Institute of Technology
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Modal Analysis

What we did on the computers last class

Sec. 4.2-4.6

Free Vibration Solution

- In Sec. 4.1 we solved the system differential matrix equation:
- by assuming:
- resulting in the equation:
- which was used to solve for natural frequencies:
- and mode shapes:

Eigenvalues and Eigenvectors

- In Linear Algebra, the Eigenvalue problem is:
  - Given:
    - Matrix **A**
    - Matrix equation $\mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v}$
  - Find solutions for:
- A lot of knowledge is available in mathematics about Eigenvalue problems.

Mapping System Equation to Eigenvalue problem

1. Solve for **L** such that $\mathbf{M} = \mathbf{L}\,\mathbf{L}^T$
   - This can be done using a "Cholesky decomposition"
   - This is like solving for the square-root of **M**.
   - Let's use '$\mathbf{M}^{1/2}$' to refer to **L**.
2. Solving for inverse of **L**:
   - $\mathbf{M}^{-1/2}$ = inverse(**L**)

Mapping System Equation to Eigenvalue problem

3. Introduce new function of time $\mathbf{q}(t)$ such that:

$$\mathbf{q}(t) = \mathbf{M}\,\mathbf{x}(t) \quad \text{or} \quad \mathbf{x}(t) = \mathbf{M}^{-1/2}\,\mathbf{q}(t)$$

4. Substituting into system differential equation and pre-multiplying by $\mathbf{M}^{-1/2}$:

Mapping System Equation to Eigenvalue problem

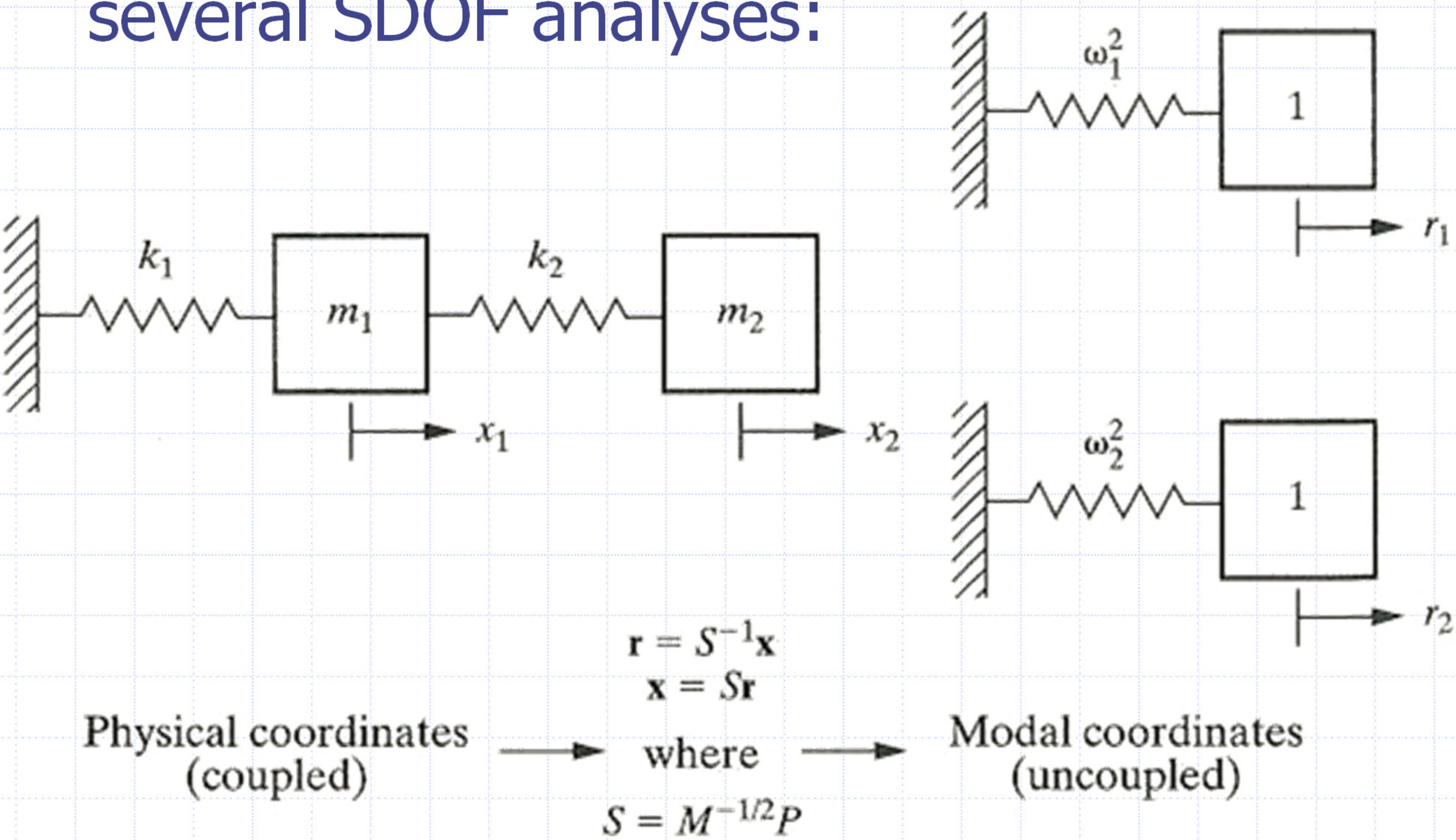
5. Assume solution $\mathbf{q}(t) = \mathbf{v}\, e^{j\omega t}$. Therefore:

