Beam Elements

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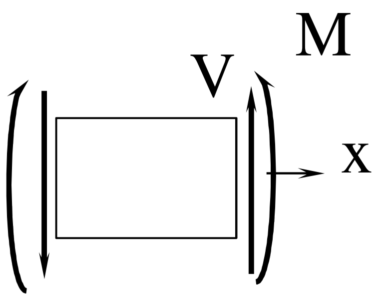
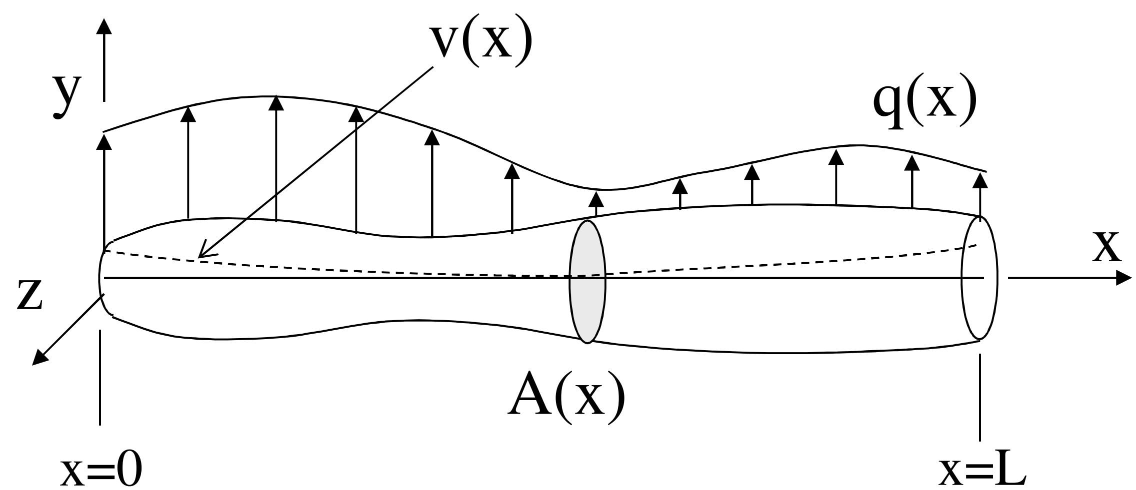
Slides from this set are adapted from B.S. Altan, Michigan Technological U.

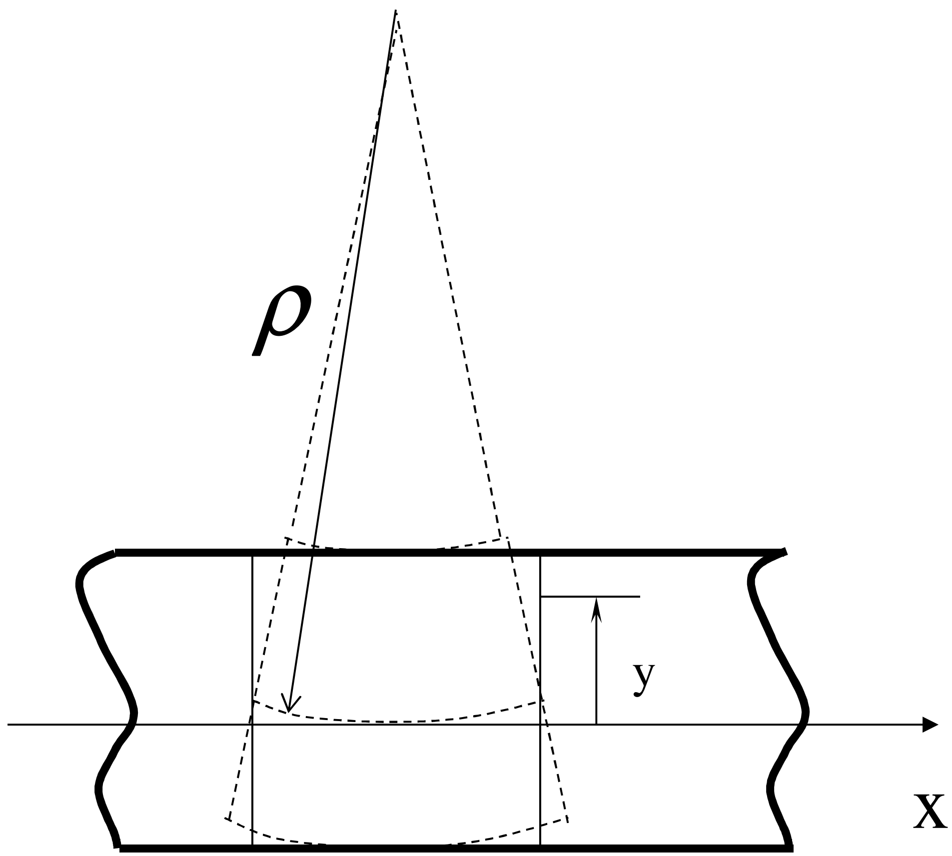
Bending of Beams – Definition of Problem

Beam with a straight axis (x-axis), with varying cross-section (A(x)), loaded transversely (q(x) in y direction) with transverse deflections (v(x) in y direction).





