EE327

Prove that $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$

$$F(s) = \int_0^\infty \sin(\omega t)\, e^{-st}\, dt$$ Use inverse Euler Equation

$$= \int_0^\infty \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) e^{-st}\, dt =$$

$$= \frac{1}{2j}\int_0^\infty \left[e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right] dt =$$

$$= \frac{1}{2j}\left[\frac{-1}{s-j\omega} e^{-(s-j\omega)t} + \frac{1}{s+j\omega} e^{-(s+j\omega)t}\right]\Bigg|_0^\infty =$$

$$= \frac{1}{2j}\left[0 + 0 - \left(\frac{-1}{s-j\omega} + \frac{1}{s+j\omega}\right)\right] = \quad (\text{if } \sigma > 0)$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] =$$

$$= \frac{1}{2j}\left[\frac{s+j\omega - (s-j\omega)}{(s-j\omega)(s+j\omega)}\right] =$$

$$= \frac{\omega}{s^2+\omega^2}$$

EE321

Prove that $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

$$\mathcal{L}\{\underbrace{e^{-at} f(t)}_{f_1(t)}\} = \int_0^\infty f_1(t) e^{-st} dt$$

$$= \int_0^\infty e^{-at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s+a)t} dt$$

Let $s_1 = s + a$

$$\int_0^\infty f(t) e^{-s_1 t} dt = F(s_1) = F(s+a)$$

Prove that $\mathcal{L}\{t f(t)\} = \frac{-d}{ds} F(s)$

Let $F(s) = \int_0^\infty f(t) e^{-st} dt$

Take the derivative

$$\frac{d}{ds} F(s) = \int_0^\infty f(t)\left(-t e^{-st}\right) dt$$

$f(t) \rightarrow$ has no "s" variable
$\rightarrow$ acts as a constant with respect to "s"

$$= -\int_0^\infty t f(t) e^{-st} dt$$

$$= -\mathcal{L}\{t f(t)\}$$

Find the Laplace transforms for the following signals

$x(t) = 5$

Since the unilateral Laplace transform is only valid for $t \geq 0$, $x(t)$ can be rewritten as

$$x(t) = 5u(t)$$

$$\mathcal{L}\{5u(t)\} = \frac{5}{s}$$

$x(t) = 5te^{-3t}$

Again, this can be rewritten as

$$x(t) = 5te^{-3t}u(t)$$

Since there is an exponential e^{-3t}
$\rightarrow (s)$ is replaced by $(s+3)$

$$5e^{-3t}u(t) \leftrightarrow \frac{5}{s+3}$$

$$5te^{-3t}u(t) \leftrightarrow \frac{5}{(s+3)^2}$$

$x(t) = 5te^{-3t}\cos(5t)\,u(t)$

$$\cos(5t) \leftrightarrow \frac{s}{s^2+25}$$

$$5e^{-3t}\cos(5t)\,u(t) \leftrightarrow \frac{5(s+3)}{(s+3)^2+5^2} = \frac{5s+15}{s^2+6s+34}$$

EE 327

Find the following Laplace Transforms. Assume all initial conditions equal 0

$\mathcal{L}\{\frac{d}{dt} t^2 e^{-3t}\}$

$$\mathcal{L}\{t^2 e^{-3t}\} = \frac{2!}{(s+3)^3} = \frac{2}{(s+3)^3}$$

$$\mathcal{L}\{\frac{d}{dt} t e^{-3t}\} = sX(s) - x(0^-) =$$

$$= s\frac{}{(s+3)^3} - x(0^-) = \frac{2s}{(s+3)^3} - x(0^-)$$

$$= \boxed{\frac{2s}{(s+3)^3}}$$ Initial Conditions = 0

$\mathcal{L}\{\left(\frac{d}{dt}\right)^3 t^2\}$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\mathcal{L}\{\left(\frac{d}{dt}\right)^3 t^2\} = s^3\left(\frac{2}{s^3}\right) - s^2 x(0^-) - s\dot{x}(0^-) - \ddot{x}(0^-) =$$

$$= \boxed{2}$$ (Initial conditions equal zero)

$\mathcal{L}\{3t^3(t-1) + e^{-5t}\}$

$$\mathcal{L}\{3t^3(t-1) + e^{-5t}\} = \mathcal{L}\{3t^4 - 3t^3 + e^{-5t}\} =$$

$$= (3)\frac{4!}{s^5} - (3)\frac{3!}{s^4} + \frac{1}{s+5} =$$

$$= \frac{3 \cdot 24}{s^5} - \frac{3 \cdot 6}{s^4} + \frac{1}{s+5} = \frac{72}{s^5} - \frac{18}{s^4} + \frac{1}{s+5} =$$

$$= \frac{72(s+5) - 18(s)(s+5) + s^5}{(s^5)(s+5)} =$$

$$= \frac{72s + 360 - 18s^2 - 90s + s^5}{(s^5)(s+5)} =$$

$$= \boxed{\frac{s^5 - 18s^2 - 18s + 360}{s^6 + 5s^5}}$$

Find the inverse Laplace transform of

$$X(s) = \frac{5s+13}{s^2+6s+5} = \frac{5s+13}{(s+1)(s+5)} = \frac{k_1}{s+1} + \frac{k_2}{s+5}$$

poles = $-1, -5$ Distinct and Real

Use the Residue Method

$$k_1 = X(s)(s+1)\Big|_{s=-1} = \frac{(5s+13)\cancel{(s+1)}}{\cancel{(s+1)}(s+5)}\Big|_{s=-1} = \frac{5s+13}{s+5}\Big|_{s=-1} =$$

$$= \frac{-5+13}{-1+5} = \frac{8}{4} = 2$$

$$k_2 = X(s)(s+5)\Big|_{s=-5} = \frac{5s+13}{s+1}\Big|_{s=-5} = \frac{-25+13}{-5+1} = \frac{-12}{-4} = 3$$

$$\therefore X(s) = \frac{2}{s+1} + \frac{3}{s+5}$$

Check $\rightarrow$ $\frac{2(s+5)+3(s+1)}{(s+1)(s+5)} = \frac{2s+10+3s+3}{(s+1)(s+5)} = \frac{5s+13}{(s+1)(s+5)}$

