NONPARAMETRIC REGRESSION ESTIMATION WITH GENERAL PARAMETRIC ERROR COVARIANCE

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Abstract. The asymptotic distribution for the local linear estimator in nonparametric regression models is established under a general parametric error covariance with dependent and heterogeneously distributed regressors. A two-step estimation procedure that incorporates the parametric information in the error covariance matrix is proposed. Sufficient conditions for its asymptotic normality are given and its efficiency relative to the local linear estimator is established. We give examples of how our results are useful in some recently studied regression models. A Monte Carlo study confirms the asymptotic theory predictions and compares our estimator with some recently proposed alternative estimation procedures.

Keywords and Phrases. local linear estimation; asymptotic normality; mixing processes.

JEL Classifications. C14, C22
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1 Introduction

Recently there has been a growing interest in the specification of nonparametric regression models in which the regression errors' correlation structure can be described parametrically. For example, Xiao et al. (2003) consider a nonparametric regression with stationary error terms that have an invertible linear process representation which encompasses all finite order ARMA(p,q) processes; Vilar-Fernández and Francisco-Fernández (2002) consider a fixed design nonparametric regression whose errors follow an AR(1) process; Lin and Carroll (2000), Ruckstuhl et al. (2000), Wang (2003) consider a nonparametric regression for panel/clustered data where the error term covariance structure follows a pre-specified parametric structure; Fan et al. (1996) consider a nonparametric regression frontier model with errors whose covariance structure follows a parametric specification proposed by Aigner et al. (1977); Smith and Kohn (2000) consider the estimation of a finite set of nonparametric regressions whose error structure follows the parametric seemingly unrelated structure proposed by Zellner (1962).

These models can be viewed as extensions of the regression literature in two related but distinct ways. First, they represent an extension of the vast Generalized Least Squares(GLS) linear and nonlinear parametric regression literatures (Gallant, 1987; White, 2001) to the nonparametric regression setting, and as such they represent improvements on the modeling of (un)conditional expectations. Second, they can be viewed as extensions of the nonparametric regression literature from the typical case where regression errors are independent and identically distributed (iid) to cases where specific parametric structures for correlation and heteroscedasticity are allowed (Severini and Staniswalis, 1994). In either case, the usefulness of these extensions in econometric and statistical practice is well recognized and documented (Pagan and Ullah, 1999; Fan and Yao, 2003). In their most general form, these regression models can be written as,

$$Y_i = m(X_i) + U_i,\ i = 1, 2, \cdots \tag{1}$$

where X_i is a vector of regressors, Y_i is a regressand and the error U_i is such that

$$E(U_i) = 0 \text{ for all } i = 1, 2, \cdots, E(U_iU_j) = \omega_{ij}(\theta_0), \theta_0 \in \Re^p, p < \infty. \tag{2}$$

The important characteristic of (**??**) is that each element of the error covariance can be expressed as a function $\omega_{ij}(\theta)$ of a finite set of parameters θ_0. Previous works on the estimation of these models have had two main objectives. The first is to establish the asymptotic properties of well known nonparametric regression estimators such as local polynomial and Nadaraya-Watson estimators under the assumed error correlation structure (Xiao et al., 2003; Vilar-Fernández and Francisco-Fernández, 2002). Although progress in this direction has been made, it is unfortunate that most asymptotic results for traditional estimators are specific to the assumed covariance structure and lack the generality that would allow their applicability under alternative parametric structures for the error correlation. A more general result under covariance structure (**??**) for the local linear estimator seems to be especially useful as this estimator has a number desirable properties, such as design adaptability, reduced bias (as compared to Nadaraya-Watson estimators), good boundary properties and mini-max efficiency (Fan, 1992; Fan, 1993; Fan and Gijbels, 1995). The first contribution of this paper is to provide a set of sufficient conditions under which the asymptotic normality of the local linear estimator can be established when the error correlation structure has the general parametric structure in (**??**). These conditions encompass a number of models proposed so far in the nonparametric literature as well as other structures that have been popular in the GLS parametric literature (Mandy and Martins-Filho, 1994).

The second objective of the existing literature is to propose estimators that by incorporating the information contained in the error covariance structure will lead to better performance - asymptotically or in finite sample - *vis a vis* the traditional estimators (Severini and Staniswalis, 1994; Lin and Carroll, 2000; Ruckstuhl et al., 2000; Wang, 2003). How to best incorporate the error covariance matrix information into local polynomial nonparametric regression estimators is still an open question. Lin and Carroll (2000) show that in typical random effects panel data models, when a standard kernel based estimator is used, it is better to estimate the regression by ignoring the correlation structure within a cluster - the "working independence" approach. An alternative kernel smoothing method proposed by Wang (2003) achieves smaller variance when the correlation structure is taken into account. However, it is not clear how to generalize this approach to the case of a general error covariance. A particularly promising approach has been the *pre whiten* method proposed by Ruckstuhl et al. (2000) and adopted by Xiao et al. (2003). However, as in the case of the local linear estimator, the asymptotic properties of this *pre whiten* estimator have been established only for specific parametric structures of the error covariance (random effects panel data and autocorrelated errors). In fact, as will be argued below, establishing the asymptotic normality of the *pre whiten* estimator in general settings could be quite difficult. Hence, in the second part of this paper we propose a new two step estimator, inspired by Ruckstuhl at al. (2000), that incorporates information contained in the error covariance structure and is asymptotically normal under fairly mild restrictions on the parametric structure of the covariance. Our estimator is an improvement over the traditional local linear estimator in that its bias is of the same order but its asymptotic distribution has strictly smaller variance.

Our results are useful from at least two perspectives. First, since our results hold for generally specified parametric covariances, they eliminate the need to repeatedly establish asymptotic normality for both estimators - local linear and the two step procedure proposed herein - under specific structures of $\omega_{ij}(\theta_0)$. Second, because both estimators are asymptotically normal and converge at similar rates establishing relative efficiency is facilitated. At their technical core, both contributions in this paper can be viewed as extensions to the results of Mack and Silverman (1982) and Masry and Fan (1997). These extensions are made possible by relying on inequalities for non stationary processes provided by Doukhan (1994) and Volkonskii and Rozanov (1959). The rest of the paper is organized as follows. Section 2 provides the general characteristics of the regression model we consider, defines the local linear estimator, gives a list of assumptions and the two main theorems necessary to establish the properties of the local linear estimator for model (**??**)-(**??**). In section 3 we define a new two step estimator based on the knowledge of $\omega_{ij}(\theta_0)$ and give sufficient conditions for obtaining its asymptotic normality. We then obtain the asymptotic equivalence of the two-step estimator based on $\omega_{ij}(\theta_0)$ and its feasible version based on an estimator $\omega_{ij}(\dot{\theta})$, where $\dot{\theta} - \theta_0 = o_p(1)$. Section 4 gives two applications of our results that illustrate how our theorems encompass and extend previous results in the literature. Sections 5 contains a Monte Carlo study that implements our two step estimator, sheds some light on its finite sample properties, and compares its performance to that of existing estimators. Section 6 provides a summary of the paper.

2 A Nonparametric Regression Model with General Parametric Covariance

Suppose there are n observations $\vec{y} = (Y_1, \cdots, Y_n)'$, $\vec{x} = (X_1, \cdots, X_n)'$ on the regressand and regressors for the model (1)-(2). The objective is to estimate the regression function $m(x)$ at some point $x \in \Re^D$, $D < n$.[1] There is a vast literature (Györfi et al., 2002) on how to proceed with estimation of m. Here, we focus our attention on the local linear estimator (LLE) which was popularized by Fan (1992) due to its well known desirable properties. Furthermore, our results for the LLE are easily extended for the also popular Nadaraya-Watson estimator. Let $e' = (1, 0)$, $1_n' = (1, \cdots, 1)$ a vector of ones of length n and $h_n > 0$ a sequence of bandwidths, then the LLE is defined as

$$\check{m}(x) = e' \left(R_x' K_x R_x\right)^{-1} R_x' K_x \vec{y} \quad (3)$$

where $R_x = (1_n, \vec{x} - 1_n x)$, $K_x = diag\left\{K\left(\frac{X_i - x}{h_n}\right)\right\}_{i=1}^{n}$. It will be convenient for our purposes to rewrite (**??**) as $\check{m}(x) = \frac{1}{nh_n}\sum_{i=1}^{n} W_n\left(\frac{x_i - x}{h_n}, x\right) Y_i$, where $W_n(z, x) = e' S_n^{-1}(x)(1, z)' K(z)$ and

$$S_n(x) = \frac{1}{nh_n}\begin{pmatrix} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right) & \sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right)\left(\frac{X_i - x}{h_n}\right) \\ \sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right)\left(\frac{X_i - x}{h_n}\right) & \sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right)\left(\frac{X_i - x}{h_n}\right)^2 \end{pmatrix} = \begin{pmatrix} s_{n,0}(x) & s_{n,1}(x) \\ s_{n,1}(x) & s_{n,2}(x) \end{pmatrix}.$$

To establish the asymptotic normality of $\check{m}(x)$ for model (**??**)-(**??**) we follow the traditional approach of breaking the problem into two parts. First, we establish the uniform convergence in probability of the components of $R_x' K_x R_x$ after a suitable normalization. This is accomplished as an application of Theorem 1 which is given below. Second, we establish the asymptotic distribution of the $R_x' K_x \vec{y}$ vector (and of the estimator itself) in Theorem 2. We now provide a list of general assumptions that will be selectively adopted in these theorems and introduce some notation. In what follows C always denotes a generic constant that may take different values in $\Re$ and the sequence of bandwidths h_n is such that, $h_n \to 0$ and $nh_n^2 \to \infty$ as $n \to \infty$.

Assumption A1. 1. Let $f_i(x)$ be the marginal density of X_i evaluated at x, with $f_i(x) < C$ for all i and x; 2. $f_i^{(d)}(x)$ is the d^{th} order derivative of $f_i(x)$ evaluated at x and we assume that $|f_i^{(1)}(x)| < C$; 3.

[1] In what follows we proceed for simplicity with the assumption that $D = 1$. *Mutatis Mutandis* all results follow for $D > 1$.

$|f_i(x) - f_i(x')| \leq C|x - x'|$ for all x, x'; 4. $f_{lkijmo}(x_l, ..., x_o)$ denotes the joint density of $X_l, ..., X_o$ evaluated at $x_l, ..., x_o$ and we assume that $f_{lkijmo}(x_l, ..., x_o) < C$ for all $x_l, ..., x_o$. 5. $\bar{f}_n(x) = n^{-1}\sum_{i=1}^n f_i(x) \to \bar{f}(x)$ as $n \to \infty$ where $0 < \bar{f}(x) < \infty$; 6. As $n \to \infty$ $0 < inf_{x \in G}|\bar{f}_n(x)| < C$ for $x \in G$ a compact set.

Assumption A2. $K(x) : \Re \to \Re$ is a symmetric bounded function with compact support S_K such that; 1. $\int K(x)dx = 1$; 2. $\int xK(x)dx = 0$; 3. $\int x^2 K(x)dx = \sigma_K^2$; 4. for all $x, x' \in S_K$ we have $|K(x) - K(x')| \leq C|x - x'|$.

Assumption A3. $\omega_{ij}(\theta_0)$ is the (i, j) element of $\Omega = E(UU')$ with $|\omega_{ij}(\theta_0)| < C$ for all i, j, $\bar{\omega}_n(\theta) = n^{-1}\sum_{i=1}^n \omega_{ii}(\theta) \to \bar{\omega}(\theta)$ as $n \to \infty$ where $0 < \bar{\omega}(\theta) < \infty$ for every θ and $\bar{\omega}_{fn}(x, \theta) = n^{-1}\sum_{i=1}^n \omega_{ii}(\theta) f_i(x) \to \bar{\omega}_f(x, \theta)$ as $n \to \infty$ where $0 < \bar{\omega}_f(x, \theta) < \infty$ for every x and θ.

Let $\{R_t\}$ be a sequence of random variables defined in a probability space (S, F, P) and $\Im_a^b$ be the σ-algebra of events generated by the random variables $\{R_t : a \leq t \leq b\}$, then $\alpha(\Im_a^b, \Im_c^d) = sup_{A \in \Im_a^b, B \in \Im_c^d}|P(A \cap B) - P(A)P(B)|$ and $\alpha(m) = sup_t \alpha(\Im_{-\infty}^t, \Im_{t+m}^\infty)$. A stochastic process is said to be α-mixing if process $\alpha(m) \to 0$ as $m \to \infty$. Then we assume,

Assumption A4. 1. $\{(X_i, U_i)'\}_{i=1,2,...}$ is an α-mixing process of size -2, which implies that $\sum_{j=1}^\infty j^a \alpha(j)^{1-\frac{2}{\delta}} < \infty$ for $\delta > 2$ and $a > 1 - 2/\delta$; 2. We denote the joint density of $(X_i, U_i)'$ by $f_{X_i,U_i}(x_i, u_i)$, the density of X_i conditional on U_i by $f_{X_i|U_i}(x)$ with $f_{X_i|U_i}(x) < C$ and the conditional density of X_i, X_j given U_i, U_j by $f_{X_iX_j|U_iU_j}(x_i, x_j)$ with $f_{X_iX_j|U_iU_j}(x_i, x_j) < C$ for all x_i, x_j; 3. There exists a sequence of positive integers satisfying $s_n \to \infty$ and $s_n = o((nh_n)^{1/2})$ such that $\left(\frac{n}{h_n}\right)^{1/2} \alpha(s_n) \to 0$ as $n \to \infty$.

Assumption A5. $m^{(d)}(x) < C$ for all x and $d = 1, 2$, where $m^{(d)}(x)$ is the d^{th} order derivative of $m(x)$ evaluated at x.

