Solve the following differential equation

$$\ddot{y} + 2\dot{y} + 5y = x$$

Given $x(t) = \sin(3t)$, $y(0)=1, \dot{y}(0) = -1$

Take the Laplace Transform

$$\underbrace{s^2 Y(s) - s\,y(0) - \dot{y}(0)}_{\text{from } \ddot{y}} + 2\underbrace{[s\,Y(s) - y(0)]}_{\text{from } \dot{y}} + 5\underbrace{Y(s)}_{\text{from } y} = X(s) = \underbrace{\frac{3}{s^2+9}}_{\sin(3t)}$$

Plug in initial conditions

$$s^2 Y(s) - s + 1 + 2s\,Y(s) - 2 + 5\,Y(s) = \frac{3}{s^2+9}$$

$$Y(s)\left[s^2 + 2s + 5\right] - s - 1 = \frac{3}{s^2+9}$$

$$Y(s)\left[s^2 + 2s + 5\right] = \frac{3}{s^2+9} + s + 1$$

$$Y(s) = \frac{3}{(s^2+9)(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}$$

$$Y(s) = \underbrace{\frac{3}{(s^2+9)((s+1)^2+4)}}_{\text{Use Partial Fraction Expansion}} + \underbrace{\frac{s+1}{(s+1)^2+4}}_{\text{Already in proper form for inverse L.T.}}$$

$$\frac{3}{(s^2+9)((s+1)^2+4)} = \frac{k_1 s + k_2}{s^2+9} + \frac{k_3 s + k_4}{(s+1)^2+4}$$

Recombine left-hand side to a single common denominator

$$\frac{3}{\cancel{(s^2+9)((s+1)^2+4)}} = \frac{(k_1 s + k_2)(s^2+2s+5) + (k_3 s + k_4)(s^2+9)}{\cancel{(s^2+9)((s+1)^2+4)}}$$

$$3 = k_1 s^3 + 2k_1 s^2 + 5k_1 s + k_2 s^2 + 2k_2 s + 5k_2 + k_3 s^3 + 9k_3 s + k_4 s^2 + 9k_4$$

$$3 = s^3\left[k_1 + k_3\right] + s^2\left[2k_1 + k_2 + k_4\right] + s\left[5k_1 + 2k_2 + 9k_3\right] + \left[5k_2 + 9k_4\right]$$

Equate Like terms of "s"

$s^3 \rightarrow \quad 0 = k_1 + k_3 \Rightarrow k_1 = -k_3$

$s^2 \rightarrow \quad 0 = 2k_1 + k_2 + k_4$

$s^1 \rightarrow \quad 0 = 5k_1 + 2k_2 + 9k_3 = 5k_1 + 2k_2 - 9k_1 = -4k_1 + 2k_2$

$\Rightarrow k_2 = 2k_1$

$s^0 \rightarrow \quad 3 = 5k_2 + 9k_4$

Plug in value of k_2 in "s^2" term

$0 = 2k_1 + k_2 + k_4 = 2k_1 + 2k_1 + k_4 \Rightarrow k_4 = -4k_1$

s^0 term $\rightarrow \quad 3 = 5k_2 + 9k_4 = 5(2k_1) + 9(-4k_1) = 10k_1 - 36k_1$

$\Rightarrow k_1 = \frac{-3}{26}, \quad k_2 = \frac{-6}{26} = \frac{-3}{13}, \quad k_3 = \frac{3}{26}$

$k_4 = \frac{12}{26} = \frac{6}{13}$

$$\therefore \frac{3}{(s^2+9)((s+1)^2+4)} = \frac{3}{26}\left[-\frac{s+2}{s^2+9} + \frac{s+4}{(s+1)^2+4}\right] =$$

$$= \frac{3}{26}\left[-\frac{s}{s^2+9} - \frac{2}{3}\frac{3}{s^2+9} + \frac{s+1}{(s+1)^2+4} + \frac{3}{2}\frac{2}{(s+1)^2+4}\right]$$