Mapping System Equation to Eigenvalue problem

- Therefore, to map the system equation to an Eigenvalue problem:
  - $\mathbf{A} =$
- After solving for λ and $\mathbf{v}$:
  - $\omega_{ni} =$
  - $\mathbf{u}_i =$

Mapping Eigenvalue problem to SDOF problem

- The Eigenvalues and Eigenvectors can be used to simplify a MDOF analysis into a several SDOF analyses:



Mapping Eigenvalue problem to SDOF problem

- Define matrix of eigenvectors:

  $\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \ldots\ \mathbf{v}_n]$

- Matrix of mode shapes is:

  $\mathbf{S} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \ldots\ \mathbf{u}_n] = \underline{\qquad}\ \mathbf{P}$

- Define modal coordinates **r** such that:

  $\mathbf{q}(t) = \mathbf{P}\ \mathbf{r}(t)$

Mapping Eigenvalue problem to SDOF problem

- Substituting **P** **r**(t) for **q**(t) in system equation and pre-multiplying by $\mathbf{P}^T$ yields:

Mapping Eigenvalue problem to SDOF problem

- We therefore have a nice set of SDOF equations:

Modal Analysis

- To solve with initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, use:

  $\mathbf{r}(0) =$

  $\dot{\mathbf{r}}(0) =$

- To get back $\mathbf{x}(t)$ from $\mathbf{r}(t)$, use:

  $\mathbf{x}(t) =$

Modal Analysis Procedure

1. Calculate $M^{-1/2}$.
2. Calculate $\widetilde{K} = M^{-1/2}KM^{-1/2}$, the mass normalized stiffness matrix.
3. Calculate the symmetric eigenvalue problem for $\widetilde{K}$ to get ω_i^2 and $\mathbf{v}_i$.
4. Normalize $\mathbf{v}_i$ and form the matrix $P = [\mathbf{v}_1 \quad \mathbf{v}_2]$.
5. Calculate $S = M^{-1/2}P$ and $S^{-1} = P^T M^{1/2}$.
6. Calculate the modal initial conditions: $\mathbf{r}(0) = S^{-1}\mathbf{x}_0, \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}_0$.
7. Substitute the components of $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ into equations (4.66) and (4.67) to get the solution in modal coordinate $\mathbf{r}(t)$.
8. Multiply $\mathbf{r}(t)$ by S to get the solution $\mathbf{x}(t) = S\mathbf{r}(t)$.

Note that S is the matrix of mode shapes and P is the matrix of eigenvectors.

Comparing System Representations

	Original Problem	Eigenvalue Prob.	Modal Problem
System D.E.	$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}$	$\boldsymbol{I}\ddot{\boldsymbol{q}} + \widetilde{\boldsymbol{K}}\boldsymbol{q} = \boldsymbol{0}$	$\boldsymbol{I}\ddot{\boldsymbol{r}} + \boldsymbol{\Lambda}\boldsymbol{r} = \boldsymbol{0}$
Form of sol.	$\boldsymbol{x}(t) = \boldsymbol{u}e^{i\omega_n t}$	$\boldsymbol{q}(t) = \boldsymbol{v}e^{i\omega_n t}$	$\boldsymbol{r}(t) = \boldsymbol{A}e^{i\omega_n t}$
After sub.	$(\boldsymbol{K} - \omega_n{}^2\boldsymbol{M})\boldsymbol{u} = \boldsymbol{0}$	$\widetilde{\boldsymbol{K}}\boldsymbol{v} = \omega_n{}^2\boldsymbol{v}$	$\boldsymbol{A}$ is from I.C.