Our assumption A1 requires the densities of regressor X_i to be smooth and bounded functions, and in the case where X_i come from heterogeneous distributions, the average of the densities must converge. This is automatically satisfied if X_i come from the same distribution, or X_i are part of a strictly stationary sequence. Assumption A2 is a standard assumption for the kernel functions in the nonparametric regression estimation. Assumption A3 ensures that the weighted average of the diagonal terms of the error covariance converge as $n \to \infty$ which is trivially met when there is a homoscedastic error structure. Under the mixing conditions imposed in A4, the temporal dependence among $\{(X_i, U_i)'\}$ will diminish as the time distance increases, which is general enough to include many interesting cases like panel data models or autoregressive model of order (p) (see section 4), while still allowing a central limit theorem to apply on the standardized summation. We impose smoothness condition on $m(x)$ in A5 so the standard Taylor approximations could carry through.

We now state Theorem 1 which is a supporting result for the main theorems that follow. All proofs are provided in Appendix 1.

Theorem 1 *Let $\{(X_i, U_i)\}_{i=1}^n$ be a stochastic sequence of vectors, $\{v_i\}_{i=1}^n$ be a uniformly bounded non stochastic sequence in $\Re$ and define*

$$s_j(x) = (nh_n)^{-1} \sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right)\left(\frac{X_i - x}{h_n}\right)^j g(U_i)v_i \text{ with } j = 0, 1, 2.$$

where $g : \Re \to \Re$ is measurable. Assume that: 1. $E(|g(U_i)|^{2+\theta}) < C$ for some $\theta > 0$ and all i; 2. $sup_{x\in G} \int |g(U_i)|^a f_{X_i,U_i}(x, U_i)dU_i < \infty$ for some $a > 1$; 3. A2 and A4. For G a compact subset of $\Re$ we have

$$sup_{x\in G}|s_j(x) - E(s_j(x))| = O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2}\right) \tag{4}$$

provided that $s, \beta > 2$ we have that $n^{(\theta+1/s)(\beta+1.5)+1.25-\beta/2}h_n^{-1.75-\beta/2}(ln(n))^{0.25+\beta/2} \to 0$.

By taking $v_i = 1$ and $g(x) = 1$ for all i and x in Theorem 1 we have that $sup_{x\in G}|s_{n,j}(x) - E(s_{n,j}(x))| = o_p(h_n^p)$ for $p > 0$ and $j = 0, 1, 2$ provided that $\frac{nh_n^{2p+1}}{ln(n)} \to \infty$. The last condition is consistent with $n^{(\theta+1/s)(\beta+1.5)+1.25-\beta/2}h_n^{-1.75-\beta/2}(ln(n))^{0.25+\beta/2} \to 0$ as $n \to \infty$ for $\theta > 0$ and $s > 2$. Consequently, if $p = 1$, $\frac{nh_n^3}{ln(n)} \to \infty$ we have that $sup_{x\in G}\frac{1}{h_n}|s_{n,j}(x) - E(s_{n,j}(x))| = o_p(1)$.

The next theorem establishes the asymptotic $\sqrt{nh_n}$ - normality for the local linear estimator under general parametric covariance structure. We stress that the importance of the result lies in the fact that the regression errors are not restricted to be (iid) or even weakly stationary. We do assume, however, that $\{X_i\}_{i=1,2,\cdots}$ and $\{U_i\}_{i=1,2,\cdots}$ are independent processes.

Theorem 2 *Let $\{(X_i, U_i)\}_{i=1}^n$ be a stochastic sequence of vectors and assume that $Y_i = m(X_i) + U_i$ for $i = 1, 2, \cdots$, $\{X_i\}_{i=1,2,\cdots}$ and $\{U_i\}_{i=1,2,\cdots}$ are independent with $E(U_i) = 0$ for all $i = 1, 2, \cdots$, $E(U_iU_j) = \omega_{ij}(\theta_0)$*

$\theta_0 \in \Re^p, p < \infty$. If we assume that A1-A5 are met and $E(|U_i|^{2+\theta}) < C$ for some $\theta > 0$ and all i, then

$$(nh_n)^{1/2}(\check{m}(x) - m(x) - B_{n,1}(x)) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}^2(x)} \int K^2(\phi)d\phi\right) \tag{5}$$

where $B_{n,1}(x) = \frac{h_n^2}{2}\sigma_K^2 m^{(2)}(x) + o_p(h_n^2)$, provided $\frac{ln(n)}{nh_n^3} \to 0$ and $h_n^2 ln(n) \to 0$.

In the case where $\{(X_i, U_i)'\}$ is an iid sequence with $f(x)$ being the marginal density for X_i and $\omega(\theta)$ the variance of U_i, the asymptotic variance is simplified to be $\frac{\omega(\theta)}{f(x)} \int K^2(\phi)d\phi$. Theorem 2 can therefore be seen as as a generalization of the classic asymptotic normality result for local linear estimation under the iid assumption. Examples in Section 4 illustrate the applicability of this general result in regression models where the error covariance has a random effects panel data structure, and an AR(p) structure.

3 Two Step Estimation - Asymptotic Normality

The estimator $\check{m}(x)$ studied in the previous section has the desirable property of being $\sqrt{nh_n}$-asymptotically normal. However, the fact that none of the information provided by the error covariance structure is used in its construction suggests that alternative estimators can provide improved performance. How to incorporate the covariance structure in defining an alternative estimator has been the subject of various papers (see, *inter alia* Severini and Staniswalis, 1994 and Lin and Carroll, 2000), but one promising approach has been a two step procedure that transforms the model to obtain spherical regression errors. The motivation behind the procedure is quite simple. Let $\Omega(\theta_0)$ be an $n \times n$ matrix with (i, j) element given by $\omega_{ij}(\theta_0)$, $P^{-1}(\theta_0)$ an $n \times n$ matrix with (i, j) element given by $v_{ij}(\theta_0)$ and $P(\theta_0)$ an $n \times n$ matrix with (i, j) element given by $p_{ij}(\theta_0)$ such that $\Omega(\theta_0) = P(\theta_0)P(\theta_0)'$. Let $\vec{m}' = (m(X_1), ..., m(X_n))$, $U' = (U_1, ..., U_n)$, I_n be the identity matrix of size n and define $Z = P^{-1}(\theta_0)\vec{y} + (I_n - P^{-1}(\theta_0))\vec{m}$. Then,

$$Z = \vec{m} + P^{-1}(\theta_0)U = \vec{m} + \varepsilon. \tag{6}$$

Given that the components of the stochastic process $\{U_i\}_{i=1,2,...}$ can be written $U_i = \sum_{j=1}^{q} p_{ij}\varepsilon_j$ where $q = 1, 2, ..., n$, if $\{\varepsilon_i\}_{i=1,2,...}$ is an independent identically distributed process with zero mean and variance σ^2 then the model described in (**??**) is the standard nonparametric regression model with spherical errors. The difficulty in dealing with such model stems from the fact that the regressand Z is not observed since

$\vec{m}$ and the components of $P^{-1}(\theta_0)$ are generally unknown - since θ_0 is unknown - and must be substituted by suitable estimates. Hence, implementation normally requires a first stage estimation in which $\check{m}(x)$ and estimators for the elements of $P^{-1}(\theta_0)$, say $P^{-1}(\dot{\theta})$ (normally using residuals $\check{U}_i = Y_i - \check{m}(X_i)$), are obtained, and a second stage in which the regressand $\hat{Z} = P^{-1}(\dot{\theta})\vec{y} + (I_n - P^{-1}(\dot{\theta}))\check{\vec{m}}$ is used in (**??**). The asymptotic properties of the resulting estimator are not known in general, but Xiao et al. (2003) have obtained $\sqrt{nh_n}$-asymptotic normality for a stationary error structure that has an invertible linear process representation $U_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$. A key feature of their structure is that the diagonal elements of $P^{-1}(\theta_0)$ are all equal to 1, a property that we will see below has important consequences in establishing the asymptotic normality of the estimator. Since this cannot be generally assumed we will propose a slightly different estimator that circumvents the difficulties we encountered with the estimator for general models.

In what follows we will restrict ourselves to stochastic processes $\{U_i\}_{i=1,2,...}$ that can be constructed from linear transformations of iid processes. Hence, we assume:

Assumption A6. The components of the stochastic process $\{U_i\}_{i=1,2,...}$ can be written as $U_i = \sum_{j=1}^{q} p_{ij}\varepsilon_j$ where $q = 1, 2, ..., n$ and $\{\varepsilon_i\}_{i=1,2,...}$ is an independent identically distributed process with zero mean and unit variance.

For economy of notation we also write p_{ij}, v_{ij}, P and P^{-1} where it is well understood that all of these variables depend on θ. Let $H = diag\{v_{ii}^{-1}\}_{i=1}^{n}$ and define $Z = HP^{-1}\vec{y} + (I_n - HP^{-1})\vec{m}$. Then,

$$Z = \vec{m} + HP^{-1}U = \vec{m} + \gamma. \tag{7}$$

Given assumption A6 $\{\gamma_i\}_{i=1,2,...}$ is an independent heterogeneous sequence with $E(\gamma) = 0$ and $E(\gamma\gamma') = H^2 = diag\{v_{ii}^{-2}\}_{i=1}^{n}$.

As above, the regression error γ_i in the transformed regression (**??**) is independent and heteroscedastic, but the vector of regressands is unknown. If $m(X_i)$ is estimated at a first stage by $\check{m}(X_i)$, then the only source of ignorance about Z is due to P^{-1} or the fact that θ_0 is unknown. Theorem 3 below we focus on establishing the asymptotic normality of the estimator

$$\hat{m}(x) = e'\left(R_x' K_x R_x\right)^{-1} R_x' K_x \check{Z} \tag{8}$$

where $\check{Z} = HP^{-1}\vec{y} - (HP^{-1} - I_n)\check{m}$, $\check{m}' = (\check{m}(X_1), ..., \check{m}(X_n))$ and for the moment we assume that θ_0, and therefore P^{-1} (and consequently H), is known.

Theorem 3 *Let $\{(X_i, U_i)\}_{i=1}^n$ be a stochastic sequence of vectors and assume that $Y_i = m(X_i) + U_i$ for $i = 1, 2, \cdots$, $\{X_i\}_{i=1,2,\cdots}$ and $\{U_i\}_{i=1,2,\cdots}$ are independent with $E(U_i) = 0$ for all $i = 1, 2, \cdots$, $E(U_iU_j) = \omega_{ij}(\theta_0)$ $\theta_0 \in \Re^p, p < \infty$. Consider the estimator $\hat{m}(x)$ described above, such that h_n is the bandwidth used in the first stage estimation and g_n is the bandwidth used in the second stage of the estimation. If we assume that A1-A6 are met and $E(|U_i|^{2+\theta}) < C$ for some $\theta > 0$ and all i, then,*

$$(ng_n)^{1/2}(\hat{m}(x) - m(x) - B_{n,1}(x)) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x, \theta_0)}{\bar{f}^2(x)} \int K^2(\phi)d\phi\right) \quad (9)$$

where $B_{n,1}(x) = \frac{g_n^2}{2}\sigma_K^2 m^{(2)}(x) + o_p(g_n^2)$, $\bar{\omega}_f(x, \theta_0) = lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f_i(x)v_{ii}^{-2}$ provided that: 1. $\frac{h_n}{g_n} \to 0$ and $g_n = O(n^{-1/5})$; 2. $sup_i \sum_{j=1,j\neq i}^n \frac{|v_{ij}|}{|v_{ii}|} = O(1)$ and $sup_i \sum_{j=1,j\neq i}^n \frac{|v_{ji}|}{|v_{jj}|} = O(1)$.

We note that difference between the variances of the asymptotic distributions of $\check{m}(x)$ and $\hat{m}(x)$ is given by,

$$lim_{n\to\infty} \frac{1}{n\bar{f}(x)^2} \sum_{i=1}^n f_i(x) \left(\omega_{ii}(\theta_0) - \frac{1}{v_{ii}^2}\right) \int K^2(\phi)d\phi. \quad (10)$$

By Theorem 12.2.10 in Graybill (1983) that $p_{ii}v_{ii} \geq 1$. Consequently,

$$p_{ii}^2 \geq \frac{1}{v_{ii}^2} \Rightarrow \omega_{ii}(\theta_o) = p_{ii}^2 + \sum_{j=1,j\neq i}^n p_{ij}^2 \geq \frac{1}{v_{ii}^2}$$

which establishes that $\hat{m}(x)$ is *efficient* relative to $\check{m}(x)$. The improvement over local linear estimation is obtained even though $\hat{m}(x)$ ignores the heteroscedastic structure of the error.

Notice also that we impose two more assumptions in Theorem 3. The first one relates to undersmoothing in the first stage regression so that the magnitude of the bias created by $\hat{m}(x)$ will be smaller than the leading bias term in the second stage. This assumption is common in two stage nonparametric regression estimation, e.g., Assumption 7 in Xiao et al. (2003), Assumption B5 in Su and Ullah (2003) and Remark 1 in Wang (2003). The second assumption is essentially uniform summability of the rows of error covariance, which is a sufficient condition used in the proof of Theorem 3 to control the order of magnitude for summation terms showing up in the second stage. Similar assumptions have been used in the literature, i.e., Assumption A.3 in Francisco-Fernandez and Vilar-Fernandez(2001) and Assumption 5 in Xiao et al. (2003).

An important part of the proof in Theorem 3 is that $\check{Z}_i = m(X_i) - \sum_{j=1,j\neq i}^{n} \frac{v_{ij}}{v_{ii}}(\check{m}(X_j) - m(X_j)) + \gamma_i$. If instead we were considering the estimator $\tilde{m}(x) = e'\left(R_x' K_x R_x\right)^{-1} R_x' K_x \check{Z}$ where $\check{Z} = P^{-1}\vec{y} - (P^{-1} - I_n)\check{m}$, then $\check{Z}_i = m(X_i) + \varepsilon_i - \sum_{j=1}^{n} v_{ij}(\check{m}(X_j) - m(X_j)) + (\check{m}(X_i) - m(X_i))$ and $B_n(x) = \frac{1}{ng_n \hat{f}_n(x)} \sum_{i=1}^{n} K\left(\frac{X_i - x}{g_n}\right) \check{Z}_i^*$ would have an extra term given by $\frac{1}{\hat{f}_n(x)} \frac{1}{ng_n} \sum_{i=1}^{n} K\left(\frac{X_i - x}{g_n}\right)(\check{m}(X_i) - m(X_i))$ which cannot easily be shown to be $o_p((ng_n)^{-1/2})$ under the general conditions we consider. By construction, whenever the diagonal elements of P^{-1} are equal to 1 this extra term does not appear even when $\check{Z} = P^{-1}\vec{y} - (P^{-1} - I_n)\check{m}$. Hence, we have the following result which we state as a Corollary to Theorem 3.

