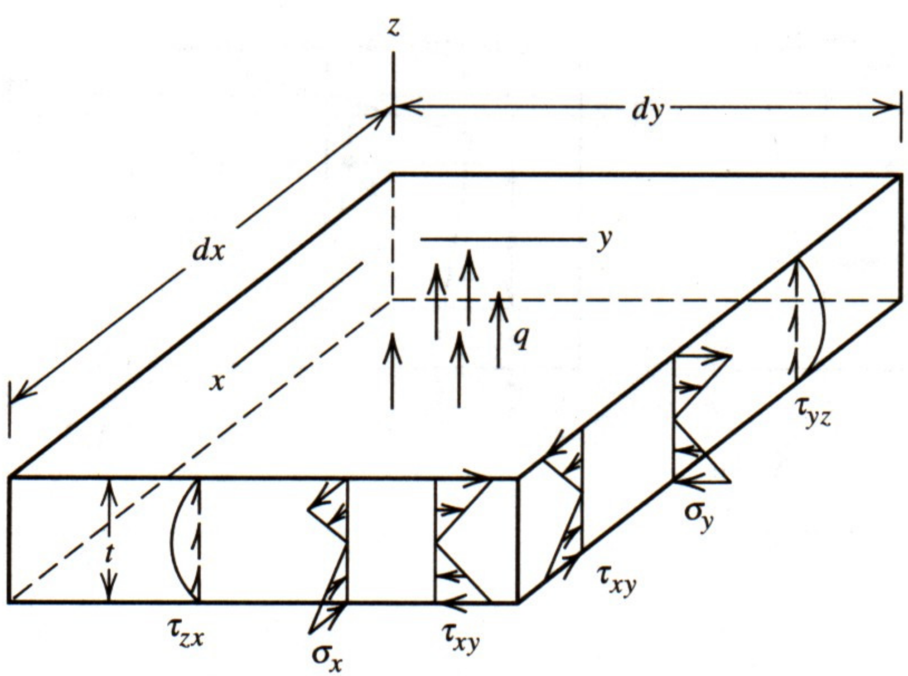
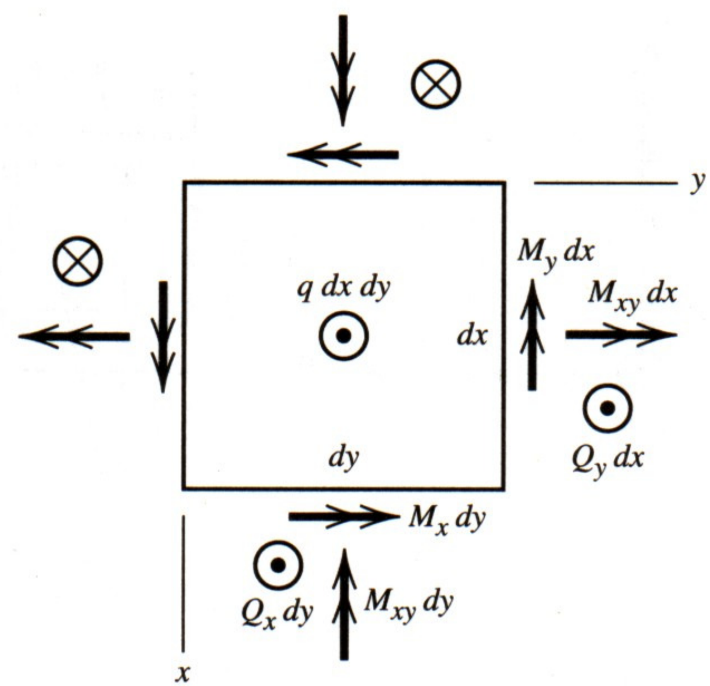
Plates and Shells

All images are from R. Cook, et al. *Concepts and Applications of Finite Element Analysis*, 1996.

