Find the z Transform of the following signals.

A. $x[n] = \delta[n] + 7\delta[n-2] - 9\delta[n-3]$

$X(z) = \sum_{n=0}^{\infty} x[n](z^{-1})^n = (1)z^{-0} + (7)z^{-2} - 9z^{-3} =$

$= 1 + 7z^{-2} - 9z^{-3}$

B. $x[n] = \left(\frac{1}{4}\right)^{n-2} u[n-2]$

$\left(\frac{1}{4}\right)^n u[n] \longleftrightarrow \frac{z}{z-\frac{1}{4}}$

Delay by n-2

$\left(\frac{1}{4}\right)^{n-2} u[n-2] \longleftrightarrow \frac{z}{z-\frac{1}{4}} z^{-2} = \frac{z^{-1}}{z-\frac{1}{4}} = \frac{1}{z(z-\frac{1}{4})} = X(z)$

C. $x[n] = \left(\frac{1}{4}\right)^n \left(\underbrace{u[n-2] - u[n-5]}_{\text{only nonzero for } n=2,3,4}\right)$

$x[n] = \left(\frac{1}{4}\right)^2 \delta[n-2] + \left(\frac{1}{4}\right)^3 \delta[n-3] + \left(\frac{1}{4}\right)^4 \delta[n-4] =$

$= \frac{1}{16}\delta[n-2] + \frac{1}{64}\delta[n-3] + \frac{1}{256}\delta[n-4]$

$X(z) = \frac{1}{16}z^{-2} + \frac{1}{64}z^{-3} + \frac{1}{256}z^{-4}$

D. $x[n] = n\,v[n]$ where $v[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{-1}{2}\right)^n u[n]$

$x[n] = n\left[\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n]\right]$

$V(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{2}} = \frac{(z)(z+\frac{1}{2}) + (z)(z-\frac{1}{2})}{z^2 - \frac{1}{4}} = \frac{z^2 + \cancel{\frac{1}{2}z} + z^2 - \cancel{\frac{1}{2}z}}{z^2 - \frac{1}{4}}$

$= \frac{2z^2}{z^2 - \frac{1}{4}}$

$X(z) = -z\frac{d}{dz}\left(\frac{2z^2}{z^2-\frac{1}{4}}\right) = -z\left[\frac{4z}{z^2-\frac{1}{4}} - \frac{2z^2}{(z^2-\frac{1}{4})^2}(2z)\right] =$

$= -z\left[\frac{4z(z^2-\frac{1}{4}) - 4z^3}{(z^2-\frac{1}{4})^2}\right] = -z\left[\frac{4z^3 - z - 4z^3}{(z^2-\frac{1}{4})^2}\right] = \frac{z^2}{(z^2-\frac{1}{4})^2}$

Find the inverse z Transform of the following signals

A. $X(z) = 5(1-z^{-1})(1+z^{-1})(1+10z^{-1})$

First, expand $X(z)$ as a polynomial

$$X(z) = 5(1-z^{-2})(1+10z^{-1}) = 5[1+10z^{-1}-z^{-2}-10z^{-3}] =$$

$$= 5 + 50z^{-1} - 5z^{-2} - 50z^{-3}$$

$$x[n] = 5\delta[n] + 50\delta[n-1] - 5\delta[n-2] - 50\delta[n-3]$$

B. $$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Re-write as a rational function of "z"

$$X(z) = \frac{z^2}{(z-\frac{1}{2})(z-1)}$$

General Form of the solution $\rightarrow$ $x[n] = k_1\left(\frac{1}{2}\right)^n u[n] + k_2 u[n]$