Bending of Beams – Definition of Problem



$$\varepsilon(x) = -\frac{y}{\rho}$$

$$\frac{1}{\rho} \cong \frac{d^2 v}{dx^2}$$

$$\sigma = E\varepsilon = -Ey\frac{d^2 v}{dx^2}$$

$$\int_A \sigma y dA = \int_A -Ey^2 \frac{d^2 v}{dx^2} dA$$

Bending of Beams – Definition of Problem

$$\int_A \sigma y \, dA = \int_A -Ey^2 \frac{d^2 v}{dx^2} dA$$

Continued from previous slide.

$$M = -EI \frac{d^2 v}{dx^2}$$

$$V = -\frac{dM}{dx} = \frac{d}{dx} EI \frac{d^2 v}{dx^2}$$

$$q(x) = \frac{dV}{dx} = \frac{d^2}{dx^2} EI \frac{d^2 v}{dx^2} \leftarrow \text{Resulting Field Equation}$$

Bending of Beams – Definition of Problem

Summary

Field equation: $$EI\frac{d^4v}{dx^4} = q(x)$$

Slope: $$\theta(x) = \frac{dv}{dx}$$

Bending Moment: $$M(x) = EI\frac{d^2v}{dx^2}$$

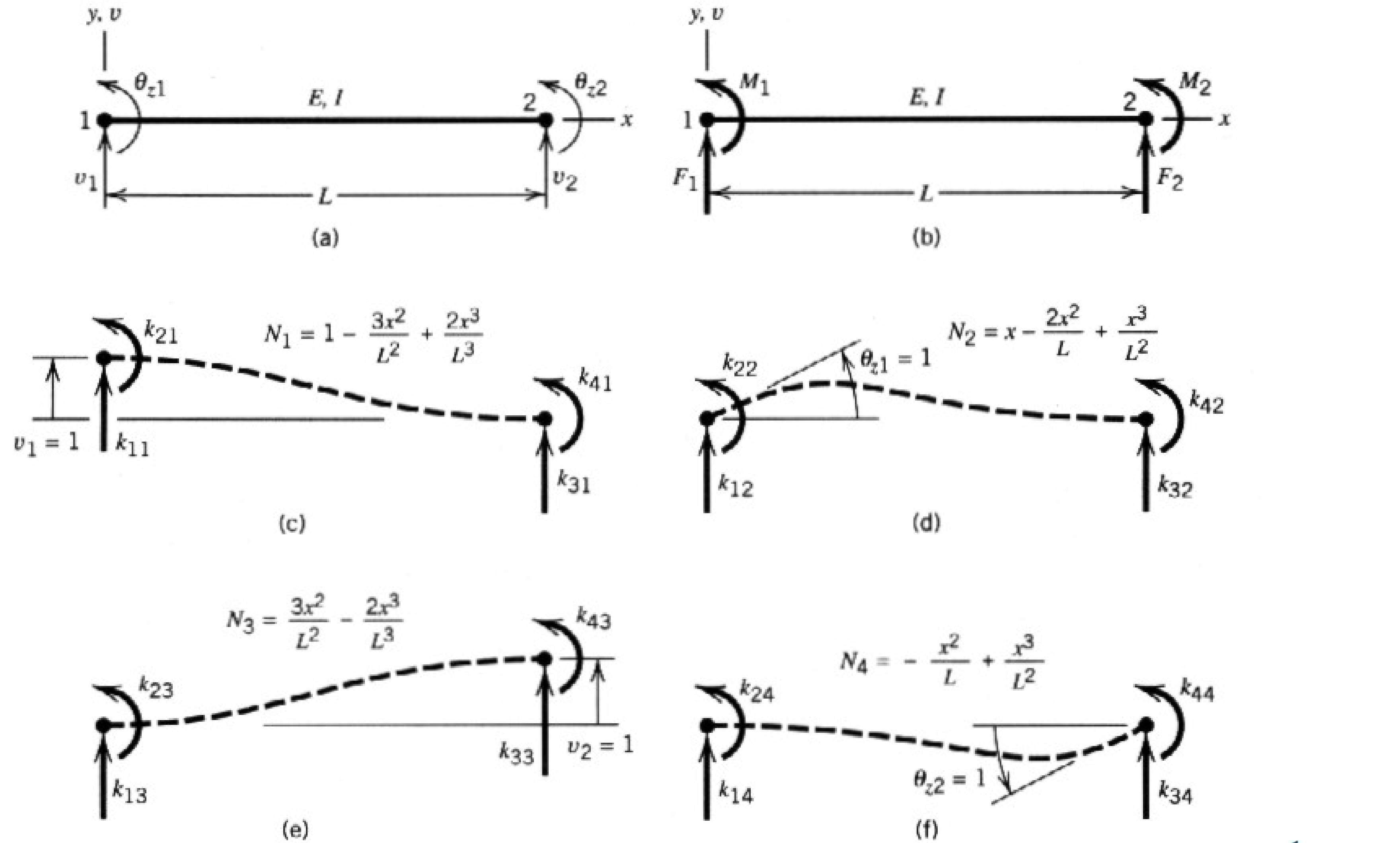
Shear Force: $$V(x) = -EI\frac{d^3v}{dx^3}$$