Plate Formulation

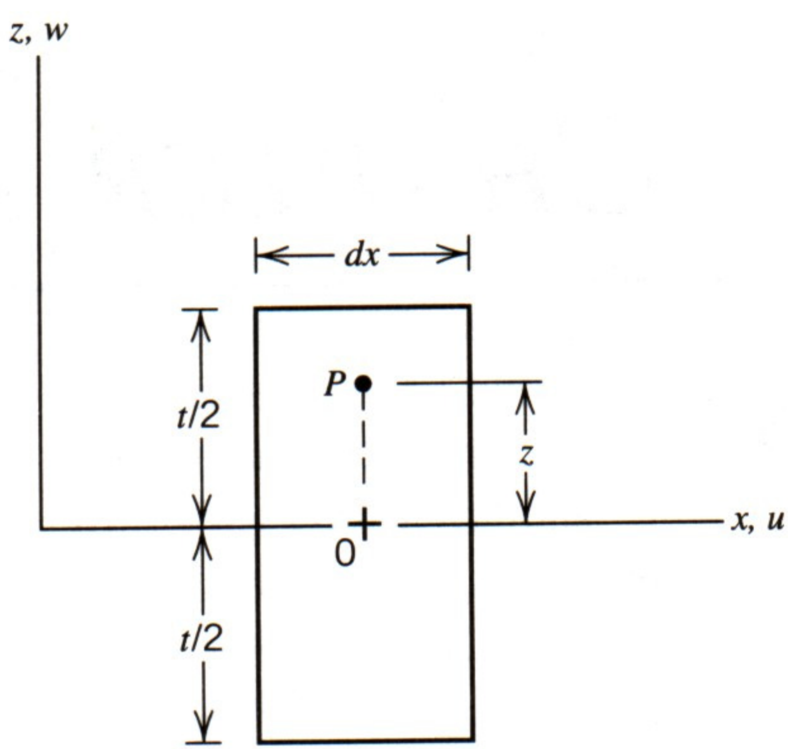
- Plates may be considered similar to beams, however:
  - Plates can bend in two directions
  - Plates are flat with a thickness (can't have an interesting cross-section)

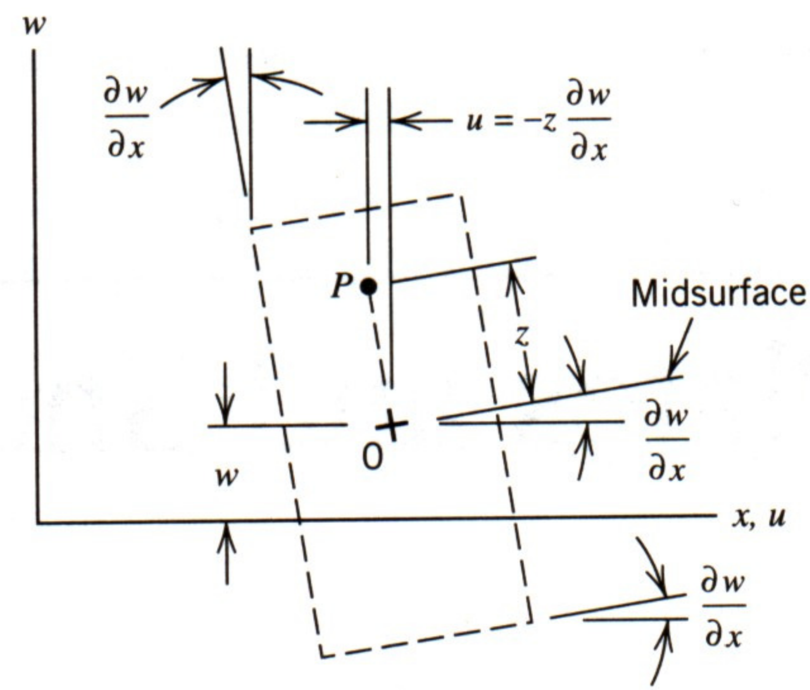




Thin Plate Formulation

- Consider a thin plate on the xy plane (z = 0), with thickness t, & neglecting shear strain.
- If we take a differential slice from plate:





Thin Plate Formulation

then:

$$w = w(x, y)$$

$$u = -z\frac{\partial w}{\partial x}$$

$$v = -z\frac{\partial w}{\partial y}$$

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}$$

$$\gamma_{yz} = \gamma_{zx} = 0$$

- Assume $\sigma_z = 0$. Therefore:

$$\begin{Bmatrix}\sigma_x \\ \sigma_y\end{Bmatrix} = -z\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu \\ \nu & 1\end{bmatrix}\begin{Bmatrix}\partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2\end{Bmatrix} \qquad \tau_{xy} = -2zG\frac{\partial^2 w}{\partial x\,\partial y}$$

Thin Plate Formulation

- These stresses give rise to moments:

$$M_x = \int_{-t/2}^{t/2} \sigma_x \, z \, dz \qquad M_y = \int_{-t/2}^{t/2} \sigma_y \, z \, dz \qquad M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} \, z \, dz$$

