} \end{bmatrix} = \begin{bmatrix} 10 & -10 & 0 & 20 & -10 & 0 \\ -10 & 10 & -10 & -10 & 20 & -10 \\ 0 & -10 & 0 & 0 & -10 & 10 \\ -20 & 10 & 0 & 10 & -10 & 0 \\ 10 & -20 & 10 & -10 & 10 & -10 \\ 0 & 10 & -10 & 0 & -10 & 0 \end{bmatrix} \text{ (N/m)}$$

$$\textbf{inv of 6x6 above} = \begin{bmatrix} 0.003448 & -0.02414 & -0.03448 & -0.04138 & -0.01034 & 0.013793 \\ -0.02414 & -0.03103 & -0.05862 & -0.01034 & -0.02759 & 0.003448 \\ -0.03448 & -0.05862 & -0.05517 & 0.013793 & 0.003448 & -0.03793 \\ 0.041379 & 0.010345 & -0.01379 & 0.003448 & -0.02414 & -0.03448 \\ 0.010345 & 0.027586 & -0.00345 & -0.02414 & -0.03103 & -0.05862 \\ -0.01379 & -0.00345 & 0.037931 & -0.03448 & -0.05862 & -0.05517 \end{bmatrix} \text{ (m/N)}$$

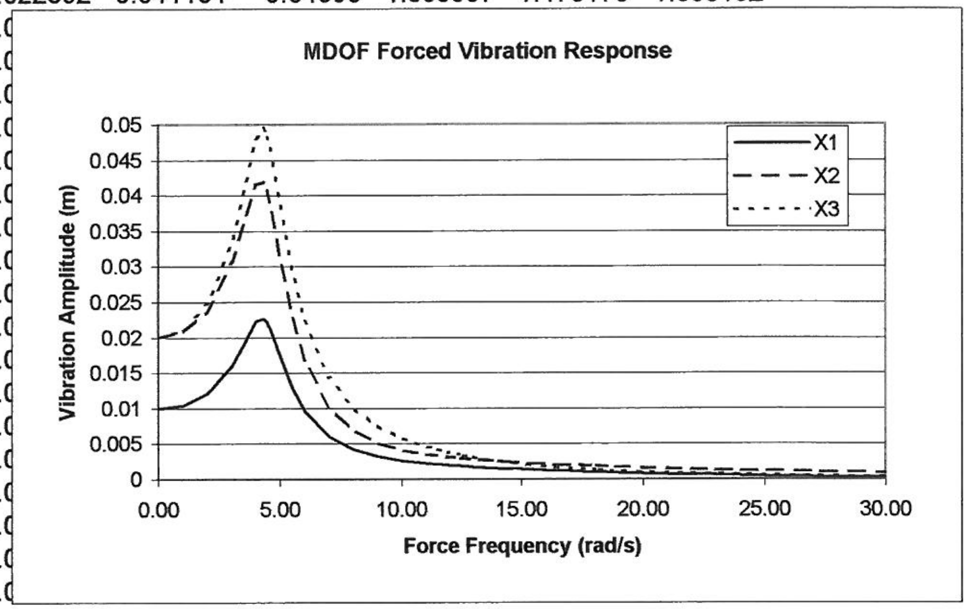
$$\begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ (N)} \qquad \text{Solution } \begin{bmatrix} \mathbf{X1} \\ \mathbf{X2} \end{bmatrix} \begin{matrix} \text{(cos)} \\ \text{(sin)} \end{matrix} = \begin{bmatrix} -0.00241 \\ -0.0031 \\ -0.00586 \\ 0.001034 \\ 0.002759 \\ -0.00034 \end{bmatrix} \text{ (m)}$$

