Find state-space equations for the following differential equations

A. $\ddot{y} + 2\dot{y} + 5y = 7v$

Let $x_1 = y \Rightarrow \dot{x}_1 = \dot{y} = x_2$

$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = -2\dot{y} - 5y + 7v$

$= -2x_2 - 5x_1 + 7v$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

B. $\ddot{y} - 2y = v$

Let $x_1 = y \Rightarrow \dot{x}_1 = \dot{y} = x_2$

$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = -2y + v$

$= -2x_1 + v$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

C. $\dddot{y} + 2\ddot{y} + 3\dot{y} + y = 5v$

Let $x_1 = y \Rightarrow \dot{x}_1 = \dot{y} = x_2$

$x_2 = \dot{y} \Rightarrow \dot{x}_2 = \ddot{y} = x_3$

$x_3 = \ddot{y} \Rightarrow \dot{x}_3 = \dddot{y} = -y - 3\dot{y} - 2\ddot{y} + 5v$

$= -x_1 - 3x_2 - 2x_3 + 5v$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Determine the transfer functions from the following State Space Model

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & 1 \\ 4 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} s+2 & -1 \\ -4 & s+3 \end{bmatrix} \frac{1}{(s+2)(s+3)-4}$$

$$H(s) = C(sI-A)^{-1}B + \cancel{D}^{0} =$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+2 & -1 \\ -4 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s^2+5s+2} =$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+2 \\ -4 \end{bmatrix} \frac{1}{s^2+5s+2} =$$

$$= \frac{(s+2)+4}{s^2+5s+2} = \frac{s+6}{s^2+5s+2}$$

Determine the transfer function from the following State-Space Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix} \frac{1}{s^2+3s+2}$$

$$H(s) = C(sI-A)^{-1}B + \cancel{D}^{\,0} =$$

$$= \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2+3s+2} = \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s^2+3s+2} =$$

$$= \frac{7s+5}{s^2+3s+2}$$

What is the matrix exponential, e^{At} ?

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s+3 & 1 \\ 2 & s \end{bmatrix} \frac{1}{s^2+3s+2}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}\right\}$$

Use P.F.E.

Given the following State-space model, determine the matrix exponential, e^{At}, and solve the state-space model. Find $x(t)$ and $y(t)$.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \qquad \text{Let } v = \text{a step input}$$

$$y = \begin{bmatrix} 5 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First, find $(sI-A)^{-1}$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{s^2+3s+2}$$

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}\right\}$$

use Partial Fraction expansion

$$G_{11} = \frac{s+3}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} \Rightarrow k_1 = G_{11}(s+1)\big|_{s=-1} = \frac{2}{1} = 2$$

$$k_2 = G_{11}(s+2)\big|_{s=-2} = -1$$

$$G_{12} = \frac{1}{(s+1)(s+2)} = \frac{k_3}{s+1} + \frac{k_4}{s+2} \Rightarrow k_3 = G_{12}(s+1)\big|_{s=-1} = 1$$

$$k_4 = G_{12}(s+2)\big|_{s=-2} = -1$$

$$G_{13} = \frac{-2}{(s+1)(s+2)} = \frac{k_5}{s+1} + \frac{k_6}{s+2} \Rightarrow k_5 = G_{13}(s+1)\big|_{s=-1} = -2$$

$$k_6 = G_{13}(s+2)\big|_{s=-2} = 2$$

$$G_{14} = \frac{s}{(s+1)(s+2)} = \frac{k_7}{s+1} + \frac{k_8}{s+2} \Rightarrow k_7 = G_{14}(s+1)\big|_{s=-1} = -1$$

$$k_8 = G_{14}(s+2)\big|_{s=-2} = 2$$

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}\right\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x(t) = \underbrace{\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}_{e^{At}} x(0) + \mathcal{L}^{-1}\left\{(sI-A)^{-1} B\, V(s)\right\}$$

$$(sI-A)^{-1} B\, V(s) = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[\tfrac{1}{s}\right] \frac{1}{(s+1)(s+2)} =$$

$$= \begin{bmatrix} 1 \\ s \end{bmatrix} \left[\tfrac{1}{s}\right] \frac{1}{(s+1)(s+2)} = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} \frac{1}{(s+1)(s+2)}$$