- Maximum stresses are therefore given by:

$$\sigma_{x,\max} = \frac{6M_x}{t^2} \left(\text{since } \sigma_x = \frac{2z}{t} \sigma_{x,\max} \right),$$

$$\sigma_{y,\max} = \frac{6M_y}{t^2},$$

$$\tau_{xy,\max} = \frac{6M_{xy}}{t^2}$$

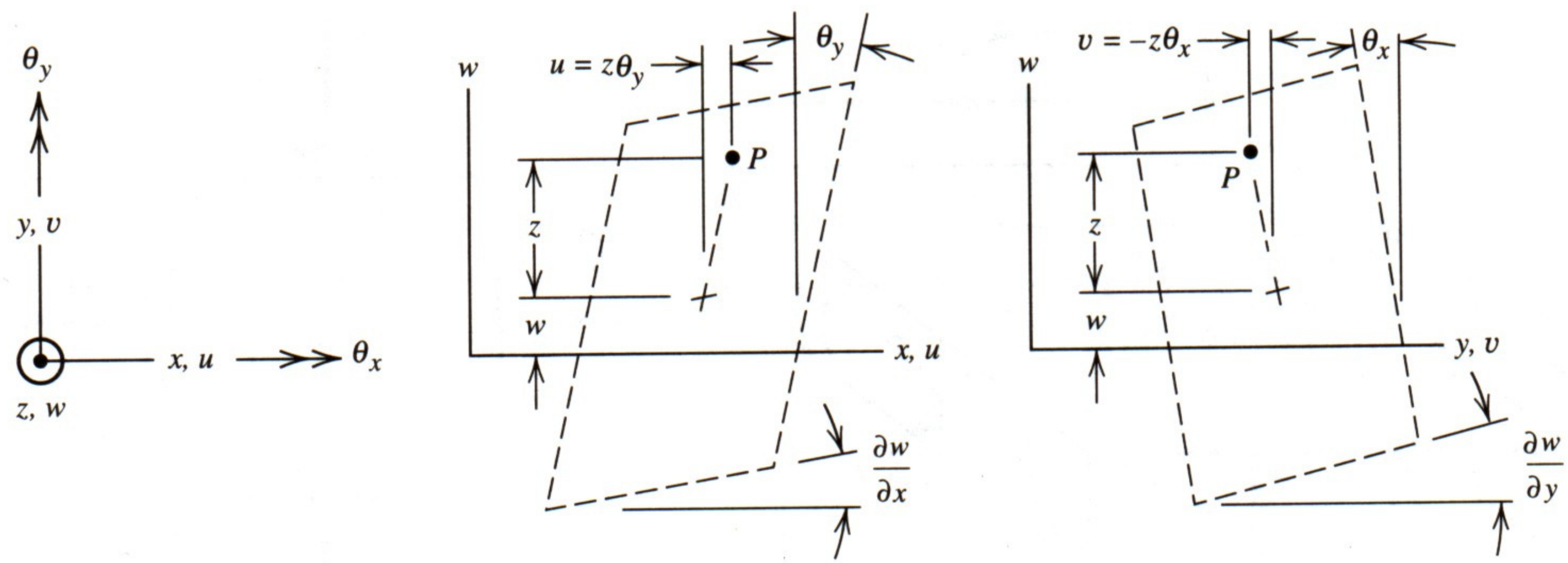
Thin Plate Formulation

- This is similar to the beam formula, but since the plate is very wide we have a situation similar to plain strain.
- For a unit width beam, flexural rigidity $D=EI=Et^3/12$.
- For a unit width plate, flexural rigidity $D=EI/(1-\nu^2)=Et^3/[12(1-\nu^2)]$.

- This thin plate theory is also called the "**Kirchhoff plate theory**."

Mindlin Plate Theory

- Mindlin Plate Theory assumes that **transverse shear** deformation also occurs.



Mindlin Plate Theory

- The deformations and strains are therefore given by:

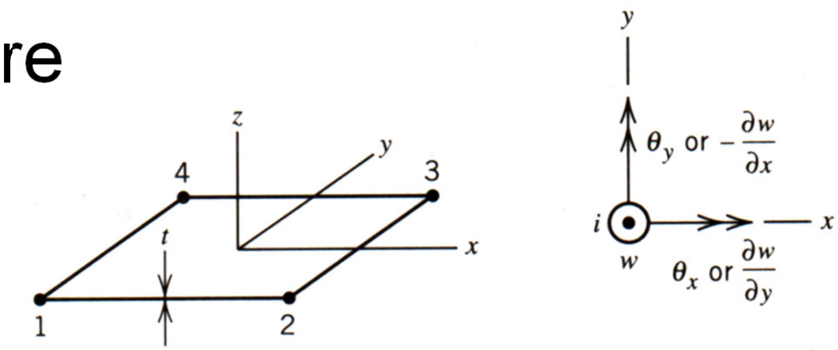
$$u = z\theta_y \qquad \varepsilon_x = z\frac{\partial \theta_y}{\partial x} \qquad \gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$v = -z\theta_x \qquad \varepsilon_y = -z\frac{\partial \theta_x}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \theta_y$$

Mindlin Plate Theory

- Mindlin plate elements are more common than Kirchhoff elements.