Corollary 1 *Let $\{(X_i, U_i)\}_{i=1}^{n}$ be a stochastic sequence of vectors and assume that $Y_i = m(X_i) + U_i$ for $i = 1, 2, \cdots$, $\{X_i\}_{i=1,2,\cdots}$ and $\{U_i\}_{i=1,2,\cdots}$ are independent with $E(U_i) = 0$ for all $i = 1, 2, \cdots$, $E(U_i U_j) = \omega_{ij}(\theta_0)$ $\theta_0 \in \Re^p, p < \infty$. Consider the estimator $\tilde{m}(x)$ described above, such that h_n is the bandwidth used in the first stage estimation and g_n is the bandwidth used in the second stage of the estimation. If we assume that A1-A6 are met and $E(|U_i|^{2+\theta}) < C$ for some $\theta > 0$ and all i. Then,*

$$(ng_n)^{1/2}(\tilde{m}(x) - m(x) - B_{n,1}(x)) \xrightarrow{d} N\left(0, \frac{1}{f(x)} \int K^2(\phi)d\phi\right) \tag{11}$$

provided that: 1. $\frac{h_n}{g_n} \to 0$ and $g_n = O(n^{-1/5})$; 2. $sup_i \sum_{j=1,j\neq i}^{n} |v_{ij}| = O(1)$ and $sup_i \sum_{j=1,j\neq i}^{n} |v_{ji}| = O(1)$; 3. $P^{-1}(\theta_0)$ is such that $v_{ii}(\theta_0) = 1$ for all i.

The use of Theorem 3 and its Corollary is restricted in practice due to the fact that the parameter θ used in defining P is generally unknown and must be estimated. Hence, we now turn our attention to a feasible estimator $\dot{m}(x) = e'\left(R_x' K_x R_x\right)^{-1} R_x' K_x \dot{Z}$ where $\dot{Z} = H(\dot{\theta})P^{-1}(\dot{\theta})\vec{y} - (H(\dot{\theta})P^{-1}(\dot{\theta}) - I_n)\check{m}$ and for which $\dot{\theta} - \theta_0 = o_p(1)$. The next theorem provides sufficient conditions under which $\sqrt{ng_n}(\dot{m}(x) - \hat{m}(x)) = o_p(1)$. As such, it gives conditions under which the the feasible estimator is asymptotically equivalent to $\hat{m}(x)$, therefore inheriting its desirable properties, namely asymptotic normality and efficiency relative to the LLE. The theorem can be viewed as an extension of the theorem in Mandy and Martins-Filho (1994) to the case of nonparametric regression.

Theorem 4 *Suppose that all assumptions in Theorem 3 are holding and assume in addition that:*

TA 4.1: $H(\theta)P^{-1}(\theta)$ has at most $W < \infty$ distinct nonzero elements for every n, denoted by $g_{wn}(\theta)$ for

$w = 1, 2, ..., W$. That is, there are $n^2 - W$ elements that are either zero or duplicates of other nonzero elements in $H(\theta)P^{-1}(\theta)$. For each w, $g_{wn}(\theta)$ converges uniformly as $n \to \infty$ to a real valued function $g_w(\theta)$ on an open set O containing θ_0, where g_w is continuous at θ_0.

TA 4.2: The number of nonzero elements in each column (and row) of $H(\theta)P^{-1}(\theta)$ is uniformly bounded by $\aleph$ as $n \to \infty$.

TA 4.3: There exists $C < \infty$ such that $\sum_{i=1}^{n} |\omega_{ij}(\theta)| < C$ for every $n = 1, 2, ...$ and $j = 1, 2, ...$

If $\dot{\theta} - \theta_0 = o_p(1)$ then we have

$$\sqrt{ng_n}(\hat{m}(x) - \dot{m}(x)) = o_p(1).$$

4 Selected Applications

In this section we provide two applications for the results we have obtained. The first deals with clustered or panel data models. Here, the asymptotic normality result we obtain for local linear and the two stage estimator is novel. The second application is for nonparametric regression models with autoregressive errors of order p, which have been studied by Vilar-Fernández and Francisco-Fernández (2002) for the case where $p = 1$ under fixed design regressors. The examples illustrate the applicability of our theorems to popular nonparametric models and reveal the ease of verifying the conditions listed in Theorems 3 and 4.

4.1 Clustered or Panel Data Models

We focus on the regression models for clustered data proposed by Ruckstuhl et al. (2000) and also studied by Wang (2003). The model is a direct extension to the nonparametric regression setting of the one-way *random effects* model that is popular in the panel data literature (Baltagi, 1995). Consider

$$Y_{ij} = m(X_{ij}) + \alpha_i + \varepsilon_{ij} \;\; i = 1, ..., N; j = 1, ..., J, \tag{12}$$

where $\{\alpha_i\}_{i=1,2,...}$ are independent with $E(\alpha_i) = 0$ and $V(\alpha_i) = \sigma_\alpha^2$ for all i; $\{\varepsilon_{ij}\}_{i,j=1,2,...}$ are independent with $E(\varepsilon_{ij}) = 0$ and $V(\varepsilon_{ij}) = \sigma_\varepsilon^2$ for all i, j and the processes $\{\alpha_i\}_{i=1,2,...}$ and $\{\varepsilon_{ij}\}_{i,j=1,2,...}$ are independent. Ruckstuhl et al. (2000) assume that $\{X_i\}_{i=1,2,...}$ where $X_i' = (X_{i1}, ..., X_{iJ})$ is an independent and identically distributed vector sequence with the marginal density of X_{ij} given by f_j.

We define $Y_i' = (Y_{i1}, ..., Y_{iJ})$, $\vec{y} = (Y_1', ..., Y_N')'$, $X_i' = (X_{i1}, ..., X_{iJ})$, $\vec{x} = (X_1', ..., X_N')'$ and $U_{ij} = \alpha_i + \varepsilon_{ij}$. Then, given the assumptions on α_i and ε_{ij} we have that for $U_i' = (U_{i1}, ..., U_{iJ})$, $E(U_i U_i') = \Sigma = \sigma_\varepsilon^2 I_J + \sigma_\alpha^2 1_J 1_J'$ and if $U = (U_1', ..., U_N')'$, $E(UU') = I_N \otimes \Sigma = \Omega(\sigma_\varepsilon^2, \sigma_\alpha^2)$. In this context we have that $\check{m}(x) = e'\left(\bar{R}_x' \bar{K}_x \bar{R}_x\right)^{-1} \bar{R}_x' \bar{K}_x \vec{y}$ where $\bar{R}_x = (1_{NJ}, \vec{x} - 1_{NJ} x)$, $\bar{K}_x = diag\left\{K\left(\frac{X_{ij}-x}{h_n}\right)\right\}_{i=1,j=1}^{N,J}$. Let $n = NJ$, then the LLE estimator can be written as $\check{m}(x) = \frac{1}{nh_n}\sum_{i=1}^N \sum_{j=1}^J W_n\left(\frac{X_{ij}-x}{h_n}, x\right) Y_{ij}$.

We assume A1.1-4 and verify that A1.5-6 hold since $\bar{f}_n(x) = \frac{1}{J}\sum_{j=1}^J f_j(x)$ and as assumed in Ruckstuhl et al. (2000) if $0 < f_j(x) < C$ we have $0 < \bar{f}_n(x) < B$. A3 is verified since $0 < \sigma_\alpha^2, \sigma_\varepsilon^2 < C$ and consequently $\frac{1}{n}\sum_{i=1}^n \omega_{ii}(\sigma_\alpha^2, \sigma_\varepsilon^2) = \sigma_\alpha^2 + \sigma_\varepsilon^2$ and $\bar{\omega}_f(x, \sigma_\alpha^2, \sigma_\varepsilon^2) = (\sigma_\alpha^2 + \sigma_\varepsilon^2)\bar{f}_n(x)$. Now, since the process $\{X_i\}$ is independent and identically distributed, $\{X_{ij}\}$ is such that $\alpha(t) = 0$ for all $t \geq J$. Similarly, since $\{\alpha_i\}$ is independent and $\{\varepsilon_{ij}\}$ is independent, we have that U_{ij} and $U_{i'j'}$ is independent for all $i \neq i'$ for all j, j' and therefore $\alpha(t) = 0$ for all $t \geq J$, verifying A4 given the independence of $\{X_i\}$ and $\{U_{ij}\}$. A6 is easily verified by the independence of $\{\alpha_i\}$ and $\{\varepsilon_{ij}\}$ and noting that $U = Pv$ where v is a vector of iid random variables with $E(v_i) = 0$ and $V(v_i) = 1$. Hence, we conclude that

$$\sqrt{ng_n}\left(\check{m}(x) - m(x) - \left(\sigma_K^2 \frac{m^{(2)}(x)}{2} g_n^2 + o_p(g_n^2)\right)\right) \xrightarrow{d} N\left(0, \frac{\sigma_\varepsilon^2 + \sigma_\alpha^2}{\frac{1}{J}\sum_{j=1}^J f_j(x)} \int K^2(\phi) d\phi\right). \quad (13)$$

From Wansbeek and Kapteyn(1983) we have that $P^{-1}(\sigma_\alpha^2, \sigma_\varepsilon^2) = I_N \otimes V^{-1/2}$ where

$$V^{-1/2} = v_d \begin{pmatrix} 1 & \frac{v_0}{v_d} & \cdots & \frac{v_0}{v_d} \\ \frac{v_0}{v_d} & 1 & \cdots & \frac{v_0}{v_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{v_0}{v_d} & \frac{v_0}{v_d} & \cdots & 1 \end{pmatrix} \quad (14)$$

where $v_d = \frac{1}{\sigma_\varepsilon} - \left(1 - \frac{\sigma_\varepsilon}{\sigma_1}\right)\frac{1}{J\sigma_\varepsilon}$ and $v_0 = -\left(1 - \frac{\sigma_\varepsilon}{\sigma_1}\right)\frac{1}{J\sigma_\varepsilon}$ and $\sigma_1 = \sqrt{J\sigma_\alpha^2 + \sigma_\varepsilon^2}$. Hence, since $0 < \sigma_\alpha^2, \sigma_\varepsilon^2 < C$ and J is finite, we have that the sum of the elements in every row and column of HP^{-1} (excluding the diagonals) is $(J-1)\frac{v_0}{v_d} < C$, which satisfies condition 2 in Theorem 3. TA 4.1 is met with $W = 2$, $g_1(\sigma_\alpha^2, \sigma_\varepsilon^2) = v_0/v_d$ and $g_2(\sigma_\alpha^2, \sigma_\varepsilon^2) = 1$ the uniform convergence is trivial as neither function depends on n and the continuity is easily verified. TA 4.2 is met with $\aleph = J$ and TA 4.3 is met since $\sum_{i=1}^n |\omega_{ij}(\theta_0)| \leq J\sigma_\alpha^2 + \sigma_\varepsilon^2$.

Consistent estimators for σ_α^2 and σ_ε^2 are given by $\dot{\sigma}_\varepsilon^2 = \frac{1}{N(J-1)}\sum_{i=1}^N \sum_{j=1}^J (Y_{ij} - \check{m}(X_{ij}) - (\bar{Y}_i - \bar{m}_i))^2$ and $\dot{\sigma}_\alpha^2 = \frac{1}{N}\sum_{i=1}^N (\bar{Y}_i - \bar{m}_i)^2 - \frac{1}{J}\dot{\sigma}_\varepsilon^2$, where $\bar{Y}_i = \frac{1}{J}\sum_{j=1}^J Y_{ij}$ and $\bar{m}_i = \frac{1}{J}\sum_{j=1}^J \check{m}(X_{ij})$.Thus, we conclude that

$$\sqrt{ng_n}\left(\dot{m}(x)-m(x)-\left(\sigma_K^2\frac{m^{(2)}(x)}{2}g_n^2+o_p(g_n^2)\right)\right)\xrightarrow{d}N\left(0,\frac{\sigma_\varepsilon^2\left(1-\frac{1}{J}(1-\frac{\sigma_\varepsilon}{\sigma_1})\right)^{-2}}{\frac{1}{J}\sum_{j=1}^J f_j(x)}\int K^2(\phi)d\phi\right). \quad (15)$$

4.2 Nonparametric Regression with AR(p) Errors

We now consider

$$Y_i = m(X_i) + U_i \text{ for } t = 1, ..., n \quad (16)$$

where $\{X_i\}$ is independent of $\{U_i\}$, satisfies assumption A1, A3 and is α-mixing of size -2. U_i is strictly stationary with $U_i = r_1U_{i-1} + r_2U_{i-2} + ... + r_pU_{i-p} + v_i$ for $i = 0, \pm 1, \pm 2, ...$ where $v_i \sim iid(0, \sigma^2)$ with probability density function $f_v(x)$. Then $\{U_i\}$ satisfies the relevant portions of A3. Pham and Tram (1985) show that $\{U_i\}$ is α-mixing with $\alpha(j) \to 0$ exponentially as $j \to \infty$, which gives $\{U_i\}$ is of size $-a$ for all $a \in \Re^+$, therefore satisfying A4.1. Hence,

$$\sqrt{ng_n}\left(\check{m}(x)-m(x)-\left(\sigma_K^2\frac{m^{(2)}(x)}{2}g_n^2+o_p(g_n^2)\right)\right)\xrightarrow{d}N\left(0,\frac{\gamma(0)}{\bar{f}}\int K^2(\phi)d\phi\right) \quad (17)$$

where $\gamma(0)$ is the variance of the AR(p) process.