$$\mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{(s)(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix} \right\}$$ Use Partial Fraction Expansion

$$G_1 = \frac{1}{(s)(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$k_1 = G_{11}\,(s)\big|_{s=0} = \frac{1}{2}$$

$$k_2 = G_{11}\,(s+1)\big|_{s=-1} = -1$$

$$k_3 = G_{11}\,(s+2)\big|_{s=-2} = \frac{1}{2}$$

$$G_2 = \frac{1}{(s+1)(s+2)} = \frac{k_4}{s+1} + \frac{k_5}{s+2}$$

$$k_4 = G_2\,(s+1)\big|_{s=-1} = 1$$

$$k_5 = G_2\,(s+2)\big|_{s=-2} = -1$$

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1} B\, V(s)\right\} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\therefore\ x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

↳ For any given initial conditions

$$y(t) = C\,x(t) + \cancel{D\,v(t)}^{\,0} = C\,x(t)$$

$$y(t) = \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} =$$

$$= \begin{bmatrix} 10e^{-t} - 5e^{-2t} - 14e^{-t} + 14e^{-2t} & 5e^{-t} - 5e^{-2t} - 7e^{-t} + 14e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{5}{2} - 5e^{-t} + \frac{5}{2}e^{-2t} + 7e^{-t} - 7e^{-2t} \end{bmatrix} =$$

$$= \begin{bmatrix} -4e^{-t} + 9e^{-2t} & -2e^{-t} + 9e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{5}{2} + 2e^{-t} - \frac{9}{2}e^{-2t} \end{bmatrix} =$$

$$= \left(-4e^{-t} + 9e^{-2t}\right)x_1(0) + \left(-2e^{-t} + 9e^{-2t}\right)x_2(0) + \frac{5}{2} + 2e^{-t} - \frac{9}{2}e^{-2t} =$$

$$= \frac{5}{2} + \left(2 - 4x_1(0) - 2x_2(0)\right)e^{-t} + \left(-\frac{9}{2} + 9x_1(0) + 9x_2(0)\right)e^{-2t}$$

Determine a state-space model for the discrete-time system given by the following difference equation.

$$y[n+3] + 5\,y[n+2] + 7\,y[n+1] + 9\,y[n] = 10\,v[n]$$

Solution

Define the states

Let $x_1[n] = y[n]$

$x_2[n] = y[n+1]$

$x_3[n] = y[n+2]$

Find the next values of the states (approximation to the derivative)

$$x_1[n+1] = y[n+1] = x_2[n]$$

$$x_2[n+1] = y[n+2] = x_3[n]$$

$$x_3[n+1] = y[n+3] = -5\,y[n+2] - 7\,y[n+1] - 9\,y[n] + 10\,v[n]$$

$$= -5\,x_3[n] - 7\,x_2[n] - 9\,x_1[n] + 10\,v[n]$$

Write the state-space equations

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -7 & -5 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix}$$

Determine a state-space model for the discrete-time system given by the following difference equation.

$$y[n] + 2y[n-1] + 3y[n-2] = v[n-2]$$

Solution

Since discrete-time systems are recursive, the difference equation can be rewritten as $y[n+2] + 2y[n+1] + 3y[n] = v[n]$

(shift the value of "n" by 2)

Define the states

Let $x_1[n] = y[n]$

$x_2[n] = y[n+1]$

Find the next values of the states (approximation to the derivative)

$$x_1[n+1] = y[n+1] = x_2[n]$$

$$x_2[n+1] = y[n+2] = -2y[n+1] - 3y[n] + v[n]$$

$$= -2x_2[n] - 3x_1[n] + v[n]$$

Write the state-space equations

$$x[n+1] = \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

Determine the transfer function of the follow discrete-time state-space model.

$$x[n+1] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[n]$$

Solution

$$H(z) = C(zI - A)^{-1}B + D$$

First, find $(zI - A)^{-1}$

$$(zI - A) = \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}$$

$$(zI - A)^{-1} = \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} \frac{1}{z^2+3z+2}$$

$$H(z) = C(zI - A)^{-1}B + \cancel{D}^{0} =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2+3z+2} =$$

$$= \begin{bmatrix} z+3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2+3z+2} = \frac{1}{z^2+3z+2}$$

$$\boxed{H(z) = \frac{1}{z^2+3z+2}}$$

Determine the transfer function of the following discrete-time state-space system.