Since the degree of the numerator equals the degree of the denominator

$$\frac{X(z)}{z} = \frac{z}{(z-\frac{1}{2})(z-1)} = \frac{k_1}{z-\frac{1}{2}} + \frac{k_2}{z-1}$$

$$k_1 = \left(\frac{X(z)}{z}(z-\tfrac{1}{2})\right)\Big|_{z=\frac{1}{2}} = \frac{z}{z-1}\Big|_{z=\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}-1} = -1$$

$$k_2 = \left(\frac{X(z)}{z}(z-1)\right)\Big|_{z=1} = \frac{z}{z-\frac{1}{2}}\Big|_{z=1} = \frac{1}{\frac{1}{2}} = 2$$

$$\frac{X(z)}{z} = \frac{-1}{z-\frac{1}{2}} + \frac{2}{z-1} = \frac{(-1)(z-1)+(2)(z-\frac{1}{2})}{(z-\frac{1}{2})(z-1)} = \frac{-z+1+2z-1}{(z-\frac{1}{2})(z-1)}$$

$$= \frac{z}{(z-\frac{1}{2})(z-1)} \quad \text{Agrees}$$

$$X(z) = -\frac{z}{z-\frac{1}{2}} + \frac{2}{z-1}$$

$$x[n] = -\left(\frac{1}{2}\right)^n u[n] + 2u[n]$$

c. $X(z) = \dfrac{3z^2 + 1}{(z - \frac{1}{4})^2}$ ← degree = 2 ← degree = 2

$$\frac{X(z)}{z} = \frac{3z^2+1}{(z)(z-\frac{1}{4})^2} = \frac{k_0}{z} + \frac{k_1}{z-\frac{1}{4}} + \frac{k_2}{(z-\frac{1}{4})^2}$$

General Form of the solution → $x[n] = k_0 + k_1 \left(\frac{1}{4}\right)^n u[n] + k_2 n \left(\frac{1}{4}\right)^n u[n]$

Residue Method

$$k_0 = \left(\frac{X(z)}{z} z\right)\Big|_{z=0} = \frac{3z^2+1}{(z-\frac{1}{4})^2}\Big|_{z=0} = \frac{1}{\frac{1}{16}} = 16$$

$$k_2 = \left(\frac{X(z)}{z}(z-\tfrac{1}{4})^2\right)\Big|_{z=\frac{1}{4}} = \frac{3z^2+1}{z}\Big|_{z=\frac{1}{4}} = \frac{\frac{3}{16}+1}{\frac{1}{4}} = \frac{3}{4} + 4 = \frac{19}{4}$$

Differentiate for $k_1$

$$k_1 = \frac{d}{dz}\left[\frac{X(z)}{z}(z-\tfrac{1}{4})^2\right]\Big|_{z=\frac{1}{4}} = \frac{d}{dz}\left[\frac{3z^2+1}{z}\right]\Big|_{z=\frac{1}{4}} =$$

$$= \frac{6z}{z} + \frac{-(3z^2+1)}{z^2}\Big|_{z=\frac{1}{4}} = 6 - 3 - \frac{1}{z^2}\Big|_{z=\frac{1}{4}} = 3 - \frac{1}{z^2}\Big|_{z=\frac{1}{4}} =$$

$$= 3 - 16 = -13$$

$$\frac{X(z)}{z} = \frac{16}{z} + \frac{-13}{(z-\frac{1}{4})} + \frac{\frac{19}{4}}{(z-\frac{1}{4})^2} = \frac{16(z-\frac{1}{4})^2 - 13z(z-\frac{1}{4}) + \frac{19}{4}z}{(z)(z-\frac{1}{4})^2} =$$

$$= \frac{16z^2 - 8z + 1 - 13z^2 + \frac{13}{4}z + \frac{19}{4}z}{(z)(z-\frac{1}{4})^2} = \frac{3z^2+1}{(z)(z-\frac{1}{4})^2}$$ Agrees

$$X(z) = 16 - 13\frac{z}{z-\frac{1}{4}} + \frac{19}{4}\frac{z}{(z-\frac{1}{4})^2}$$

$$x[n] = 16\delta[n] - 13\left(\tfrac{1}{4}\right)^n u[n] + \tfrac{19}{4} n \left(\tfrac{1}{4}\right)^n u[n]$$

D. $X(z) = \dfrac{z}{z^2 + 4z + 8}$

roots $\rightarrow$ $\dfrac{-8 \pm \sqrt{16-32}}{2} = \dfrac{-8 \pm \sqrt{-16}}{2} = -\dfrac{8}{2} \pm j\dfrac{4}{2} = -4 \pm j2$

$X(z) = \dfrac{k_1}{(z+4+j2)} + \dfrac{k_2}{(z+4-j2)}$ where $k_2 = k_1^*$

$k_1 = \left[X(z)\,(z+4+j2)\right]\Big|_{z=-4-j2} = \dfrac{z}{(z+4-j2)}\Big|_{z=-4-j2} =$

