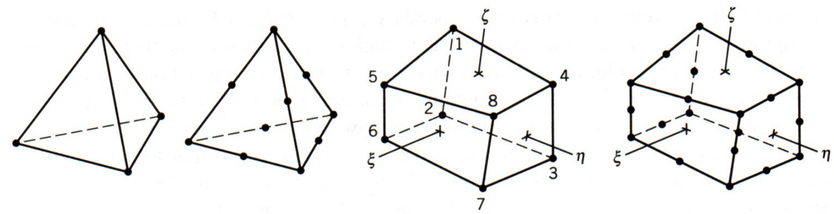
Solid Elements



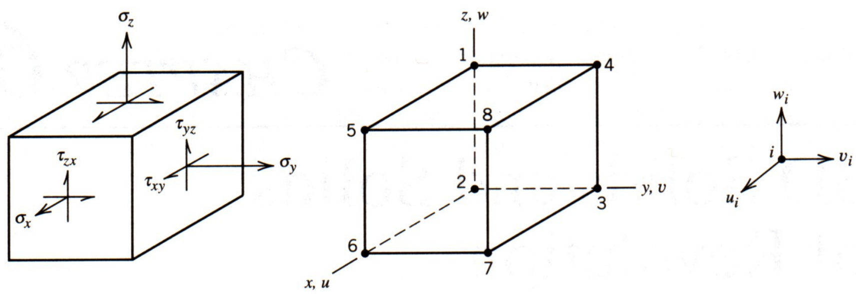
All images are from R. Cook, et al. *Concepts and Applications of Finite Element Analysis*, 1996.

Introduction

- Problems of beam bending, plane stress, plates, etc. may be considered as special cases of 3D solids.
- So why not use 3D solids all the time?

Introduction

- Stress can vary in all three directions
- Nodes are located in 3 space
- Nodes have displacements in 3 directions



Stress-strain relations

- Stress-strain are now related by a 6x6 matrix

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} (1-\nu)c & \nu c & \nu c & 0 & 0 & 0 \\ & (1-\nu)c & \nu c & 0 & 0 & 0 \\ & & (1-\nu)c & 0 & 0 & 0 \\ & & & G & 0 & 0 \\ & \text{symmetric} & & & G & 0 \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$c = \frac{E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)}$$

Stress-strain relations

- To get strain from stress, use:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Strain-Displacement Relations

- If strains are small:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

- These are the most general equations. In deriving previous elements we made assumptions about some of these values (e.g., in plane strain we assumed ε_z=0)

Displacement Interpolation

- Displacements within an element are interpolated from nodal displacements using **u**=**Nd**, as before, however there are now interpolations in three directions.

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}$$

General Formula for k

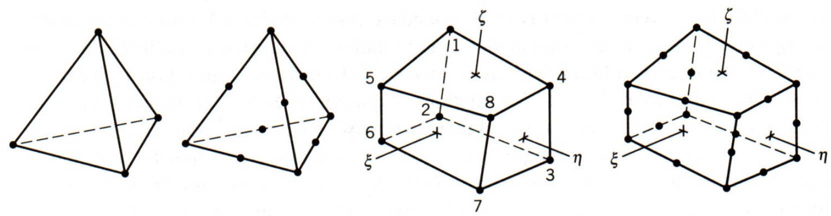
- The general, energy-based formula for **k** is the same as for previous elements, except that there are more terms in the integration due to the bigger matrices.

$$\underset{6\times1}{\boldsymbol{\varepsilon}} = \underset{6\times3n}{\mathbf{B}}\ \underset{3n\times1}{\mathbf{d}}$$

$$\underset{3n\times3n}{\mathbf{k}} = \int\int\int \mathbf{B}^T\ \underset{6\times6}{\mathbf{E}}\ \mathbf{B}\ dx\ dy\ dz$$

3D Solid Elements

- The 3D solid elements are analogous to planar counterparts:
  - 3-Node Triangle → 4-Node Tetrahedron
  - 6-Node Triangle → 10-Node Tetrahedron
  - 4-Node Quadrilateral → 8-Node Hexahedron
  - 8-Node Quadrilateral → 20-Node Hexahedron



4-Node Tetrahedron

- As with the 3-Node Triangle, it is only accurate when strains are almost constant over the element span.

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z$$

$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 z$$

$$w = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} z$$

Trilinear Hexahedron

- Is also called the 8 node “brick” element.

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 z + \beta_5 xy + \beta_6 yz + \beta_7 zx + \beta_8 xyz$$

$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} z + \beta_{13} xy + \beta_{14} yz + \beta_{15} zx + \beta_{16} xyz$$

$$w = \beta_{17} + \beta_{18} x + \beta_{19} y + \beta_{20} z + \beta_{21} xy + \beta_{22} yz + \beta_{23} zx + \beta_{24} xyz$$

- Solving for shape functions:

$$u = \sum N_i u_i \qquad v = \sum N_i v_i \qquad w = \sum N_i w_i$$

$$N_i = \tfrac{1}{8}(1 \pm \xi)(1 \pm \eta)(1 \pm \zeta)$$

- Isoparametrically: $$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, |\mathbf{J}|\, d\xi\, d\eta\, d\zeta$$

Distributed Loading

- The work equivalent nodal loads for a constant distributed pressure p are as shown.