$$x[n+1] = \begin{bmatrix} 0 & 1 \\ -8 & 9 \end{bmatrix} x[n] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 4 & 1 \end{bmatrix} x[n]$$

Solution

$$H(z) = C(zI - A)^{-1}B + D$$

$$(zI - A) = \begin{bmatrix} z & -1 \\ 8 & z-9 \end{bmatrix}$$

$$(zI - A)^{-1} = \begin{bmatrix} z-9 & 1 \\ -8 & z \end{bmatrix} \frac{1}{z^2 - 9z + 8}$$

$$H(z) = C(zI - A)^{-1}B + \cancel{D}^{0} =$$

$$= \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} z-9 & 1 \\ -8 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{z^2 - 9z + 8} =$$

$$= \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} z-9 \\ -8 \end{bmatrix} \frac{1}{z^2 - 9z + 8} =$$

$$= \frac{4z - 36 - 8}{z^2 - 9z + 8} = \frac{4z - 42}{z^2 - 9z + 8}$$

Determine the characteristic equation, the eigenvalues, and the stability of the following state-space model

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Solution

Find the determinant of $(zI - A)$

$$|zI - A| = \begin{vmatrix} z & -1 \\ 1 & z-1 \end{vmatrix} = (z)(z-1) - (-1)(1) = z^2 - z + 1$$

Characteristic Equation

$$\text{Eigenvalues} = \frac{1 \pm \sqrt{1-4}}{2} = 0.5 \pm j\frac{\sqrt{3}}{2} = e^{\pm j\frac{\pi}{3}}$$

Magnitude of the eigenvalues = 1

$\Rightarrow$ Marginally Stable

Determine the characteristic equation, the eigenvalues, and the stability of the following state space model

$$A = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$$

Solution

$$|zI - A| = \begin{vmatrix} z & -0.5 \\ -0.5 & z \end{vmatrix} = z^2 - 0.25 \qquad \text{Characteristic Equation}$$

$$\text{Eigenvalues} = \frac{0 \pm \sqrt{0 - (4)(-0.25)}}{2} = \pm \frac{1}{2}$$

These eigenvalues lie within the unit circle.

$\Rightarrow$ Stable

Determine the characteristic equation, the eigenvalues, and the stability of the following discrete-time state-space model.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution

Expand around the first row

$$|zI - A| = \begin{vmatrix} z & -1 & -1 \\ -1 & z & -1 \\ -1 & -1 & z \end{vmatrix} = z\begin{vmatrix} z & -1 \\ -1 & z \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & z \end{vmatrix} + (-1)\begin{vmatrix} -1 & z \\ -1 & -1 \end{vmatrix} =$$

$$= (z)(z^2 - 1) + (-z - 1) - (1 + z) =$$

$$= z^3 - 3z - 2 \quad \leftarrow \text{Characteristic Equation}$$

$$= (z)(z-1)(z+1) - (z+1) - (z+1) =$$

$$= (z+1)((z)(z-1) - 2) =$$

$$= (z+1)(z^2 - z - 2) =$$

$$= (z+1)(z+1)(z-2)$$

Eigenvalues $= -1, -1, 2$

Lies outside the unit circle

Unstable

Determine the first five values of the states and the output of the following discrete-time state-space system

$$x[n+1] = \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + [0]\, v[n]$$

given a unit pulse input and initial states of

$$x[0] = \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution

Input is a unit pulse $\rightarrow$ $v[n] = \delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

$x[0] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ From initial states

$$y[0] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + [0]\, v[0] =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [0]\,[1] = 1$$

Next State

$$x[n+1] = x[1] = \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[0] =$$

$\hookrightarrow n=0$

$$= \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] =$$

$$= \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

First iteration $\rightarrow$ Let $n = 1$

Next state

$$x[2] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v[1] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0] =$$

$$= \begin{bmatrix} 9 \\ 31 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 31 \end{bmatrix}$$

$$y[1] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[1] \\ x_2[1] \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} = 2$$

Second iteration $\rightarrow$ Let $n = 2$

$$x[3] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 31 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0] = \begin{bmatrix} 31 \\ 111 \end{bmatrix}$$

$$y[2] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 31 \end{bmatrix} = 9$$