$= \dfrac{-4-j2}{-j4} = \dfrac{1}{j} + \dfrac{1}{2} = \dfrac{1}{2} - j$

$k_2 = k_1^* = \dfrac{1}{2} + j$

$$X(z) = \frac{\frac{1}{2} - j}{z+4+j2} + \frac{\frac{1}{2}+j}{z+4-j2} = \frac{\left(\frac{1}{2}-j\right)(z+4-j2) + \left(\frac{1}{2}+j\right)(z+4+j2)}{z^2+4z+8} =$$

$$= \frac{\frac{1}{2}z \cancel{+2} \cancel{-j} \cancel{-jz} \cancel{-j4} \cancel{-2} + \frac{1}{2}z \cancel{+2} \cancel{+j} \cancel{+jz} \cancel{+j4} \cancel{-2}}{z^2+4z+8} =$$

$$= \frac{z}{z^2+4z+8} \quad \text{Agrees}$$

$$X(z) = z^{-1}\underbrace{\frac{(z)\left(\frac{1}{2}-j\right)}{z+4+j2}}_{\left(\frac{1}{2}-j\right)(-4-j2)^n u[n]} + z^{-1}\underbrace{\frac{(z)\left(\frac{1}{2}+j\right)}{z+4-j2}}_{\left(\frac{1}{2}+j\right)(-4+j2)^n u[n]}$$

$$x[n] = \left(\tfrac{1}{2}-j\right)(-4-j2)^{n-1}u[n-1] + \left(\tfrac{1}{2}+j\right)(-4+j2)^{n-1}u[n-1]$$

Combine complex terms

does not matter +/-

pole 1 $\rightarrow$ magnitude $= |-4+j2| = \sqrt{(-4)^2 + (+2)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

negative in front of the real part

angle $= \angle(-4-j2) = \pi - \tan^{-1}\dfrac{2}{4} = \pi - \tan^{-1}\dfrac{1}{2} = 2.6779$ rad

$k_1^* \rightarrow$ magnitude $= \left|\frac{1}{2}+j\right| = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2} = \sqrt{\frac{1}{4}+1} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$

angle $= \angle\left(\frac{1}{2}+j\right) = \tan^{-1}\dfrac{1}{\frac{1}{2}} = \tan^{-1}(2) = 1.1071$

delay

$x[n] = 2|k_1|r^{n-1}\cos(\omega(n-1) + \angle k_1)u[n-1] = \sqrt{5}\,(2\sqrt{5})^{n-1}\cos(2.6779(n-1) + 1.1071)u[n-1]$

Find transfer functions for the following difference equations

A. $y[n] + 0.5\,y[n-1] = 2x[n]$

$$Y(z) + 0.5\,z^{-1}\,Y(z) = 2X(z)$$

$$Y(z)\left[1 + 0.5\,z^{-1}\right] = 2\,X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1+0.5z^{-1}} = \frac{2z}{z+0.5}$$

B. $y[n] + 2y[n-2] = 2x[n] - x[n-1]$

$$Y(z) + 2z^{-2}\,Y(z) = 2X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 + 2z^{-2}\right] = X(z)\left[2 - z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - z^{-1}}{1 + 2z^{-2}} = \frac{2z^2 - z}{z^2 + 2}$$

Find the final value of the following discrete-time signals.

A. $X(z) = \dfrac{(z-1)(z+0.2)}{(z-0.3)(z+0.4)}$

$$x[\infty] = \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)\,X(z) = \lim_{z\to 1} \frac{(z-1)^2(z+0.2)}{(z-0.3)(z+0.4)} = 0$$

B. $X(z) = \dfrac{z^3+z+1}{(z^2-0.25z+0.1)(z-1)}$

$$x[\infty] = \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)\,X(z) = \lim_{z\to 1} \frac{\cancel{(z-1)}(z^3+z+1)}{(z^2-0.25z+0.1)\cancel{(z-1)}} =$$

$$= \frac{(1^3+1+1)}{[1^2-(0.25)(1)+(0.1)]} = \frac{3}{0.85} = 3.5294$$