Total output

$$Y(s) = \frac{3}{26}\left[-\frac{s}{s^2+9} - \frac{2}{3}\frac{3}{s^2+9} + \frac{s+1}{(s+1)^2+4} + \frac{3}{2}\frac{2}{(s+1)^2+4}\right] + \frac{s+1}{(s+1)^2+4} =$$

$$= \frac{3}{26}\left[-\frac{s}{s^2+9} - \frac{2}{3}\frac{3}{s^2+9} + \frac{3}{2}\frac{2}{(s+1)^2+4}\right] + \frac{29}{26}\frac{s+1}{(s+1)^2+4}$$

Take the inverse Laplace transform

$$y(t) = -\frac{3}{26}\cos(3t) - \frac{1}{13}\sin(3t) + \frac{9}{52}e^{-t}\sin(2t) + \frac{29}{26}e^{-t}\cos(2t), \quad t \geq 0$$

Find the final value of the following signals

a) $X(s) = \dfrac{10s}{(s+1)(s+2)^2}$ poles $= -1, -2, -2 \le 0$ Final Value Theorem Applies

Use the final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s X(s) = \lim_{s\to 0} \frac{10s^2}{(s+1)(s+2)^2} = 0$$

b) $Y(s) = \dfrac{10s}{s^2+2^2} = \dfrac{10s}{s^2+4}$ poles $= \pm j2$

Real part of the pole lies on the imaginary axis
→ this is a sinusoidal signal
→ cannot find the final value (there is none)
→ keeps oscillating

c) $Z(s) = \dfrac{5(s^2-2s+4)}{(s)(s+1)(s+2)(s+3)}$ poles $= 0, -1, -2, -3 \le 0$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sZ(s) = \lim_{s\to 0} \frac{5(s^2-2s+4)}{(s+1)(s+2)(s+3)} =$$

$$= \frac{5(4)}{(1)(2)(3)} = \frac{20}{6} = \frac{10}{3}$$

Determine the transfer function of the following systems

a) $\ddot{y} + 4\dot{y} + 4y = 2\dot{x} - x$

Take the Laplace transform

$$s^2Y(s) - s\,y(0) - \dot{y}(0) + 4[sY(s) - y(0)] + 4Y(s) = 2[sX(s) - x(0)] - X(s)$$

Set initial conditions to zero

$$s^2Y(s) + 4sY(s) + 4Y(s) = 2sX(s) - X(s)$$

$$Y(s)[s^2 + 4s + 4] = X(s)[2s - 1]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s-1}{s^2+4s+4} = 2\,\frac{s - \frac{1}{2}}{(s+2)^2}$$

b) $\ddot{v} + \dot{v} + 5v = x$

$\dot{y} + y = 5v$

Take the Laplace transform with I.C.'s set to zero

$$s^2V(s) + sV(s) + 5V(s) = X(s)$$

$$sY(s) + Y(s) = 5V(s)$$

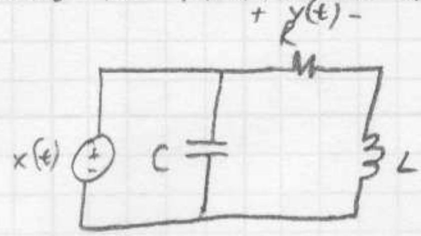
$$\rightarrow V(s)[s^2 + s + 5] = X(s)$$

$$Y(s)[s+1] = 5V(s)$$

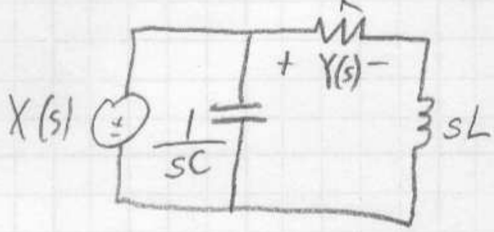
$$Y(s)[s+1][s^2+s+5] = 5\underbrace{V(s)[s^2+s+5]}_{X(s)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{5}{(s+1)(s^2+s+5)}$$

Determine the transfer function of the following system



Use the impedance method → convert everything to the Laplace domain



Use the voltage divider

$$Y(s) = X(s)\,\frac{R}{R + sL}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{R}{sL + R}$$