Third iteration → Let $n=3$

$$x[4] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 31 \\ 111 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0] = \begin{bmatrix} 111 \\ 395 \end{bmatrix}$$

$$y[3] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 31 \\ 111 \end{bmatrix} = 31$$

Fourth iteration → Let $n=4$

$$x[5] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 111 \\ 395 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 395 \\ 1407 \end{bmatrix}$$

$$y[4] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 111 \\ 395 \end{bmatrix} = 111$$

Fifth iteration → Let $n=5$

$$x[6] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 395 \\ 1407 \end{bmatrix} = \begin{bmatrix} 1407 \\ 5011 \end{bmatrix}$$

$$y[5] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 395 \\ 1407 \end{bmatrix} = 395$$

This system is obviously unstable! It grows without bound in response to a unit impulse.

Determine the first five values of the states and the output of the following discrete-time state-space system

$$x[n+1] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} x[n] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v[n]$$

$$y[n] = \begin{bmatrix} 1 & 1 \end{bmatrix} x[n]$$

in response to a step input and the initial states

$$x[0] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Sketch the states and the output over the first five values.

Solution

From the initial states

$$x[0] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Output at $n=0$ $\Rightarrow$ $y[0] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4$

Next state

$$x[1] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{[1]}_{\text{Step input}} =$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

First iteration → Let $n = 1$

$$x[2] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{x[1]} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{[1]}_{v[1]} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y[1] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Next iteration → Let $n = 2$

$$x[3] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$y[2] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$$

Next iteration → Let $n = 3$

$$x[4] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$y[3] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1$$

Next iteration → Let $n = 4$

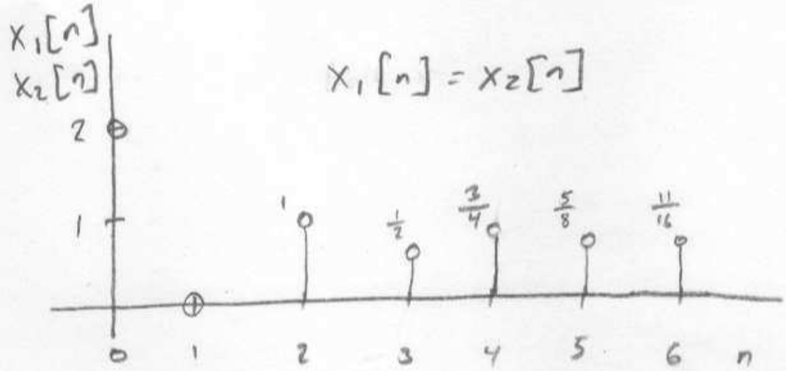
$$x[5] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} -\frac{3}{8} \\ -\frac{3}{8} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{8} \\ \frac{5}{8} \end{bmatrix}$$

$$y[4] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} = 1.5$$

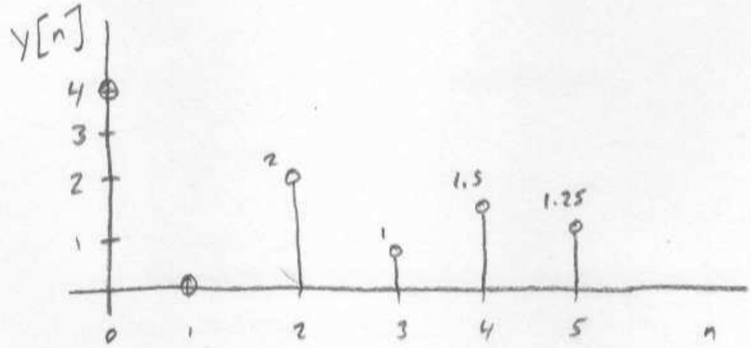
Next iteration → Let $n = 5$

$$x[6] = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{8} \\ \frac{5}{8} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} -\frac{5}{16} \\ -\frac{5}{16} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{16} \\ \frac{11}{16} \end{bmatrix}$$

$$y[5] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{8} \\ \frac{5}{8} \end{bmatrix} = \frac{10}{8} = \frac{5}{4} = 1.25$$



Oscillating to a steady-state value



Oscillating to a steady-state value

This is a stable system.