- The displacement interpolation is given by:

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \mathbf{Nd}$$

N_i can be the same shape functions as for Q4 and Q8 quadrilateral elements.

Support Conditions

- Support Conditions are similar to those for beams:

Edge condition	Prescribed d.o.f.	Natural condition
Clamped	$w = \theta_n = \theta_s = 0$	None
Simply supported	$w = 0$	$M_n = 0$
Free	None	$Q = M_n = M_{ns} = 0$

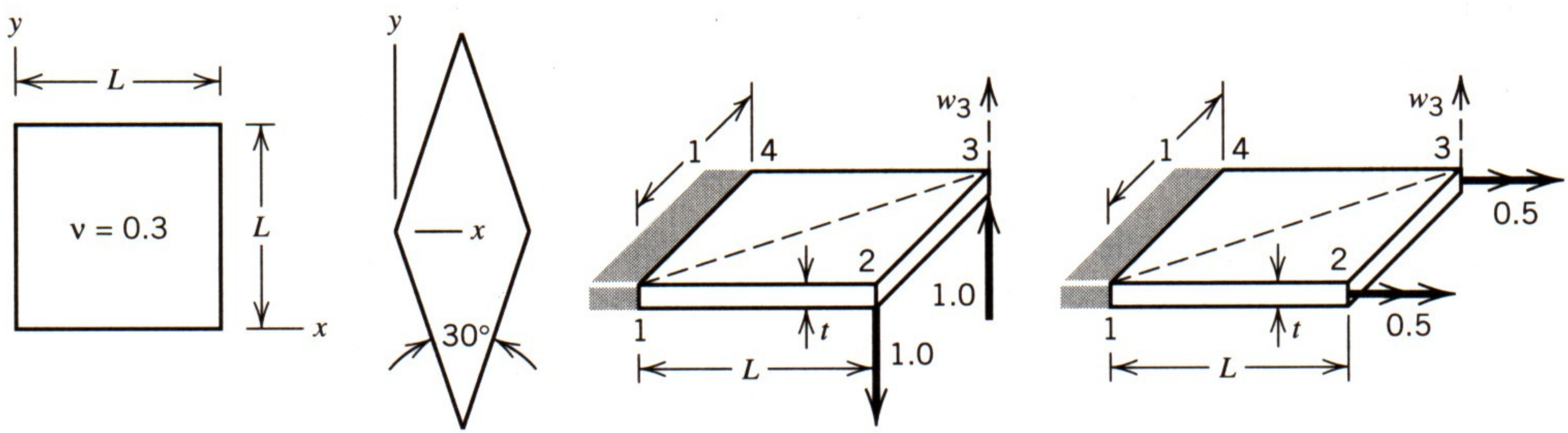
θ_n, M_n – rotation and moment normal to edge

θ_s, M_s– rotation and moment perpendicular to edge

For Mindlin plates, do not restrain θ_n, to avoid accuracy problems.

Test Cases

- For plate elements, patch tests and single element tests should include the cases shown:



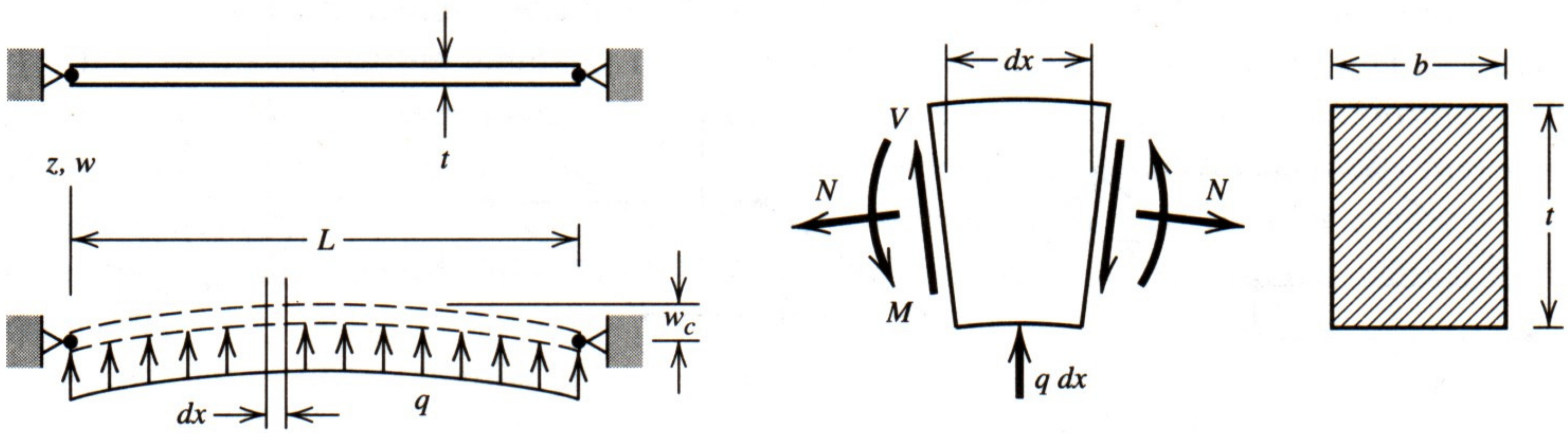
Test Cases

- Plate elements must be able to show constant σ_x, σ_y and τ_{xy} at each z level to pass a patch test. They must pass the test for constant M_x, M_y and M_{xy}.

- Many element formulations perform poorly for these tests.

Large Displacements and Membrane Forces

- A beam with fixed supports will exhibit "**string action**" axial forces as shown.



- If we consider both string action and bending stresses, a beam can carry a distributed load of:

$$q = q_s + q_b \approx \frac{Ebt^4}{L^4}\left[21.3\left(\frac{w_c}{t}\right)^3 + 6.40\left(\frac{w_c}{t}\right)\right]$$

Large Displacements and Membrane Forces

- A similar situation arises with plates, however basic plate elements are not set up to handle "**membrane**" forces.
- If w/t is large (e.g., greater than 0.1), a **non-linear analysis** must be performed using **shell elements**, which do handle membrane forces.
- In general, tensile membrane forces will have a stiffening effect and compressive membrane forces will decrease stiffness.

Shell Finite Elements

- Shell elements are different from plate elements in that:
  - They carry **membrane AND bending** forces
  - They can be curved
- The most simple shell element **combines** a bending element with a membrane element.
  - E.g., combines a **plate element** and a **plane stress** element.
  - These elements are flat, therefore it is important that elements are not all coplanar where they meet at a node.

Shell Finite Elements

- Curved shell elements can be derived using “shell theory.”
- “Isoparametric” shell elements can also be obtained by starting with a solid element and reducing degrees of freedom.

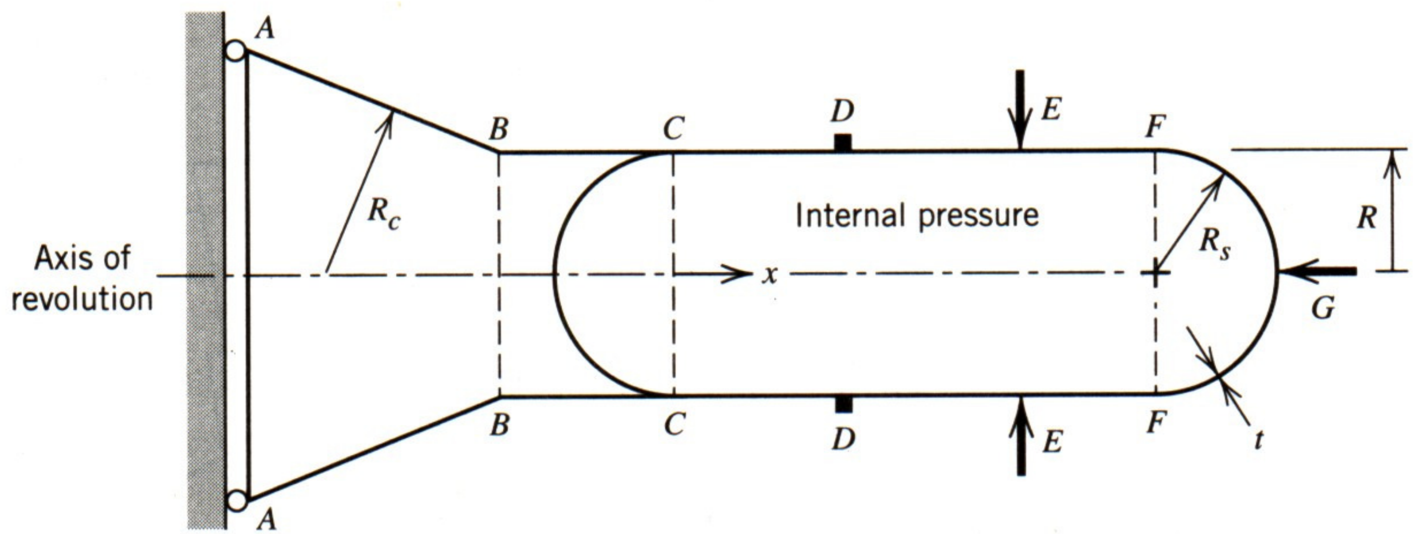
- Thin shell behavior varies widely between formulations and should be tested before use.