Following Mandy and Martins-Filho (1994) we note that since $0 < \sigma^2 < C$ we can find a matrix $p \times p$ lower triangular matrix A such that

$$AE((u_1, ..., u_p)'(u_1, ..., u_p))A' = \sigma^2 I_p \text{ and}$$

$$P^{-1}(\theta_0) = \begin{pmatrix} & A & & | & 0 & \cdots & \cdots & 0 \\ ---- & ---- & ---- & | & ---- & ---- & ---- & \vdots \\ -r_p & \cdots & -r_1 & | & 1 & 0 & \cdots & 0 \\ 0 & -r_p & \cdots & | & -r_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & | & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & | & & -r_p & \cdots & -r_1 \;\; 1 \end{pmatrix} \quad (18)$$

where $\theta_0 = (r_1, r_2, ..., r_p, \sigma^2)$. Since there are a finite number of bounded nonzero elements in each column and row of $P^{-1}(\theta_0)$, conditions 1 and 2 in Theorem 2 are automatically met. Also, since P^{-1} is a lower triangular matrix where all elements that lie more than p positions away from the main diagonal are zero, verifying TA 4.2 with $\aleph = p + 1$. Also, there are at most $W = p(p+1)/2 + (p+1)$ distinct functions in P^{-1}, all of which are independent of n for $n \geq W$ (implying uniform convergence trivially) and continuous at θ_0 since the operations involved in obtaining A are continuous when $0 < \sigma^2 < C$. This verifies TA 4.1.

To verify TA 4.3 we note that an $AR(p)$ process can be written as a p-dimensional VAR(1) process $e_i = Re_{i-1} + \varepsilon_i$, where $e_i = (U_{i-p+1}...U_i)'$, $\varepsilon_i = (0, ...0, v_i)'$, and

$$R = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ & & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ r_p & r_{p-1} & \cdots & \cdots & r_2 & r_1 \end{pmatrix} \tag{19}$$

If the process is strictly stationary then the absolute eigenvalues of R are less than one, and also $E(e_i e_j') = R^{|i-j|}E(e_t e_t')$ for arbitrary t. From the definition of e_i, the sum $\sum_{i=1}^n |E(U_i U_j)|$ is the lower right element of $\sum_{i=1}^n |E(e_i e_j')|$ where the absolute value is taken element-wise. But,

$$\sum_{i=1}^n |E(e_i e_j')| \leq 2\sum_{i=1}^n |E(e_i e_0')| \leq 2\left(\sum_{i=0}^n |R^i|\right)|E(e_0 e_0')|$$

and re-writing $|R^i|$ in Jordan Canonical form yields,

$$\sum_{i=1}^n |E(e_i e_j')| \leq 2|J|\left(\sum_{i=0}^n |\Lambda^i|\right)|J^{-1}||E(e_0 e_0')|$$

where Λ is a diagonal matrix involving the eigenvalues of R and J is a fixed matrix depending only on R. Since the absolute eigenvalues are less than one $\sum_{i=0}^\infty |\Lambda_i|$ converges, which verifies TA 4.3.

Consistent estimators $\dot{r}_i$ for r_i, $i = 1, ..., p$ can be obtained (see Vilar-Fernández and Francisco-Fernández, 2002) by defining residuals $\check{U}_i = Y_i - \check{m}(X_i)$ and performing least squares estimation on the following artificial regression,

$$\check{U}_i = r_1\check{U}_{i-1} + r_2\check{U}_{i-2} + ... + r_p\check{U}_{i-p} + \check{v}_i \text{ for } i = p+1, p+2, ...$$

where $\check{v}_i$ is an arbitrary regression error. Hence, we conclude

$$\sqrt{ng_n}\left(\dot{m}(x) - m(x) - \left(\sigma_K^2 \frac{m^{(2)}(x)}{2} g_n^2 + o_p(g_n^2)\right)\right) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\bar{f}} \int K^2(\phi)d\phi\right). \tag{20}$$

5 Monte Carlo Study

In this section, we perform a Monte Carlo study to implement our two step estimator, henceforth referred to as 2SLL, and illustrate its finite sample performance. We consider a one-way random effects panel data and an AR(2) parametric covariance structures, under which the asymptotic properties of 2SLL and of LLE are provided in the previous section.

For panel data structure, the data generating process (DGP) is given by (12), where the univariate pseudo random variable X_{ij} is generated independently from an uniform distribution with support $[-2, 2]$. The pseudo random variable α_i is independently generated from a normal distribution with zero mean and variance $\sigma_\alpha^2 = 4$, and ϵ_{ij} is independently generated from a standard normal distribution. We investigate three function specifications for $m(x)$: $m_1(x) = sin(0.75x)$, $m_2(x) = 0.5 + \frac{exp(-4x)}{1+exp(-4x)}$ and $m_3(x) = 1 - 0.9exp(-2x^2)$. $m_1(x)$ was used by Fan (1992) to illustrate the advantage of LLE over Nadaraya-Watson and Gasser-Müller estimators, and $m_2(x)$ and $m_3(x)$ were used by Martins-Filho and Yao (2006) to model the volatility of financial asset returns. All specifications for $m(\cdot)$ are nonlinear and twice differentiable. We fix $J = 2$ and consider three sample sizes $n = 100, 150$ and 200.

For the AR(2) structure, the DGP is given by (16), where the univariate pseudo random variable X_i is generated independently from an uniform distribution with support $[-2, 2]$. For the error $U_i = r_1 U_{i-1} + r_2 U_{i-2} + v_i$, we set $r_1 = 0.5, r_2 = -0.4$ and generate the pseudo random variable v_i independently from a standard normal distribution. It is straightforward to verify that for this choice of parameters $\{U_i\}$ is a stationary process. The same three functional forms for $m(\cdot)$ given above are adopted. We consider three sample sizes $n = 100, 200$, and 400.

The implementation of our 2SLL estimator requires the selection of bandwidth sequences h_n and g_n. We select the bandwidth $\hat{g}_n$ using the *rule-of-thumb* data driven plug-in method of Ruppert et al. (1995) and let $\hat{h}_n = (nJ)^{-\frac{1}{10}} \hat{g}_n$ in the panel data model and $\hat{h}_n = n^{-\frac{1}{10}} \hat{g}_n$ in the AR(2) model. An Epanechnikov kernel is utilized throughout the simulations. We note that the choice of bandwidth and kernel satisfies the requirements in Theorems 2 and 3.

For comparison purpose, we include in our simulations several estimators proposed in the extant literature. Ullah and Roy (1998), Lin and Carroll (2000) and Henderson and Ullah (2005) consider the panel data model and local linear estimators based on transformed observations to incorporate the information contained in error covariance structure in a specific fashion. Their estimators are defined as

$$\hat{\delta}_i(x) = e'(R_x'W_r(x)R_x)^{-1}R_x'W_r(x)\vec{y}$$

for $i = 1, 2$ and $W_1(x) = (P^{-1})'K_xP^{-1}$ and $W_2(x) = K_x^{-\frac{1}{2}}\Omega^{-1}K_x^{-\frac{1}{2}}$. Essentially, $\hat{\delta}_1(x)$ is a LLE on the transformed observations $K_x^{-\frac{1}{2}}P^{-1}\vec{y}$ and $K_x^{-\frac{1}{2}}P^{-1}\vec{x}$, while $\hat{\delta}_2(x)$ is obtained using transformed observations $P^{-1}K_x^{-\frac{1}{2}}\vec{y}$ and $P^{-1}K_x^{-\frac{1}{2}}\vec{x}$. Henderson and Ullah (2005) provide feasible versions of $\hat{\delta}_i(x)$ by estimating the unknowns in Ω consistently. Henceforth, we refer to $\hat{\delta}_i(x)$ as HUi and their feasible versions as FHUi. We note that their estimators for the parameters in the covariance matrix coincide with those provided in section 4.1. For the panel data structure, we also consider the two step estimator proposed by Ruckstuhl et al. (2000), henceforth referred to as RWC, which is more efficient than the local linear estimator, and follow their suggestion to set $\tau = \sigma_\epsilon$. Note that if we set $\tau = \frac{1}{v_d}$, then RWC coincides with 2SLL. The unknown parameters in Ω are estimated as described in section 4.1.

For the AR(2) error structure, we consider the two step estimator proposed in Vilar-Fernández and Francisco-Fernández (2002), henceforth referred to as VFF. Their estimator is defined for AR(1) model and they show that under fixed design, VFF outperforms the LLE in finite sample. We consider VFF under a random design with an AR(2) covariance structure, where

$$P^{-1} = \begin{pmatrix} \left(\frac{(1+r_2)(1+r_1-r_2)(1-r_1-r_2)}{1-r_2}\right)^{\frac{1}{2}} & 0 & 0 & \cdots & \cdots & 0 \\ -\frac{r_1\sqrt{1-r_2^2}}{1-r_2} & \sqrt{1-r_2^2} & 0 & 0 & \cdots & 0 \\ -r_2 & -r_1 & 1 & 0 & \cdots & 0 \\ 0 & -r_2 & -r_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & -r_2 & -r_1 & 1 \end{pmatrix}$$

Since H in 2SLL is a diagonal matrix with the diagonal element being the reciprocal of that in P^{-1}, we observe that VFF differs from 2SLL only in the treatment of the first two observations, hence the estimators are asymptotically equivalent. Hence, we expect the estimators will have similar finite sample performance, which is confirmed in the Monte Carlo study. Although HUi were initially proposed for a panel data error structure, it is straightforward to adapt it to the AR(2) structure. We follow the procedures in section 4.2 to estimate the unknown parameters in Ω.

In total, for the panel data structure we consider nine estimators: LLE, four infeasible estimators where we utilize the true covariance matrix parameters which are available in the simulation study - HU1, HU2, RWC, 2SLL, and four feasible estimators - FHU1, FHU2, FRWC, and F2SLL, where we attach the letter "F" in front of the acronyms to indicate the unknown parameters in the covariance matrix are estimated. For the AR(2) error structure we consider nine estimators: LLE, HU1, HU2, VFF, 2SLL, FHU1, FHU2, FVFF and F2SLL. All the estimators, except 2SLL and F2SLL, are implemented with bandwidth $\hat{g}_n$ described previously. For each experiment design, we perform 1000 repetitions, evaluate $m(x)$ at twenty equally spaced points over the support interval for the regressor (X) and obtain the average bias, standard deviation and root mean squared error of each estimator. To avoid evaluation over areas of support where data are sparse, we exclude the lower and upper 5% of the support interval. The results are reported in Tables 1 and 2 (Appendix 2) for the panel data error structure and AR(2) structure, respectively.

As the sample size increases, across all experiment designs, all estimators generally perform better in terms of averaged standard deviation, root mean squared error and bias, where exceptions occur in bias, whose magnitude is much smaller. This confirms the asymptotic results in Section 4, and agrees with the consistency of the alternative estimators. In terms of the relative performance measured by standard deviation and root mean squared error, when panel data and infeasible estimators are considered, we observe that 2SLL consistently performs the best, followed closely by RWC estimator. For all three functional forms considered, we notice the reduction of standard deviation and root mean squared error by 2SLL and RWC over LL are well over 15%. These results are consistent with our Theorem 3, as well as Theorem 4 in Ruckstuhl et al. (2000), which suggests that two-step estimation properly accounting for the covariance information can improve upon classical local linear estimator. LLE carries similar standard deviation and root mean squared error as HU2, but both LLE and HU2 always outperform the HU1 estimator. Hence, HUi estimators do not seem to provide gains in terms of efficiency over LLE, at least under the panel data error specification. When the AR(2) model is considered, across all specifications for $m(x)$, VFF and 2SLL perform similarly and outperform all the other alternatives. The improvement in efficiency from both estimators against LLE is over 10%. Again this is consistent with our Theorem 3 as well as the comments above regarding the similarity in the two estimators. In addition, our results indicate that the simulation results in Vilar-Fernández and Francisco-Fernández (2002) carry through in the case of the DGP we specify.

For the AR(2) error structure, both HU1 and HU2 estimators outperform the LLE, with HU1 outperforming HU2. The asymptotic distributions for the HUi estimators under an AR(p) structure are unknown, but based on our simulations these might be viable alternatives. As we expected, the feasible estimators perform slightly worse than the infeasible estimators, where exceptions occur for the HUi estimators under the panel data error structure. We notice that the extra burden in computing the unknown parameter is minimal since the increase in magnitude of average standard deviation and root mean squared error is small. Consequently, the observations regarding the relative performance among alternative estimators are largely maintained as those for their infeasible versions. This observation gives support for our Theorem 4 in that feasible 2SLL, obtained by estimating the unknown parameters of the covariance matrix, is asymptotically equivalent to its infeasible version and outperforms the traditional LLE.

6 Summary

In this paper we provide sufficient conditions for the asymptotic normality of the local linear estimator proposed by Fan (1992) in regression models where the regression error has a non spherical parametric covariance structure and the regressors are dependent and heterogeneously distributed. In this context, it seems natural to define an alternative estimator that incorporates the parametric covariance structure in an attempt to reduce the variance of the asymptotic distribution. We propose a two step estimator that incorporates the parametric information given by the error covariance and provide sufficient conditions for obtaining its asymptotic distribution. A feasible version of the two step estimator that substitutes true parameter values with consistent estimators is shown to be $\sqrt{ng_n}$ asymptotically equivalent in probability to the two step estimator under some easily verified conditions. A Monte Carlo study reveals that the asymptotic results for our estimator are confirmed in finite samples and that our estimator can outperform previously proposed estimators.