Bending of Beams – Definition of Problem

Boundary Conditions: Four boundary conditions are necessary to solve a bending problem. Boundary conditions can be:

deflection:	$v(0), v(L)$
slope:	$\theta(0), \theta(L)$
bending Moment:	$M(0), M(L)$
shear force:	$V(0), V(L)$

Beam Element – Shape Functions



- Recall that shape functions are used to interpolate displacements.

Beam Element – Shape Functions

$$v = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \mathbf{N}\mathbf{d}$$

- There are two degrees of freedom (displacements) at each node: v and θ_z.
- Each shape function corresponds to one of the displacements being equal to 'one' and all the other displacements equal to 'zero'.
- Note that everything we do in this course assumes that the displacements are small.

Beam Element – Shape Functions

$$\sigma = E\varepsilon = -Ey\frac{d^2v}{dx^2}$$

$$= -Ey\left[\frac{d^2\mathbf{Nd}}{dx^2}\right] = -Ey\left[\frac{d^2\mathbf{N}}{dx^2}\right]\mathbf{d}$$

$$= -Ey\mathbf{Bd}$$

$$\mathbf{B} = \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & -\frac{4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix}$$

- Recall that the **B** row vector is used to interpolate stresses & strains.

Beam Element – Formal Derivation

- The formal beam element stiffness matrix derivation is much the same as the bar element stiffness matrix derivation. From the minimization of potential energy, we get the formula:

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI\,\mathbf{B}\,dx$$

- As with the bar element, the strain energy of the element is given by $\frac{1}{2}\mathbf{d}^T\mathbf{k}\mathbf{d}$.

Beam Element – Formal Derivation

The result is:

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

which operates on $\mathbf{d} = [v_1, \theta_{z1}, v_2, \theta_{z2}]^T$.

Beam Element – Formal Derivation

- The moment along the element is given by:

$$M = EI \frac{d^2 v}{dx^2} = EI\,\mathbf{Bd}$$

- The stress is given by:

$$\sigma_x = -\frac{My}{I} = -yE\mathbf{Bd}$$

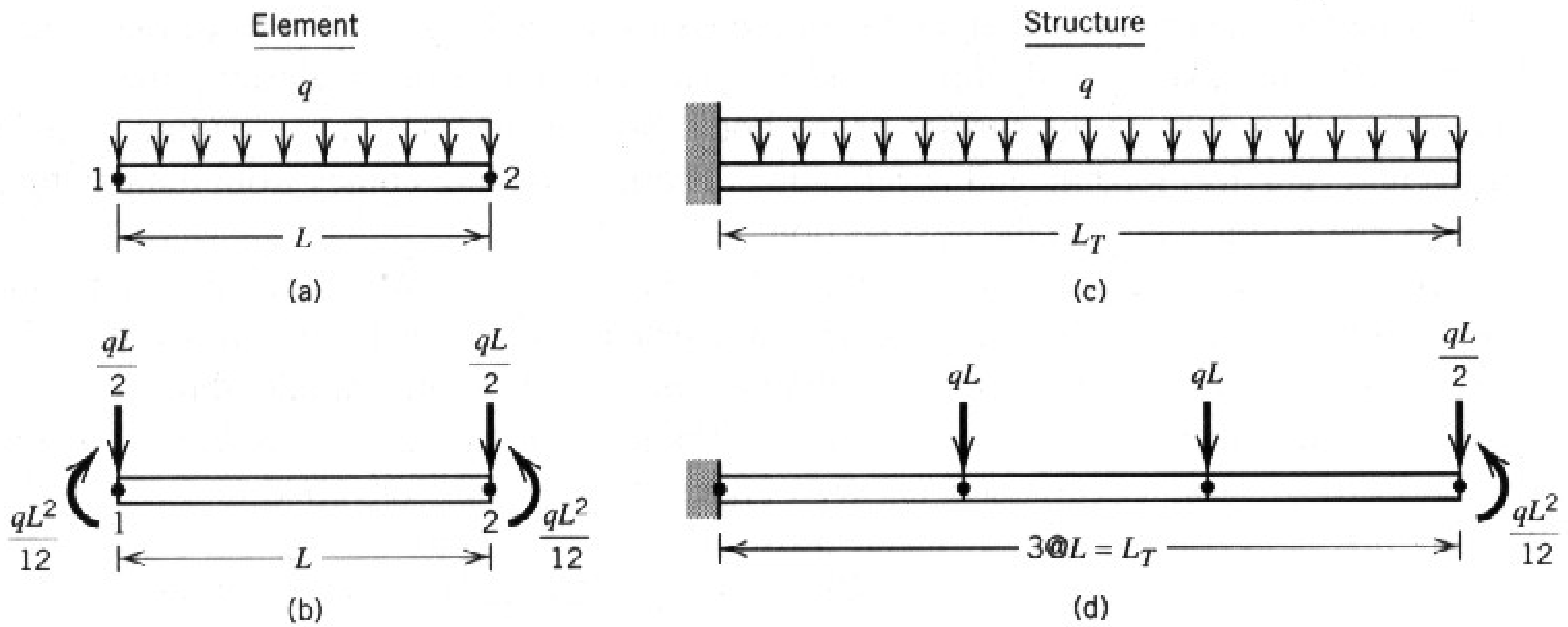
Beam Element w/Axial Stiffness

- If we combine the bar and beam stiffness matrices, we get a general beam stiffness matrix with axial stiffness.

$$\mathbf{k} = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \end{matrix}$$

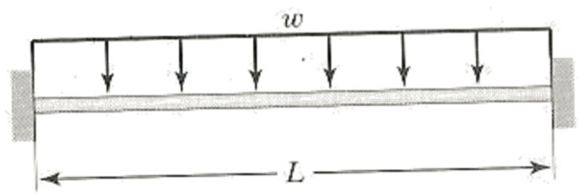
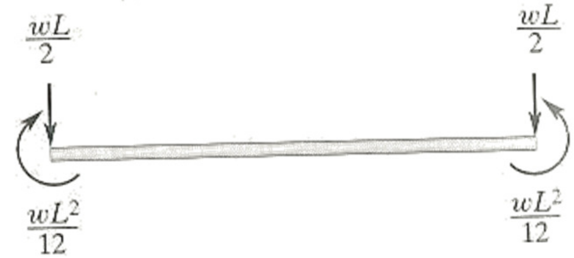
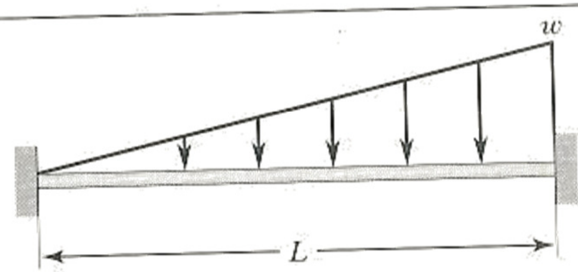
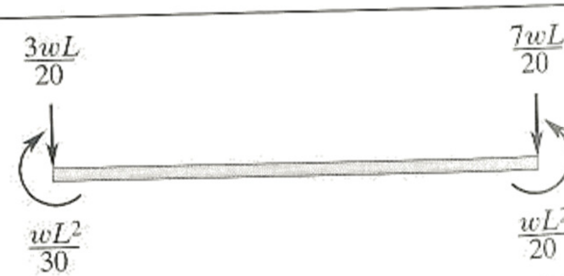
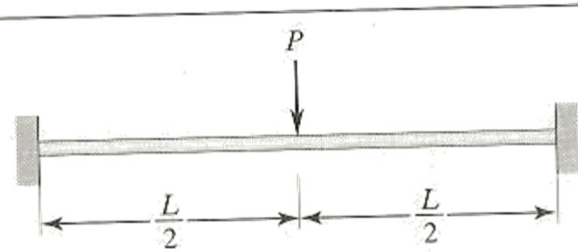
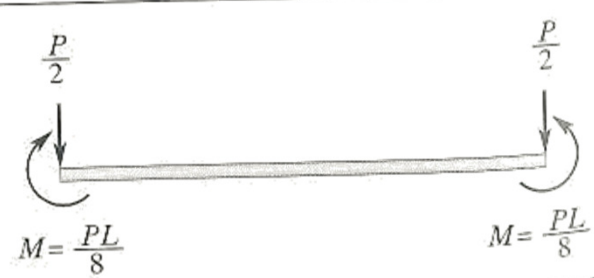
Uniformly Distributed Loads

Laterally:



Equivalent Loadings

TABLE 4.2 Equivalent nodal loading of beams

Loading	Equivalent Nodal Loading
	
	
	

Orientating Element in 3-D Space

- Transformation matrices are used to transform the equations in the element coordinate system to the global coordinate system, as was shown for the bar element.