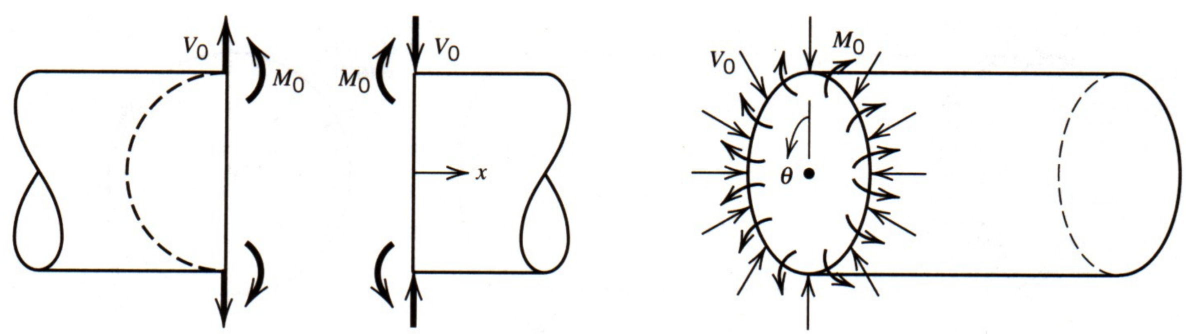
Shells and Shell Theory

- A thin shell structure can carry high loads if membrane stresses predominate.
- However, localized bending stresses will appear near load concentrations or geometric discontinuities.

Shells and Shell Theory

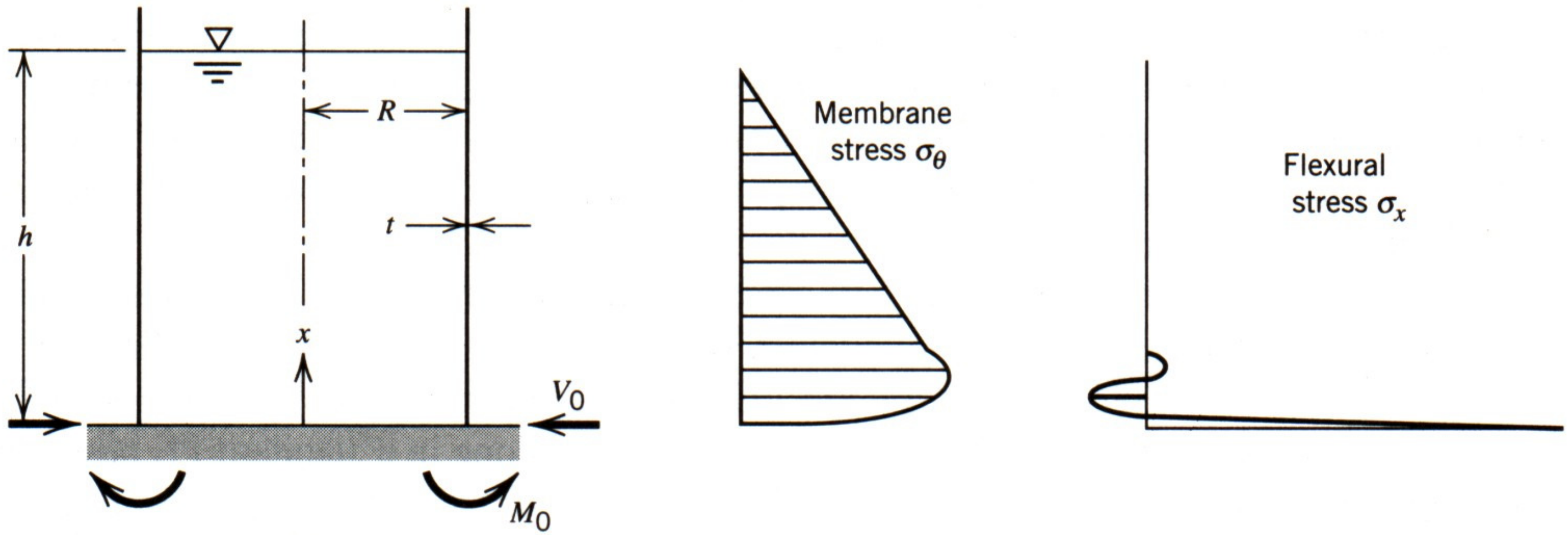
- Localized bending stresses appear in many different situations:





Shells and Shell Theory

- A thin-walled cylindrical tank has high bending (flexural) stresses at the base.



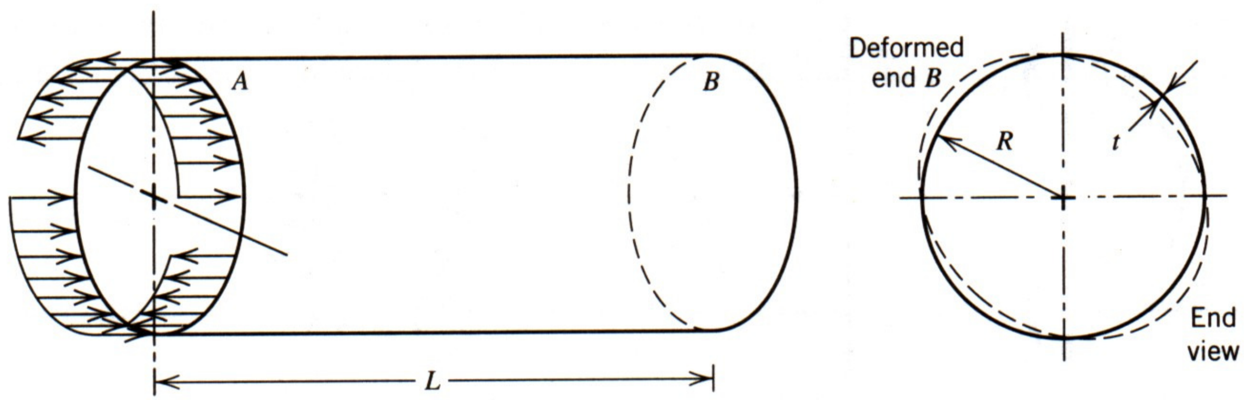
- Use a finer mesh where there are discontinuities or abrupt changes in the structure.

Shells and Shell Theory

- For a cylindrical shell of radius R and thickness t, the localized bending dies out after a distance λ:

$$\lambda = \left[\frac{3(1-\nu^2)}{R^2 t^2} \right]^{1/4}$$

- Membrane stresses do not die out.