Appendix 1

Theorem 1: *Proof* We prove the case where $j = 0$. Similar arguments can be used for $j = 1, 2$. Let $B(x_0, r) = \{x \in \Re : |x - x_0| < r\}$ for $r \in \Re^+$. G compact implies that there exists $x_0 \in G$ such that $G \subseteq B(x_0, r)$. Therefore for all $x, x' \in G$, $|x - x'| < 2r$. Let $h_n > 0$ be such that $h_n \to 0$ as $n \to \infty$ where $n \in \{1, 2, 3 \cdots\}$. For any n by the Heine-Borel Theorem there exists a finite collection of sets $\left\{B\left(x_k, \left(\frac{n}{h_n^2}\right)^{-1/2}\right)\right\}_{k=1}^{l_n}$ such that $G \subset \cup_{k=1}^{l_n} B\left(x_k, \left(\frac{n}{h_n^2}\right)^{-1/2}\right)$ for $x_k \in G$ with $l_n < \left(\frac{n}{h_n^2}\right)^{1/2} r$. The proof has three steps.

(1) We show that

$$sup_{x \in G}|s_0(x) - E(s_0(x))| \leq max_{1 \leq k \leq l_n}|s_0(x) - E(s_0(x))| + C(nh_n^2)^{-1/2},$$

(2) Let $s_0^B(x) = (nh_n)^{-1} \sum_{i=1}^n K\left(\frac{X_i - x}{h_n}\right) g(U_i) v_i I(|g(U_i)| \leq B_n)$ where $B_1 \leq B_2 \leq ...$ such that $\sum_{i=1}^\infty B_i^{-s} < \infty$ for some $s > 0$ and $I(\cdot)$ is the indicator function. We show that

$$sup_{x \in G}|s_0(x) - s_0^B(x) - E(s_0(x) - s_0^B(x))| = O_{as}(B_n^{1-s}),$$

(3) Let $0 < \Delta < \infty$, $\beta > 2$ and $\varepsilon_n = \left(\frac{nh_n}{ln(n)}\right)^{-1/2} \Delta$, we show that

$$P\left(max_{1 \leq k \leq l_n}\left|s_0^B(x_k) - E(s_0^B(x_k))\right| \geq \varepsilon_n\right) = O(B_n^{\beta+1.5} n^{1.25-\beta/2} h_n^{-1.75-\beta/2}(ln(n))^{0.25+\beta/2}).$$

Step 1: For $x \in B\left(x_k, \left(\frac{n}{h_n^2}\right)^{-1/2}\right)$,

$$\begin{aligned} |s_0(x) - s_0(x_k)| &= \left|\frac{1}{nh_n}\sum_{i=1}^n \left(K\left(\frac{X_i - x}{h_n}\right) - K\left(\frac{X_i - x_k}{h_n}\right)\right) g(U_i) v_i\right| \\ &\leq \frac{1}{nh_n}\sum_{i=1}^n C\left|\frac{x_k - x}{h_n}\right| |g(U_i)v_i| \text{ by A2.5.} \\ &\leq \frac{1}{h_n^2} C \left(\frac{n}{h_n^2}\right)^{-1/2} \frac{1}{n}\sum_{i=1}^n |g(U_i)v_i| \leq C(nh_n^2)^{-1/2}\frac{1}{n}\sum_{i=1}^n |g(U_i)| \end{aligned}$$

By the measurability of g and A4 $\{|g(U_i)|\}_{i=1,2,...}$ is α-mixing of size -2. Furthermore, given that $E(|U_i|^{2+\theta}) < C$ for some $\theta > 0$ and all i, we have from McLeish's LLN (see White, 2001, p.49) that $\frac{1}{n}\sum_{i=1}^n |g(Y_i)| - \frac{1}{n}\sum_{i=1}^n E(|g(Y_i)|) = o_p(1)$ and since $\frac{1}{n}\sum_{i=1}^n E(|g(U_i)|) < C$ we have $|s_0(x) - s_0(x_k)| \leq C(nh_n^2)^{-1/2}$ and similarly, $E(|s_0(x) - s_0(x_k)|) \leq C(nh_n^2)^{-1/2}$. Combining the two results, $sup_{x\in G}|s_0(x) - E(s_0(x))| \leq max_{1\leq k\leq l_n}|s_0(x_k) - E(s(x_k))| + 2C(nh_n^2)^{-1/2}$.

Step 2: $sup_{x\in G}|s_0(x) - s_0^B(x) - E(s_0(x) - s_0^B(x))| \leq T_1 + T_2$, where $T_1 = sup_{x\in G}|s_0(x) - s_0^B(x)|$ and $T_2 = sup_{x\in G}|E(s_0(x) - s_0^B(x))|$. We show that $T_1 = o_{as}(1)$ and $T_2 = O(B_n^{1-s})$ for $s > 0$. $T_1 = sup_{x\in G}\left|(nh_n)^{-1}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)g(U_i)v_iI(|g(U_i)| > B_n)\right|$. By the Borel-Cantelli Lemma for any $\epsilon > 0$ and for all m satisfying $m' < m < n$ we have $P(|g(U_m)| \leq B_n) > 1-\epsilon$ and by Chebyshev's Inequality and the increasing nature of the B_i sequence, for $n > N \in \Re$ we have, $P(|g(U_i)| < B_n) > 1-\epsilon$ for $i < m'$. Hence, for $n > max\{N, m\}$ we have that for all $i \leq n$, $P(|g(U_i)| < B_n) > 1-\epsilon$ and therefore $I(|g(U_i)| > B_n) = 0$ with probability 1, which gives $T_1 = o_{as}(1)$.

$$\begin{aligned} E(s_0(x) - s_0^B(x)) &= \frac{1}{nh_n}\sum_{i=1}^n \int\int_{|g(U_i)|>B_n} K\left(\frac{X_i - x}{h_n}\right) g(U_i)v_i f_{X_i,U_i}(X_i,U_i)dX_idU_i \\ &\leq \frac{C}{n}\sum_{i=1}^n sup_{x\in G}\int_{|g(U_i)|>B_n} |g(U_i)|f_{X_i,U_i}(x,U_i)dU_i \end{aligned}$$

By Hölder's inequality, for $s > 1$,

$$\int_{|g(U_i)|>B_n} |g(U_i)|f_{X_i,U_i}(x,U_i)dU_i \leq \left(\int |g(U_i)|^s f_{X_i,U_i}(x,U_i)dU_i\right)^{1/s}\left(\int I(|g(U_i)| > B_n)f_{X_i,U_i}(x,U_i)dU_i\right)^{1-1/s}$$

where the first integral after the inequality is uniformly bounded by assumption and since $f_{X_i|U_i}(x) < C$, we have by Chebyshev's Inequality $\left(\int I(|g(U_i)| > B_n)f_{X_i,U_i}(x,U_i)dU_i\right)^{1-1/s} \leq C(P(|g(U_i)| > B_n))^{1-1/s} \leq CB_n^{1-s}$. Hence, $T_2 = O(B_n^{1-s})$.

Step 3: $P\left(max_{1\leq k\leq l_n}\left|s_0^B(x_k) - E(s_0^B(x_k))\right| \geq \varepsilon_n\right) \leq \sum_{i=1}^{l_n} P\left(\left|s_0^B(x_k) - E(s_0^B(x_k))\right| \geq \varepsilon_n\right)$ and let $s_0^B(x_k) - E(s_0^B(x_k)) = \frac{1}{n}\sum_{i=1}^n Z_i$ where

$$Z_i = \frac{1}{h_n}K\left(\frac{X_i - x_k}{h_n}\right)g(U_i)v_iI(|g(U_i)| \leq B_n) - E\left(\frac{1}{h_n}K\left(\frac{X_i - x_k}{h_n}\right)g(U_i)v_iI(|g(U_i)| \leq B_n)\right)$$

By the uniform bound on v_i, A2 and $|g(U_i)|I(|g(U_i)| \leq B_n) \leq B_n$ we have that $|Z_i| \leq Ch_n^{-1}B_n$. Let $||Z_i||_\infty = inf\{a : P(Z_i > a) = 0\}$, then $sup_{1\leq i\leq n}||Z_i||_\infty \leq C\frac{B_n}{h_n}$. Then, from Theorem 1.3 in Bosq(1996) we have that for each $q = 1, 2, ..., [n/2]$

$$P\left(\frac{1}{n}\left|\sum_{i=1}^n Z_i\right| > \varepsilon_n\right) \leq 4exp\left(\frac{-\varepsilon_n^2 q}{8v^2(q)}\right) + 22\left(1 + \frac{4CB_n}{\varepsilon_n h_n}\right)^{1/2} q\alpha\left(\left[\frac{n}{2q}\right]\right)$$

where $v^2(q) = \frac{2}{p^2}\sigma^2(q) + \frac{CB_n\varepsilon_n}{2h_n}$, $p = n/2q$,

$$\sigma^2(q) = max_{0\leq j\leq 2q-1} E\left((([jp]+1-jp)Z_{[jp]+1} + Z_{[jp]+2} + ... + Z_{[(j+1)p]} + ((j+1)p - [(j+1)p])Z_{[(j+1)p+1]})^2 \right)$$

and $[a]$ denotes the integer part of $a \in \Re$. We first note that $\frac{h_n}{p}\sigma^2(q) = O(1)$. To see this note that,

$$\sigma^2(q) \leq max_{0\leq j\leq 2q-1} \left(\sum_{[jp]<i\leq[(j+1)p+1]} E(Z_i^2) + 2 \sum_{\substack{[jp]+1\leq l\leq[(j+1)p] \\ l<i}} \sum_{[jp]+1<i\leq[(j+1)p+1]} |E(Z_l Z_i)| \right).$$

Given A4.2 and $E(|g(U_i)|^{2+\theta}) < C$ for some $\theta > 0$ and all i we have after some simple algebra

$$\sum_{[jp]<i\leq[(j+1)p+1]} E(Z_i^2) \leq O(p/h_n).$$

Using Theorem(3)1 in Doukhan (1994), for $\delta > 2$ we have that $|E(Z_i Z_l)| \leq C h_n^{-2+2/\delta}(\alpha(i-l))^{1-2/\delta}$. Now, for any l such that $[jp]+1 \leq l \leq [(j+1)p]$ we have that $\sum_{[jp]+1<i\leq[(j+1)p+1]} |E(Z_l Z_i)| \leq \sum_{i=1}^{p*-1} |E(Z_l Z_{l+i})| + \sum_{i=1}^{p*-1} |E(Z_l Z_{l-i})|$ where $p* = [(j+1)p+1] - [jp] + 1$. Letting d_n be a sequence of integers such that $d_n h_n \to 0$ we can write

$$\sum_{i=1}^{p*-1} |E(Z_l Z_{l+i})| = \sum_{i=1}^{d_n} |E(Z_l Z_{l+i})| + \sum_{i=d_n+1}^{p*-1} |E(Z_l Z_{l+i})| = J_1 + J_2$$

and it can be easily shown that $J_1 = o(h_n^{-1})$ and $J_2 = O(h_n^{-1})$. Similarly we obtain $\sum_{i=1}^{p*-1} |E(Z_l Z_{l-i})| = O(h_n^{-1})$. Combining the results on the variance and covariances we have that $\frac{h_n}{p}\sigma^2(q) \leq C$ for n sufficiently large. Hence, we have that $p h_n v^2(q) \leq C + C p B_n \varepsilon_n$ and choosing $p = (B_n\varepsilon_n)^{-1}$ we have that for n sufficiently large $p h_n v^2(q) \leq C$. Then, $4exp\left(\frac{-\varepsilon_n^2 q}{8v^2(q)}\right) \leq 4exp\left(\frac{-\varepsilon_n^2 n h_n}{16C}\right) \leq 4n^{-\frac{\Delta^2}{16C}}$. Now,

$$22\left(1 + \frac{4CB_n}{\varepsilon_n h_n}\right)^{1/2} q\alpha\left(\left[\frac{n}{2q}\right]\right) = 22\left(\frac{B_n}{\varepsilon_n}\right)^{1/2} h^{-1/2}\left(\frac{h_n\varepsilon_n}{B_n} + 4C\right)^{1/2} q\alpha\left(\left[\frac{n}{2q}\right]\right)$$

and since $\frac{h_n\varepsilon_n}{B_n} \to 0$ as $n \to \infty$ we have that for n large enough and by A4, for $\beta > 2$

$$\begin{aligned} 22\left(1 + \frac{4CB_n}{\varepsilon_n h_n}\right)^{1/2} q\alpha\left(\left[\frac{n}{2q}\right]\right) &\leq C\left(\frac{B_n}{\varepsilon_n}\right)^{1/2} h_n^{-1/2}\frac{n}{2p}[p]^{-\beta} \\ &\leq C n h_n^{-1/2} B_n^{\beta+1.5}\varepsilon_n^{\beta+0.5} \end{aligned}$$

Thus, $P\left(max_{1\leq k\leq l_n}\left|s_0^B(x_k) - E(s_0^B(x_k))\right| \geq \varepsilon_n\right) < \frac{Cn^{1/2}}{h_n}\left(4n^{-\frac{\Delta^2}{16C}} + C n h_n^{-1/2} B_n^{\beta+1.5}\varepsilon_n^{\beta+0.5}\right)$ and if Δ is chosen such that $\frac{\Delta^2}{16C} > 1$ the first term in the summation to the right of the inequality is negligible and we have that $P\left(max_{1\leq k\leq l_n}\left|s_0^B(x_k)-E(s_0^B(x_k))\right|\geq\varepsilon_n\right)<CB_n^{\beta+1.5}(ln(n))^{0.25+\beta/2}n^{1.25-\beta/2}h_n^{-1.75-\beta/2}$ and therefore

$$P\left(max_{1\leq k\leq l_n}\left|s_0^B(x_k)-E(s_0^B(x_k))\right|\right)=O(B_n^{\beta+1.5}(ln(n))^{0.25+\beta/2}n^{1.25-\beta/2}h_n^{-1.75-\beta/2}).$$

Lastly, if $B_n\approx n^{1/s+\theta}$ for $s>2,\theta>0$ we have that $sup_{x\in G}|s_0(x)-s_0^B(x)-E(s_0(x)-s_0^B(x))|=o(n^{-1/2})$ and if $n^{(\theta+1/s)(\beta+1.5)+1.25-\beta/2}h_n^{-1.75-\beta/2}(ln(n))^{0.25+\beta/2}\to 0$ as $n\to\infty$, then

$$P\left(max_{1\leq k\leq l_n}\left|s_0^B(x_k)-E(s_0^B(x_k))\right|\geq\varepsilon_n\right)=O_p(1)$$

which completes the proof.