$$\widetilde{\boldsymbol{K}} = \boldsymbol{M}^{-1/2}\boldsymbol{K}\boldsymbol{M}^{-1/2} \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \omega_{n1}{}^2 & 0 & 0 \\ 0 & \omega_{n2}{}^2 & 0 \\ 0 & 0 & \omega_{n3}{}^2 \end{bmatrix}$$

$$\boldsymbol{u}_i = \boldsymbol{M}^{-1/2}\boldsymbol{v}_i \qquad \boldsymbol{r}(0) = \boldsymbol{S}^{-1}\boldsymbol{x}(0)$$

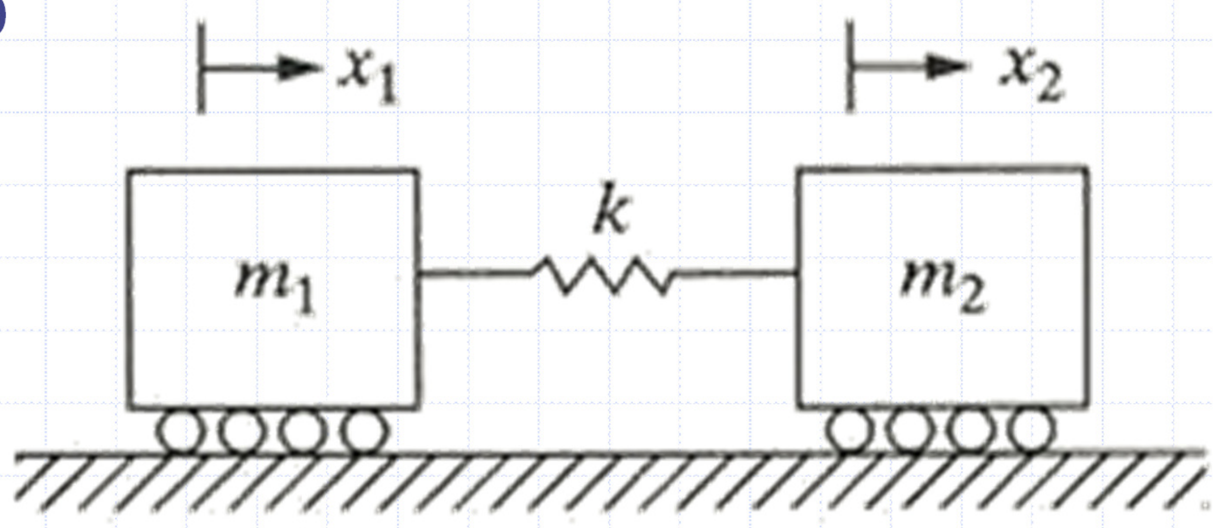
$$\boldsymbol{S} = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3] \qquad \boldsymbol{x}(t) = \boldsymbol{S}\boldsymbol{r}(t)$$

“Nodes” of a Mode

- These are places where the mode shape is zero.
- Not a good place to mount a sensor or actuator for body motion.
- Good place to mount devices that shouldn’t receive or transmit vibrations at the given natural frequency.

Rigid-Body Modes

- Appear as natural frequencies with value of zero



- Require special treatment when evaluating motion from initial conditions (see p. 314 in text)

Viscous Damping

- It is relatively difficult to model individual dampers in a Modal Analysis.
- Some "tricks" are available:
  - "Modal damping" (apply damping ζ_i to system equation for each mode **in modal coordinates** r(t))
  - "Proportional damping" (**C** = α**M** + β**K**, with α and β chosen freely)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \qquad i = 1, 2, \ldots, n$$

Forced Response

- Forces can be mapped to modal equations

1. $$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = B\mathbf{F}(t)$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{bmatrix}$$

2. $$I\ddot{\mathbf{q}}(t) + \widetilde{C}\ddot{\mathbf{q}}(t) + \widetilde{K}\mathbf{q}(t) = M^{-1/2}B\mathbf{F}(t)$$

$$\widetilde{C} = M^{-1/2}CM^{-1/2}.$$

3. $$\ddot{\mathbf{r}}(t) + \text{diag}[2\zeta_i\omega_i]\dot{\mathbf{r}}(t) + \Lambda\mathbf{r}(t) = P^T M^{-1/2}B\mathbf{F}(t)$$

$$\ddot{r}_i(t) + 2\zeta_i\omega_i\dot{r}_i(t) + \omega_i^2 r_i(t) = f_i(t)$$