Agrees

$$\therefore x(t) = 2e^{-t} + 3e^{-5t}, \quad t \geq 0$$

Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s^2+4)(s^2-4)} = \frac{1}{(s^2+4)(s+2)(s-2)} = \frac{k_1}{s+2} + \frac{k_2}{s-2} + \frac{k_3 s + k_4}{s^2+4}$$

poles $= \pm 2, \pm j2$ → 2 Distinct, Real poles ; 2 Complex Conjugate Poles

Use the residue method for k_1 and k_2

$$k_1 = F(s)(s+2)\Big|_{s=-2} = \frac{1}{(s^2+4)(s-2)}\Big|_{s=-2} = \frac{1}{(4+4)(-2-2)} = -\frac{1}{32}$$

$$k_2 = F(s)(s-2)\Big|_{s=2} = \frac{1}{(s^2+4)(s+2)}\Big|_{s=2} = \frac{1}{32}$$

Use the brute force method for k_3 and k_4
→ Recombine over a common denominator

$$F(s) = \frac{k_1(s-2)(s^2+4) + k_2(s+2)(s^2+4) + (k_3 s + k_4)(s+2)(s-2)}{(s+2)(s-2)(s^2+4)} =$$

$$= \frac{k_1(s^3-2s^2+4s-8) + k_2(s^3+2s^2+4s+8) + (k_3 s+k_4)(s^2-4)}{(s+2)(s-2)(s^2+4)} =$$

$$= \frac{-\frac{1}{32}(s^3-2s^2+4s-8) + \frac{1}{32}(s^3+2s^2+4s+8) + k_3 s^3 + k_4 s^2 - 4k_3 s - 4k_4}{(s+2)(s-2)(s^2+4)} =$$

$$= \frac{\frac{1}{32}(4s^2+16) + k_3 s^3 + k_4 s^2 - 4k_3 s - 4k_4}{(s+2)(s-2)(s^2+4)} =$$

$$= \frac{s^3(k_3) + s^2\left(k_4 + \frac{1}{8}\right) + s(-4k_3) + \left(\frac{1}{2} - 4k_4\right)}{(s+2)(s-2)(s^2+4)}$$

→ Equate Numerators

$$1 = s^3(k_3) + s^2\left(k_4 + \frac{1}{8}\right) + s(-4k_3) + \left(\frac{1}{2} - 4k_4\right)$$

→ Equate like terms of "s"

$s^3 \rightarrow \quad 0 = k_3 \quad \Rightarrow k_3 = 0$
$s^2 \rightarrow \quad 0 = k_4 + \frac{1}{8} \quad \Rightarrow k_4 = -\frac{1}{8}$
Already found all coefficients, so no need to go further

$$F(s) = -\frac{1}{32}\frac{1}{s+2} + \frac{1}{32}\frac{1}{s-2} - \frac{1}{8}\frac{1}{s^2+4} \rightarrow \frac{1}{16}\frac{2}{s^2+4}$$

$$f(t) = \frac{1}{32}\left(-e^{-2t} + e^{2t}\right) + \frac{1}{16}\sin 2t, \quad t \geq 0$$

Find the inverse Laplace transform of

$$G(s) = \frac{1}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{(s+2)^2}$$

Use the residue method for k_1 and k_3

$$k_1 = G(s)(s+1)\Big|_{s=-1} = \frac{\cancel{(s+1)}}{\cancel{(s+1)}(s+2)^2}\Big|_{s=-1} = 1$$

$$k_3 = G(s)(s+2)^2\Big|_{s=-2} = \frac{1}{s+1}\Big|_{s=-2} = -1$$

Take the differential to find k_2

$$k_2 = \left[\frac{d}{ds}\left[G(s)(s+2)^2\right]\right]\Big|_{s=-2} = \left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right]\Big|_{s=-2} =$$

$$= (-1)\frac{1}{(s+1)^2}\Big|_{s=-2} = \frac{(-1)}{(-1)^2} = -1$$

Check $\rightarrow$ $$\frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} = \frac{(s+2)^2 - (s+1)(s+2) - (s+1)}{(s+1)(s+2)^2} =$$

$$= \frac{s^2+4s+4 - s^2 - 3s - 2 - s - 1}{(s+1)(s+2)^2} = \frac{1}{(s+1)(s+2)^2}$$

Agrees

Alternate method $\rightarrow$ brute force method

$$G(s) = \frac{k_1(s+2)^2 + k_2(s+1)(s+2) + k_3(s+1)}{(s+1)(s+2)^2} =$$

$$= \frac{(1)(s^2+4s+4) + (k_2)(s^2+3s+2) + (-1)(s+1)}{(s+1)(s+2)^2} =$$

$$= \frac{s^2(k_2+1) + s(3k_2+4-1) + (2k_2+4-1)}{(s+1)(s+2)^2}$$

Equate numerators and like terms of "s"

$$s^2 \rightarrow 0 = k_2 + 1 \Rightarrow k_2 = -1$$ Agrees

$$G(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$g(t) = e^{-t} - e^{-2t} - te^{-2t}, \quad t \geq 0$$