Theorem 2: *Proof* Note that $m(x)=\frac{1}{nh_n}\sum_{i=1}^n W_n\left(\frac{X_i-x}{h_n},x\right)(m(x)+m^{(1)}(x)(X_i-x))$ and put $S(x)=\begin{pmatrix}\bar{f}_n(x) & 0\\ 0 & \sigma_K^2\bar{f}_n(x)\end{pmatrix}$. Then $\check{m}(x)-m(x)=\frac{1}{nh_n}\sum_{i=1}^n W_n\left(\frac{X_i-x}{h_n},x\right)Y_i^*$, where $Y_i^*=Y_i-m(x)-m^{(1)}(x)(X_i-x)$. Let $A_n(x)=\frac{1}{h_n}\left(e'\left(S_n(x)^{-1}-S(x)^{-1}\right)^2e\right)^{1/2}$, $D_n(x)=\check{m}(x)-m(x)-\frac{1}{nh_n\bar{f}_n(x)}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)Y_i^*$. Then,

$$\begin{aligned}|D_n(x)| &= \frac{1}{nh_n}\left|e'(S_n^{-1}(x)-S^{-1}(x))\begin{pmatrix}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)Y_i^*\\ \sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)\left(\frac{X_i-x}{h_n}\right)Y_i^*\end{pmatrix}\right|\\ &\leq h_nA_n(x)\frac{1}{nh_n}\left(\left|\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)Y_i^*\right|+\left|\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)\left(\frac{X_i-x}{h_n}\right)Y_i^*\right|\right)\end{aligned}$$

by Hölder's Inequality. Under the conditions of Theorem 1 $sup_{x\in G}|s_{n,j}(x)-E(s_{n,j}(x))|=o_p(h_n)$ for $j=0,1,2$ provided that $\frac{nh_n^3}{ln(n)}\to\infty$. Now, $sup_{x\in G}\left|s_{n,2}(x)-\sigma_K^2\bar{f}_n(x)\right|\leq sup_{x\in G}|s_{n,2}(x)-E(s_{n,2}(x))|+sup_{x\in G}\left|E(s_{n,2}(x))-\sigma_K^2\bar{f}_n(x)\right|$, but

$$sup_{x\in G}\left|E(s_{n,2}(x))-\sigma_K^2\bar{f}_n(x)\right|\leq\frac{1}{n}\sum_{i=1}^n\int\phi^2K(\phi)|f_i(x+h_n\phi)-f_i(x)|d\phi\leq h_nC\sigma_K^2$$

given A1 and A2. Therefore, $sup_{x\in G}\left|s_{n,2}(x)-\sigma_K^2\bar{f}_n(x)\right|\leq o_p(h_n)+O(h_n)=O_p(h_n)$ and similar arguments give $sup_{x\in G}\left|s_{n,0}(x)-\bar{f}_n(x)\right|=O_p(h_n)$ and $sup_{x\in G}|s_{n,1}(x)|=O_p(h_n)$. As a result, $A_n(x)=O_p(1)$ uniformly in G. We now turn our attention to $B_n(x)=\frac{1}{nh_n\bar{f}_n(x)}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)Y_i^*$. Since, $Y_i^*=m(X_i)-m(x)-m^{(1)}(x)(X_i-x)+U_i$ and K has a bounded support $Y_i^*=\frac{1}{2}m^{(2)}(x)(X_i-x)^2+U_i+o_p(h_n^2)$ and

$$B_n(x)=\frac{h_n^2}{\bar{f}_n(x)}\frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)\frac{1}{2}m^{(2)}(x)\left(\frac{X_i-x}{h_n}\right)^2+\frac{1}{\bar{f}_n(x)}\frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{X_i-x}{h_n}\right)U_i$$

$$+ \quad o(h_n^2)\frac{1}{\bar{f}_n(x)}\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right) = B_{n,1}(x) + B_{n,2}(x) + B_{n,3}(x)$$

We examine each $B_{n,j}(x)$ for $j = 1, 2, 3$ separately.

$$\begin{aligned} B_{n,3}(x) &= \frac{1}{\bar{f}_n(x)}\left(\left(\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right) - \bar{f}_n(x)\right) + \bar{f}_n(x)\right) o(h_n^2) \text{ and} \\ |B_{n,3}(x)| &\leq \frac{1}{\bar{f}_n(x)}\left(\left|\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h_n}\right) - \bar{f}_n(x)\right| + \bar{f}_n(x)\right) o(h_n^2) \end{aligned}$$

Since $\bar{f}_n(x) \to \bar{f}(x)$ as $n \to \infty$, $|B_{n,3}(x)| \leq (O_p(h_n) + 1)o(h_n^2) = o_p(h_n^2)$. Furthermore, if $inf_{x\in G}|\bar{f}_n(x)| > 0$ as $n \to \infty$, $sup_{x\in G}|B_{n,3}(x)| = o_p(h_n^2)$. $B_{n,1}(x) = \frac{m^{(2)}(x)h_n^2}{2\bar{f}_n(x)} s_{n,2}(x)$ and therefore by Theorem 1, given that $inf_{x\in G}|\bar{f}_n(x)| > 0$ as $n \to \infty$

$$sup_{x\in G}|B_{n,1}(x) - \frac{h_n^2}{2}\sigma_K^2 m^{(2)}(x)| \quad \leq \quad C\frac{h_n^2}{2inf_{x\in G}\bar{f}_n(x)} sup_{x\in G}|s_{n,2}(x) - \sigma_K^2 \bar{f}_n(x)| = O_p(h_n^3).$$

Hence $B_{n,1}(x) = \frac{h_n^2}{2}\sigma_K^2 m^{(2)}(x) + o_p(h_n^2)$ uniformly in G.

Let $Z_i = \frac{1}{h_n}K\left(\frac{X_i - x}{h_n}\right)U_i$ then $B_{n,2}(x) = \frac{1}{\bar{f}_n(x)}\frac{1}{n}\sum_{i=1}^{n} Z_i$. Since the processes $\{X_i\}_{i=1}^n$ and $\{U_i\}_{i=1}^n$ are independent and $E(U_i) = 0$, $E(Z_i) = 0$. Now note that $V(Z_i) = \frac{1}{h_n^2}E\left(K^2\left(\frac{X_i - x}{h_n}\right)\right)E(U_i^2) = \frac{1}{h_n}\omega_{ii}(\theta_0)\int K^2(\phi)f_i(x + h_n\phi)d\phi$. Since $|\omega_{ii}(\theta_0)| < C$ and $f_i(x) < C$ we have that $h_nV(Z_i) \leq C\int K^2(\phi)d\phi$ and $sup_i h_nV(Z_i) = O(1)$. We now consider

$$\sum_{j=1, i\neq j}^{n} |cov(Z_i, Z_j)| = \sum_{j=1, i\neq j}^{n} |E(Z_i, Z_j)| \leq \sum_{j=1}^{n} |E(Z_i, Z_{i+j})| + \sum_{j=1}^{n} |E(Z_i, Z_{i-j})|.$$

First write $\sum_{j=1}^{n} |E(Z_i, Z_{i+j})| = \sum_{j=1}^{d_n-1} |E(Z_i, Z_{i+j})| + \sum_{j=d_n}^{n} |E(Z_i, Z_{i+j})| = J_{n,1} + J_{n,2}$, where d_n is a sequence of integers such that $d_n \to \infty$ and $d_nh_n \to 0$. Then,

$$\begin{aligned} J_{n,1} &= \sum_{j=1}^{d_n-1} \frac{1}{h_n^2}\left|EK\left(\frac{X_i - x}{h_n}\right)K\left(\frac{X_{i+j} - x}{h_n}\right)U_iU_{i+j}\right| \\ &= \sum_{j=1}^{d_n-1} |\omega_{i,i+j}(\theta_0)| \int K(\phi_1)K(\phi_2)f_{i,i+j}(x + h_n\phi_1, x + h_n\phi_2)d\phi_1 d\phi_2 \\ &\leq C\sum_{j=1}^{d_n-1}\left(\int K(\phi_1)d\phi_1\right)^2 = C(d_n - 1) \leq Cd_n. \end{aligned}$$

Since $d_nh_n \to 0$ we have that $h_nJ_{n,1} \leq Cd_nh_n = o(1)$ and $J_{n,1} = o(h_n^{-1})$. Given that $K(\cdot)$ is measurable we have that Z_i is $\sigma(X_i, U_i)$ measurable, where $\sigma(X_i, U_i)$ is the σ-algebra generated by (X_i, U_i). By Theorem

3(1) in Doukhan (1994) with $p = q = \delta > 2$ we have

$$|E(Z_i, Z_{i+j})| \leq 8E(|Z_i|^\delta)E(|Z_{i+j}|^\delta)\alpha(\sigma(X_i, U_i), \sigma(X_{i+j}, U_{i+j}))^{1-\frac{2}{\delta}}$$

where $\alpha(\sigma(X_i, U_i), \sigma(X_{i+j}, U_{i+j})) = sup_{A\in\sigma(X_i,U_i),B\in\sigma(X_{i+j},U_{i+j})}|P(A\cap B) - P(A)P(B)|$. Now define $\mathcal{F}^i_{-\infty} = \sigma(\cdots, X_{i-1}, U_{i-1}, X_i, U_i)$, $\mathcal{F}^\infty_{i+j} = \sigma(X_{i+j}, U_{i+j}, X_{i+j+1}, U_{i+j+1}, \cdots)$ and $\alpha(j) = sup_i\alpha(\mathcal{F}^i_{-\infty}, \mathcal{F}^\infty_{i+j})$. Then, $\alpha(\sigma(X_i, U_i), \sigma(X_{i+j}, U_{i+j})) \leq \alpha(j)$. Also,

$$\begin{aligned}
E|Z_i|^\delta &= E(|U_i|^\delta)h_n^{-\delta+1}\frac{1}{h_n}E\left(K^\delta\left(\frac{X_i - x}{h_n}\right)\right) \\
&= E(|U_i|^\delta)h_n^{-\delta+1}\int K^\delta(\phi)f_i(x + h_n\phi)d\phi \\
&\leq CE(|U_i|^\delta)h_n^{-\delta+1}\int K^\delta(\phi)d\phi \text{ by A1} \\
&\leq Ch_n^{-\delta+1}
\end{aligned}$$

Similarly $E|Z_{i+j}|^\delta \leq Ch_n^{-\delta+1}$ and we have $|E(Z_i, Z_{i+j})| \leq 8(Ch_n^{-\delta+1})^{2/\delta}\alpha(j)^{1-\frac{2}{\delta}} = Ch_n^{-2+\frac{2}{\delta}}\alpha(j)^{1-\frac{2}{\delta}}$. Hence, $J_{n,2} \leq Ch_n^{-2+\frac{2}{\delta}}\sum_{j=d_n}^\infty \alpha(j)^{1-\frac{2}{\delta}}$ and since $j \geq d_n$ we have that for some $a > 1 - \frac{2}{\delta} > 0$, $\frac{j^a}{d_n^a} \geq 1$ and $J_{n,2} \leq Ch_n^{-2+\frac{2}{\delta}}d_n^{-a}\sum_{j=d_n}^\infty j^a\alpha(j)^{1-\frac{2}{\delta}}$. But, $\sum_{j=d_n}^\infty j^a\alpha(j)^{1-\frac{2}{\delta}} \to 0$ by A4 as $n \to \infty$. Now, $h_n^{\frac{2}{\delta}-1}d_n^{-a} = \left((h_n d_n^{\frac{a\delta}{\delta-2}})^{1-\frac{2}{\delta}}\right)^{-1}$ and choosing d_n such that $h_n^{1-\frac{2}{\delta}}d_n^a = 1$ the right hand side of the last equality is equal to 1 and we have $J_{n,2} = o(h_n^{-1})$. This is obviously consistent with $d_nh_n \to 0$ in the sense that $\frac{a\delta}{\delta-2} > 1 \Rightarrow a > 1 - \frac{2}{\delta}$. Furthermore, it is easily seen from the developments above that $sup_i|J_{n,1}| + sup_i|J_{n,2}| = o(h_n^{-1})$ and $h_n sup_i\sum_{j=1}^n |E(Z_iZ_{i+j})| = o(1)$. Similar arguments show that $\sum_{j=1}^n |E(Z_iZ_{i-j})| = o(h_n^{-1})$ and $h_n sup_i\sum_{j=1}^n |E(Z_iZ_{i+j})| = o(1)$. Hence, combining results we have $\sum_{j=1,i\neq j}^n |cov(Z_i, Z_j)| = o(h_n^{-1})$ and $sup_i\sum_{j=1,i\neq j}^n |cov(Z_i, Z_j)| = o(h_n^{-1})$. Now, observe that $V\left(\frac{1}{n}\sum_{i=1}^n Z_i\right) = \frac{1}{n^2}\sum_{i=1}^n E(Z_i^2) + \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1,j\neq i}^n E(Z_iZ_j) = V_{n,1} + V_{n,2}$.

$$\begin{aligned}
V_{n,1} &= \frac{1}{n^2}\sum_{i=1}^n \frac{1}{h_n}\omega_{ii}(\theta_0)\int K^2(\phi)(f_i(x + h_n\phi) - f_i(x))d\phi + \frac{1}{n^2}\sum_{i=1}^n \frac{1}{h_n}\omega_{ii}(\theta_0)\int K^2(\phi)f_i(x)d\phi \\
&= V_{n,1}^1 + V_{n,1}^2
\end{aligned}$$