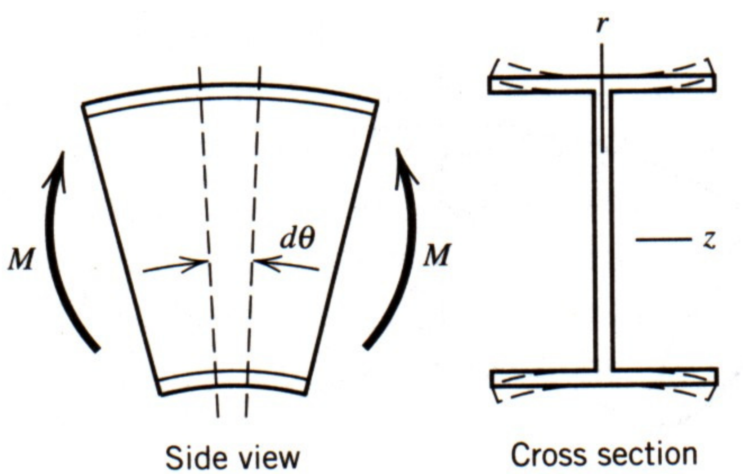


Using Shell Elements to Model Beams

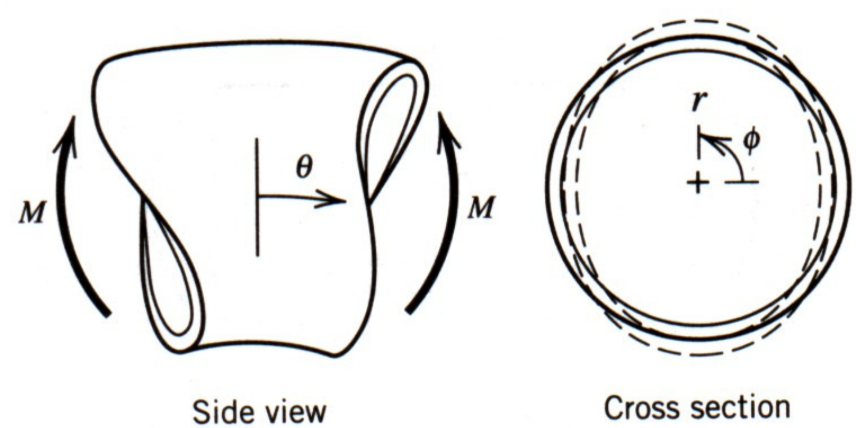
- To do a proper FE analysis, the analyst must understand how the structure is *likely* to behave and how elements are *able* to behave.
- In some cases it is more appropriate use shell elements rather than beam elements.

Using Shell Elements to Model Beams

- A curved I-beam reacts to moments as shown, therefore shell elements would be more accurate than beam elements.
- Pipe bends react to moments as shown. Use shell elements or specialized beam elements with correction factors.



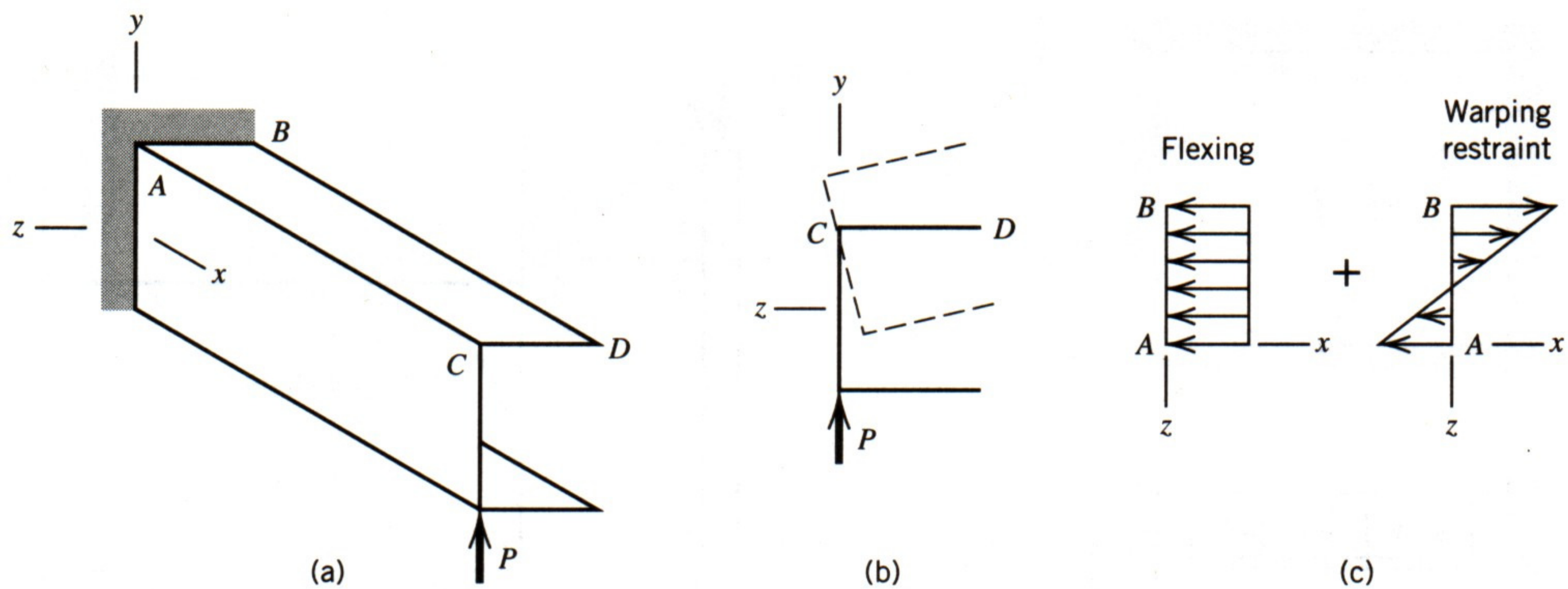
Curved I beam



Pipe bend

Using Shell Elements to Model Beams

- If the load is not applied directly below the "shear center", the channel will twist. Use shell elements instead of beam elements.



Using Shell Elements to Model Beams

- If beam flanges are wide, $\sigma_x = My/I$ is not accurate. Beam elements will not give accurate results.
- In this case, plate/shell elements should be used.