Solution

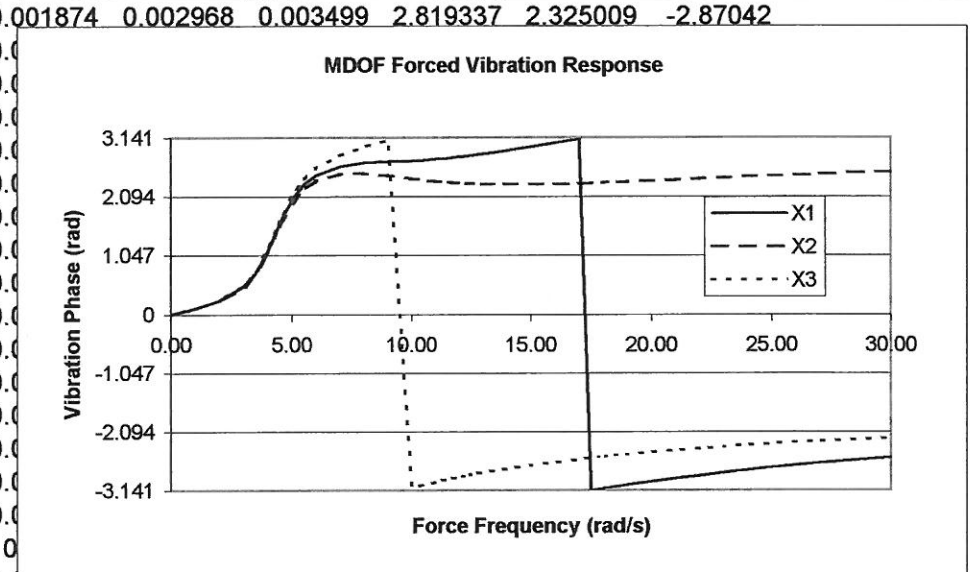
	X amplitude (m)			X phase (rad)		
ω (rad)	X_1	X_2	X_3	X_1	X_2	X_3
10	0.002626	0.004152	0.005872	2.736701	2.41495	-3.08284

	X amplitude (m)			X phase (rad)		
ω (rad)	X_1	X_2	X_3	X_1	X_2	X_3
0.00	0.01	0.02	0.02	0	0	0
1.00	0.010468	0.020833	0.021042	0.104877	0.10438	0.10538
2.00	0.012128	0.023789	0.02474	0.245025	0.241104	0.249104

3.00	0.01597	0.030624	0.033368	0.500389	0.487471	0.514464
3.50	0.019159	0.036235	0.040636	0.738088	0.71789	0.760739
4.00	0.022288	0.04152	0.048065	1.115671	1.08605	1.149963
4.30	0.022689	0.041865	0.049449	1.407757	1.371384	1.450724
4.40	0.022392	0.041184	0.04898	1.508987	1.470173	1.555152
4.45	0.(					
4.50	0.(					
4.60	0.(					
5.00	0.(					
5.50	0.(					
6.00	0.(					
7.00	0.(					
8.00	0.(					
9.00	0.(					
10.00	0.(					
10.50	0.(					
11.00	0.(					
11.30	0.(					
11.50	0.(					
11.70	0.(					
11.90	0.(					
12.00	0.(					
12.20	0.(					



12.40	0.001874	0.002968	0.003499	2.819337	2.325009	-2.87042
12.45	0.(					
12.50	0.(					
12.60	0.(					
12.80	0.(					
13.00	0.(					
13.50	0.(					
14.00	0.(					
15.00	0.(					
16.00	0.(					
17.00	0.(					
17.50	0.(					
17.80	0.(					
18.00	0.(					
18.05	0.(					
18.10	0.(					
18.20	0					



18.50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
18.70	0.000938	0.001782	0.001211	-3.06036	2.351046	-2.51309
19.00	0.000908	0.001745	0.001161	-3.04198	2.355972	-2.50119
20.00	0.000816	0.001631	0.001012	-2.98284	2.373052	-2.46369
21.00	0.000734	0.001527	0.000887	-2.92703	2.390796	-2.42916
22.00	0.000661	0.001432	0.000782	-2.87453	2.408852	-2.39718
23.00	0.000596	0.001345	0.000693	-2.82523	2.426963	-2.36743
24.00	0.000539	0.001265	0.000617	-2.77896	2.44494	-2.33966
25.00	0.000489	0.001191	0.000552	-2.73554	2.462645	-2.31366
27.00	0.000404	0.00106	0.000446	-2.6565	2.496875	-2.26628
30.00	0.000308	0.000898	0.000332	-2.55473	2.544539	-2.20487