By the Lipschitz condition on $f_i(x)$ and A2 $|V_{n,1}^1| \leq C\frac{1}{n^2}\sum_{i=1}^n \omega_{ii}(\theta_0)$ and therefore $nh_n|V_{n,1}^1| \leq \frac{Ch_n}{n}\sum_{i=1}^n \omega_{ii}(\theta_0)$

and by A3 we have $nh_n|V_{n,1}^1| = O(h_n)$. Also,

$$nh_nV_{n,1}^2 = \int K^2(\phi)d\phi \frac{1}{n}\sum_{i=1}^{n} f_i(x)\omega_{ii}(\theta_0) \to \bar{\omega}_f(x,\theta_0)\int K^2(\phi)d\phi.$$

Hence, $\frac{h_n}{n}\sum_{i=1}^{n} E(Z_i^2) = \bar{\omega}_f(x,\theta_0)\int K^2(\phi)d\phi + O(h_n)$. Now,

$$nh_n\left|\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n} E(Z_iZ_j)\right| \leq \frac{1}{n}\sum_{i=1}^{n} h_n sup_i \sum_{j=1,i\neq j}^{n} |E(Z_iZ_j)| = o(1)$$

where the last equality follows from our previous results. Hence, we have that

$$V\left(\sqrt{nh_n}\frac{1}{n}\sum_{i=1}^{n} Z_i\right) = \bar{\omega}_f(x,\theta_0)\int K(\phi)d\phi + O(h_n) + o(1). \tag{21}$$

We now consider $B_{n,2}(x)$. Here we adopt the method first proposed by Bernstein (1927) and adopted by Masry and Fan (1997) to partition the sums in large and small blocks. First, partition the set $\{1,\cdots,n\}$ into $2k_n+1$ subsets with *large* blocks of size r_n and *small* blocks of size s_n and $k_n = \left[\frac{n}{r_n+s_n}\right]$. Let $Z_{n,i} = \sqrt{h_n}Z_{i+1}$ for $i = 0, 1, \cdots, n-1$ so that $B_{n,2}(x) = \frac{1}{\bar{f}_n(x)}\frac{1}{n}\sum_{i=1}^{n} Z_i$ and $\sqrt{nh_n}\frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} Z_{n,i}$. Now let

$$\begin{aligned}
\eta_j &= \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} Z_{n,i} \text{ for } 0 \leq j \leq k_n - 1 \\
\xi_j &= \sum_{i=j(r_n+s_n)+r_n}^{(j+1)(r_n+s_n)-1} Z_{n,i} \text{ for } 0 \leq j \leq k_n - 1 \\
\zeta_j &= \sum_{i=k_n(r_n+s_n)}^{n-1} Z_{n,i}
\end{aligned}$$

and write, $\sqrt{nh_n}\frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{\sqrt{n}}\left(\sum_{j=0}^{k_n-1}\eta_j + \sum_{j=0}^{k_n-1}\xi_j + \zeta_j\right) = \frac{1}{\sqrt{n}}(Q_n' + Q_n'' + Q_n''')$. We show that $E\left(\left(\frac{1}{\sqrt{n}}Q_n''\right)^2\right) \to 0$, $E\left(\left(\frac{1}{\sqrt{n}}Q_n'''\right)^2\right) \to 0$, then the asymptotic distribution of $B_{n,2}(x)$ is determined by $\frac{1}{\sqrt{n}}Q_n'$. Note that $E\left(\left(\frac{1}{\sqrt{n}}Q_n''\right)^2\right) = \frac{1}{n}E\left(\left(\sum_{j=0}^{k_n-1}\xi_j\right)^2\right) = \frac{1}{n}\sum_{j=0}^{k_n-1} E\left(\xi_j^2\right) + \frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{l=0,l\neq j}^{k_n-1} E(\xi_j\xi_l)$ and by A4 there exists $q_n \to \infty$ such that $q_ns_n = o((nh_n)^{1/2})$, $q_n\left(\frac{n}{h_n}\right)^{1/2}\alpha(s_n) = o(1)$. Then defining $r_n = \left[\frac{(nh_n)^{1/2}}{q_n}\right]$ as $n \to \infty$ we have $\frac{s_n}{r_n} = \frac{o((nh_n)^{1/2})/q_n}{[(nh_n)^{1/2}/q_n]} \to 0$, $\frac{r_n}{n} = \left[\frac{(nh_n)^{1/2}}{q_n}\right]\frac{1}{n} \to 0$, $\frac{r_n}{(nh_n)^{1/2}} = \left[\frac{(nh_n)^{1/2}}{q_n}\right]\frac{1}{(nh_n)^{1/2}} \to 0$, $\frac{n}{r_n}\alpha(s_n) = \frac{n\alpha(s_n)}{\left[\frac{(nh_n)^{1/2}}{q_n}\right]} \approx \left(\frac{n}{h_n}\right)^{1/2} q_n\alpha(s_n) \to 0$. Since $\xi_j = \sum_{i=j(r_n+s_n)+r_n}^{(j+1)(r_n+s_n)-1} Z_{n,i}$ we have,

$$\frac{1}{n}\sum_{j=0}^{k_n-1} E(\xi_j^2) = \frac{h_n}{n}\left(\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n} E(Z^2_{j(r_n+s_n)+r_n+\theta}) + \sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}\sum_{\delta=1,\delta\neq\theta}^{s_n} E(Z_{j(r_n+s_n)+r_n+\theta}Z_{j(r_n+s_n)+r_n+\delta})\right)$$

But $\frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}E(Z^2_{j(r_n+s_n)+r_n+\theta}) \leq \frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}h_n sup_i E(Z_i^2) \leq C\frac{1}{n}k_n s_n \leq C\frac{s_n}{r_n+s_n} = o(1)$. Also, since $sup_i\sum_{j=1,i\neq j}^{n}|cov(Z_i,Z_j)| = o(h_n^{-1})$,

$$\left|\frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}\sum_{\delta=1,\delta\neq\theta}^{s_n}E(Z_{j(r_n+s_n)+r_n+\theta}Z_{j(r_n+s_n)+r_n+\delta})\right| \leq \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}\sum_{\delta=1,\delta\neq\theta}^{s_n}|cov(Z_{j(r_n+s_n)+r_n+\theta},Z_{j(r_n+s_n)+r_n+\delta})|$$

$\leq \frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{s_n}h_n sup_{j(r_n+s_n)+r_n+\theta}\sum_{l=1,l\neq j(r_n+s_n)+r_n+\theta}^{n}|cov(Z_{j(r_n+s_n)+r_n+\theta},Z_l)| = o(1)\frac{k_n}{s_n}s_n \leq o(1)\frac{s_n}{r_n+s_n} = o(1)$ and therefore $\frac{1}{n}E\left(\left(\sum_{j=0}^{k_n-1}\xi_j\right)\right)^2 = o(1)$. Now, $\xi_j\xi_l = h_n\sum_{\theta=1}^{s_n}\sum_{\delta=1}^{s_n}Z_{j(r_n+s_n)+r_n+\delta}Z_{l(r_n+s_n)+r_n+\theta}$ and consequently

$$\left|\frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{l=0,l\neq j}^{k_n-1}E(\xi_j\xi_l)\right| \leq \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{l=0,l\neq j}^{k_n-1}\sum_{\delta=1}^{s_n}\sum_{\theta=1}^{s_n}|E(Z_{j(r_n+s_n)+r_n+\delta}Z_{l(r_n+s_n)+r_n+\theta})|$$

and since $j\neq l$ the distance between the indexes must be greater than r_n as $|j(r_n+s_n)+r_n+\delta-(l(r_n+s_n)+r_n+\theta)| \geq r_n+1 > r_n$. Thus,

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{l=0,l\neq j}^{k_n-1}E(\xi_j\xi_l)\right| &\leq 2\frac{h_n}{n}\sum_{i=1}^{n-r_n}\sum_{j=i+r_n}^{n}|E(Z_iZ_j)| \leq 2\frac{h_n}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|E(Z_iZ_j)| \\ &= \frac{h_n}{n}\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}|E(Z_iZ_j)| \leq \frac{1}{n}\sum_{i=1}^{n}h_n sup_i\sum_{j=1,j\neq i}^{n}|cov(Z_i,Z_j)| = o(1)\end{aligned}$$

Combining the results above we have that $E\left(\left(\frac{1}{\sqrt{n}}Q_n''\right)^2\right) = o(1)$. We now turn our attention to the Q_n''' term.

$$\begin{aligned}E\left(\left(\frac{1}{\sqrt{n}}Q_n'''\right)^2\right) &= \frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}E(Z^2_{n,i}) + \frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}\sum_{j=k_n(r_n+s_n),i\neq j}^{n-1}E(Z_{n,i}Z_{n,j}) \\ &= \frac{h_n}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}E(Z^2_{i+1}) + \frac{h_n}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}\sum_{j=k_n(r_n+s_n),i\neq j}^{n-1}E(Z_{i+1}Z_{j+1}).\end{aligned}$$

Given $sup_i h_n E(Z_i^2) \leq C$ we have that $\frac{h_n}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}E(Z^2_{i+1}) \leq \frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}sup_i h_n E(Z_i^2) = Cn^{-1}(n-k_n(r_n+s_n)) = o(1)$, since by construction $n-k_n(r_n+s_n) \leq r_n+s_n$ and therefore $n^{-1}(n-(r_n+s_n)) \leq n^{-1}(r_n+s_n) = o(1)$. Now,

$$\frac{h_n}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}\sum_{j=k_n(r_n+s_n),i\neq j}^{n-1}E(Z_{i+1}Z_{j+1}) \leq \frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1}h_n\sum_{j=k_n(r_n+s_n),i\neq j}^{n-1}|cov(Z_{i+1},Z_{j+1})|$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1} sup_i h_n \sum_{j=1, i\neq j}^{n} |cov(Z_i, Z_j)| \\
&\leq o(1)\frac{1}{n}(n - k_n(r_n+s_n)) = o(1)
\end{aligned}
$$

and by combining the results above we have $E\left(\left(\frac{1}{\sqrt{n}}Q_n'''\right)^2\right) = o(1)$. We now turn our attention to the Q_n' term. $\eta_j = \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} Z_{n,i}$ for $0 \leq j \leq k_n - 1$ and by construction $\eta_j = h_n^{1/2}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} Z_{i+1}$. Now let $\mathcal{F}_i^j$ be the σ-algebra generated by the random variables $\{X_t, U_t : i \leq t \leq j\}$, i.e., $\mathcal{F}_i^j = \sigma(X_i, U_i, \cdots, X_j, U_j)$ so that η_j is $\mathcal{F}_{j(r_n+s_n)+1}^{j(r_n+s_n)+r_n}$ measurable. Note that $j(r_n+s_n)+1-((j-1)(r_n+s_n)+r_n) = s_n+1$ and if we define $V_j = exp(it\eta_j)$, by Lemma 1.1 in Volkonskii and Rozanov(1959) we have,

$$
\left| E\left(\prod_{j=0}^{k_n-1} V_j\right) - \prod_{j=0}^{k_n-1} E(V_j)\right| = \left| E\left(exp(it\sum_{j=0}^{k_n-1}\eta_j)\right) - \prod_{j=0}^{k_n-1} E(exp(it\eta_j))\right| \leq 16(k_n-1)\alpha(s_n+1). \quad (22)
$$

$(k_n-1)\alpha(s_n+1) \leq \frac{n}{r_n+s_n}\alpha(s_n+1) = \frac{n}{r_n(1+\frac{s_n}{r_n})}\alpha(s_n+1)$ and since by construction $\frac{s_n}{r_n} \to 0$, $\frac{n}{r_n}\alpha(s_n) \to 0$ we have that $16(k_n-1)\alpha(s_n+1) \to 0$. Thus, by Corollary 14.1 in Jacod and Protter(2002) $\{\eta_j\}_{0\leq j\leq k_n-1}$ forms a sequence which is independent as $n \to \infty$. Now, $\eta_j = h_n^{1/2}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} Z_{i+1}$ and

$$
\begin{aligned}
\frac{1}{n}\sum_{j=0}^{k_n-1} E(\eta_j^2) &= \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1}\sum_{l=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} E(Z_{i+1}Z_{l+1}) \\
&= \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} E(Z_{i+1}^2) + \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1}\sum_{l=j(r_n+s_n), i\neq l}^{j(r_n+s_n)+r_n-1} E(Z_{i+1}Z_{l+1}) \\
&= I_{n,1} + I_{n,2}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|I_{n,2}| &= \left|\frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{r_n}\sum_{\delta=1,\delta\neq\theta}^{r_n} E(Z_{j(r_n+s_n)+\theta}Z_{j(r_n+s_n)+\delta})\right| \\
&\leq \frac{h_n}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{r_n}\sum_{\delta=1,\delta\neq\theta}^{r_n} |cov(Z_{j(r_n+s_n)+\theta}, Z_{j(r_n+s_n)+\delta})| \\
&\leq \frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{\theta=1}^{r_n} h_n sup_{j(r_n+s_n)+\theta}\sum_{l=1, l\neq j(r_n+s_n)+\theta}^{n} |cov(Z_{j(r_n+s_n)+\theta}, Z_l)| \\
&= o(1)\frac{k_n r_n}{n} \leq o(1)\frac{r_n}{r_n+s_n} = o(1).
\end{aligned}
$$

For the term $I_{n,1}$ note that $E(Z_i^2) = \frac{1}{h_n}\omega_{ii}(\theta_0)\int K^2(\phi)f_i(x+h_n\phi)d\phi$ and from Taylor's expansion $|f_i(x+$

$h_n\phi) - f_i(x)| \le O(h_n)$. Therefore,

$$\begin{aligned} I_{n,1} &= \frac{h_n}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} \Big(\frac{1}{h_n} \omega_{i+1,i+1}(\theta_0) \int K^2(\phi)(f_{i+1}(x+h_n\phi) - f_{i+1}(x))d\phi \\ &+ \frac{1}{h_n} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) \int K^2(\phi)d\phi \Big) = I_{n,11} + I_{n,12} \end{aligned}$$

looking at the last two terms separately we have,

$$\begin{aligned} |I_{n,11}| &\le \frac{h_n}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} \frac{1}{h_n} \omega_{i+1,i+1}(\theta_0) \int K^2(\phi) |f_{i+1}(x+h_n\phi) - f_{i+1}(x)| d\phi \\ &\le O(h_n) \int K^2(\phi)d\phi \frac{1}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} \omega_{i+1,i+1}(\theta_0) \end{aligned}$$

and since $\frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} \omega_{i+1,i+1}(\theta_0) \le n^{-1}\sum_{i=1}^{n} \omega_{ii}(\theta_0) \to \bar{\omega}(\theta_0)$ as $n \to \infty$ we have that $|I_{n,11}| = O(h_n)$.

$$\begin{aligned} I_{n,12} &= \int K^2(\phi)d\phi \frac{1}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) = \int K^2(\phi)d\phi \frac{1}{n} \sum_{i=1}^{n} \omega_{ii}(\theta_0) f_i(x) - \\ &\left(\frac{1}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)+r_n}^{(j+1)(r_n+s_n)-1} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) + \frac{1}{n} \sum_{i=k_n(r_n+s_n)}^{n-1} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) \right) \int K^2(\phi)d\phi \end{aligned}$$

Now, $n^{-1}\sum_{i=1}^{n} \omega_{ii}(\theta_0) f_i(x) \to \bar{\omega}_f(x,\theta_0) < \infty$ by A3 and since $|\omega_{ii}(\theta_0)|, f_i(x) < C$,

$$\frac{1}{n} \sum_{j=0}^{k_n-1} \sum_{i=j(r_n+s_n)+r_n}^{(j+1)(r_n+s_n)-1} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) \le C \frac{s_n}{r_n+s_n} \to 0.$$

Similarly, $\frac{1}{n}\sum_{i=k_n(r_n+s_n)}^{n-1} \omega_{i+1,i+1}(\theta_0) f_{i+1}(x) \to 0$. Combining the above results we have that $I_{n,1} = \bar{\omega}_f(x,\theta_0)\int K^2(\phi)d\phi + o(1) + O(h_n)$, and given that $I_{n,2} = o(1)$ we conclude that

$$\frac{1}{n} \sum_{j=0}^{k_n-1} E(\eta_j^2) = \bar{\omega}_f(x,\theta_0) \int K^2(\phi)d\phi + o(1) + O(h_n).$$

Now let $\frac{1}{\sqrt{n}}Q_n' = \sum_{j=0}^{k_n-1} Z_{jn}$ where $Z_{jn} = \frac{1}{(nh_n)^{1/2}} \sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} K\left(\frac{X_{i+1}-x}{h_n}\right) U_{i+1}$ and $S_n^2 = \sum_{j=0}^{k_n-1} E(Z_{jn} - E(Z_{jn}))^2$, where $S_n^2 = \sum_{j=0}^{k_n-1} \frac{1}{n} E(\eta_j^2) \to \bar{\omega}_f(x,\theta_0)\int K^2(\phi)d\phi$ as $n \to \infty$. We first observe that if we define $W_n = \frac{1}{S_n}\frac{1}{\sqrt{n}}Q_n'$ and let $\psi_{W_n}(\lambda) = E(exp(i\lambda W_n))$ be the characteristic function of W_n we have,

$$|\psi_{W_n}(\lambda) - exp(-\lambda^2/2)| \le \left| E\left(exp(i\lambda \sum_{j=0}^{k_n-1} \frac{1}{n^{1/2}S_n} \eta_j) \right) - \prod_{j=0}^{k_n-1} E(exp(i\lambda \frac{1}{n^{1/2}S_n} \eta_j)) \right|$$

$$+\left|\prod_{j=0}^{k_n-1} E(exp(i\lambda\frac{1}{n^{1/2}S_n}\eta_j) - exp(-\lambda^2/2)\right| = A_1 + A_2$$

But $A_1 = o(1)$ by the result on equation (**??**) and $A_2 = o(1)$ by Lindeberg's CLT (Theorem 23.6 in Davidson, 1994), which is implied by Liapounov's condition. Hence,

$$\sum_{j=0}^{k_n-1} \frac{Z_{jn}}{S_n} \xrightarrow{d} N(0,1) \text{ as } n \to \infty \text{ provided that } lim_{n\to\infty} \sum_{j=0}^{k_n-1} E\left|\frac{Z_{jn}}{S_n}\right|^{2+\delta} = 0 \text{ for some } \delta > 0.$$

$$\begin{aligned}\sum_{j=0}^{k_n-1} E\left|\frac{Z_{jn}}{S_n}\right|^{2+\delta} &= (S_n^2)^{-1-\delta/2}(nh_n)^{-\delta/2}\frac{1}{nh_n}\sum_{j=0}^{k_n-1} E\left|\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1} K\left(\frac{X_{i+1}-x}{h_n}\right)U_{i+1}\right|^{2+\delta} \\ &\leq (S_n^2)^{-1-\delta/2}(nh_n)^{-\delta/2}2^{1+\delta}\frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1}\frac{1}{h_n}E\left|K\left(\frac{X_{i+1}-x}{h_n}\right)U_{i+1}\right|^{2+\delta}\end{aligned}$$

by the c_r inequality. Furthermore, $\frac{1}{h_n}E\left|K\left(\frac{X_{i+1}-x}{h_n}\right)U_{i+1}\right|^{2+\delta} = \frac{1}{h_n}E\left(K\left(\frac{X_{i+1}-x}{h_n}\right)\right)E\left|U_{i+1}\right|^{2+\delta}$ and given that $E\left|U_{i+1}\right|^{2+\delta} < C$ we have that

$$\frac{1}{h_n}E\left|K\left(\frac{X_{i+1}-x}{h_n}\right)U_{i+1}\right|^{2+\delta} \leq C\int K^{2+\delta}(\phi)f_{i+1}(x+h_n\phi)d\phi < C$$

by A2. Therefore,

$$\frac{1}{n}\sum_{j=0}^{k_n-1}\sum_{i=j(r_n+s_n)}^{j(r_n+s_n)+r_n-1}\frac{1}{h_n}E\left|K\left(\frac{X_{i+1}-x}{h_n}\right)U_{i+1}\right|^{2+\delta} \leq C\frac{r_n}{r_n+s_n} \to C$$

and since $S_n^2 \to \bar{\omega}_f(\theta_0,x)\int K^2(\phi)d\phi$ as $nh_n \to \infty$ we have $lim_{n\to\infty}\sum_{j=0}^{k_n-1} E\left|\frac{Z_{jn}}{S_n}\right|^{2+\delta} = 0$.

Finally, combining the results of $\frac{Q'_n}{\sqrt{n}}$, $\frac{Q''_n}{\sqrt{n}}$ and $\frac{Q'''_n}{\sqrt{n}}$ we conclude that $(nh_n)^{1/2}B_{n,2}(x) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{f(x)^2}\int K^2(\phi)d\phi\right)$ as $n \to \infty$. Together with $B_{n,1}(x) = \frac{h_n^2}{2}\sigma_K^2 m^{(2)}(x) + o_p(h_n^2)$ gives,

$$\left(\frac{1}{(nh_n)^{1/2}\bar{f}_n(x)}\sum_{i=1}^{n} K\left(\frac{X_i-x}{h_n}\right)Y_i^* - B_{n,1}(x)\right) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}(x)^2}\int K^2(\phi)d\phi\right) \text{ as } n \to \infty.$$

Now, we note from our previous results on $B_{n,1}(x)$, $B_{n,3}(x)$ and by applying Theorem 1 to $\bar{f}_n(x)B_{n,2}(x)$ with $g(U_i) = U_i$, $j = 0$ and $v_i = 1$ for all i we have, $\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i-x}{h_n}\right)Y_i^* = O_p(h_n^2) + O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2}\right)$ and $\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{X_i-x}{h_n}\right)\left(\frac{X_i-x}{h_n}\right)Y_i^* = O_p(h_n^2) + O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2}\right)$ uniformly in G. Hence,

$$(nh_n)^{1/2}|D_n(x)| \leq (nh_n)^{1/2}O_p(h_n^3) + (nh_n)^{1/2}O_p\left(\left(\frac{h_n ln(n)}{n}\right)^{1/2}\right).$$

Now, provided that $h_n^2 ln(n) = o(1)$ the right hand side of the inequality is $o(1)$ and we have

$$(nh_n)^{1/2}\left(\check{m}(x) - m(x) - B_{n,1}(x)\right) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}(x)^2}\int K^2(\phi)d\phi\right) \text{ as } n \to \infty.$$

Theorem 3: *Proof* Let $\check{Z}_i$ be the i^{th} component of the vector $\check{Z}$. Note that $\hat{m}(x) - m(x) = \frac{1}{ng_n}\sum_{i=1}^n W_n\left(\frac{X_i - x}{g_n}, x\right)\check{Z}_i^*$, where $\check{Z}_i^* = \check{Z}_i - m(x) - m^{(1)}(x)(X_i - x)$. Let $A_n(x) = \frac{1}{g_n}\left(e'\left(S_n(x)^{-1} - S(x)^{-1}\right)^2 e\right)^{1/2}$, $D_n(x) = \hat{m}(x) - m(x) - \frac{1}{ng_n\bar{f}_n(x)}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^*$. As in Theorem 1

$$\begin{aligned}
|D_n(x)| &= \frac{1}{nh_n}\left|e'(S_n^{-1}(x) - S^{-1}(x))\begin{pmatrix} \sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^* \\ \sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^* \end{pmatrix}\right| \\
&\leq g_n A_n(x)\frac{1}{ng_n}\left(\left|\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^*\right| + \left|\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^*\right|\right).
\end{aligned}$$

and $A_n(x) = O_p(1)$ uniformly in G. We now turn our attention to $B_n(x) = \frac{1}{ng_n\bar{f}_n(x)}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\check{Z}_i^*$. Since, $\check{Z}_i = m(X_i) - \sum_{j=1, j\neq i}^n \frac{v_{ij}}{v_{ii}}(\check{m}(X_j) - m(X_j)) + \gamma_i$ we have

$$\begin{aligned}
B_n(x) &= \frac{1}{\bar{f}_n(x)}\frac{1}{ng_n}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\frac{m^{(2)}(x)}{2}(X_i - x)^2 + \frac{1}{\bar{f}_n(x)}\frac{1}{ng_n}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\gamma_i \\
&+ o(g_n^2)\frac{1}{\bar{f}_n(x)}\frac{1}{ng_n}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right) - \frac{1}{\bar{f}_n(x)}\frac{1}{ng_n}\sum_{i=1}^n K\left(\frac{X_i - x}{g_n}\right)\sum_{\substack{j=1 \\ j\neq i}}^n \frac{v_{ij}}{v_{ii}}(\check{m}(X_j) - m(X_j)) \\
&= B_{n,1}(x) + B_{n,2}(x) + B_{n,3}(x) - B_{n,4}(x)
\end{aligned}$$

We examine each $B_{n,j}(x)$ for $j = 1,2,3,4$ separately. From Theorem 2 $B_{n,1}(x) = \frac{g_n^2}{2}\sigma_K^2 m^{(2)}(x) + o_p(g_n^2)$, $B_{n,3}(x) = o_p(g_n^2)$ uniformly in G. Also, from Theorem 2, $(ng_n)^{1/2}B_{n,2}(x) \to N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}(x)^2}\int K^2(\phi)d\phi\right)$ where $\bar{\omega}_f(x,\theta_0) = lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n f_i(x)v_{ii}^{-2}$. We now examine $B_{n,4}(x)$. From the definition of Y_i^* and Theorem 2

$$\begin{aligned}
\check{m}(X_j) - m(X_j) &= \frac{1}{nh_n\bar{f}_n(X_j)}\sum_{l=1}^n K\left(\frac{X_l - X_j}{h_n}\right)\left(m(X_l) - m(X_j) - m^{(1)}(X_j)(X_l - X_j)\right) \\
&+ \frac{1}{nh_n\bar{f}_n(X_j)}\sum_{l=1}^n K\left(\frac{X_l - X_j}{h_n}\right)U_l + O_p(h_n^3) + O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2}h_n\right)
\end{aligned}$$

and therefore we can write $B_{n,4}(x) = B_{n,41}(x) + B_{n,42}(x) + B_{n,43}(x)$ where,

$$\begin{aligned}
B_{n,41}(x) &= \frac{1}{n^2 g_n h_n\bar{f}_n(x)}\sum_{i=1}^n\sum_{\substack{j=1 \\ j\neq i}}^n\sum_{l=1}^n \frac{v_{ij}}{v_{ii}}\frac{1}{\bar{f}_n(X_j)}K\left(\frac{X_i - x}{g_n}\right)K\left(\frac{X_l - X_j}{h_n}\right)(m(X_l) - m(X_j) \\
&- m^{(1)}(X_j)(X_l - X_j))
\end{aligned}$$

$$B_{n,42}(x) = \frac{1}{n^2 g_n h_n \bar{f}_n(x)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{l=1}^{n} \frac{v_{ij}}{v_{ii}} \frac{1}{\bar{f}_n(X_j)} K\left(\frac{X_i - x}{g_n}\right) K\left(\frac{X_l - X_j}{h_n}\right) U_l$$

$$B_{n,43}(x) = \frac{1}{n g_n \bar{f}_n(x)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{v_{ij}}{v_{ii}} K\left(\frac{X_i - x}{g_n}\right) \left(O_p(h_n^3) + O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2} h_n\right)\right).$$

We look at each of these terms separately. Note that

$$B_{n,41}(x) = \frac{1}{n g_n \bar{f}_n(x)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{v_{ij}}{v_{ii}} K\left(\frac{X_i - x}{g_n}\right) \left\{ \frac{1}{n h_n \bar{f}_n(X_j)} \sum_{l=1}^{n} K\left(\frac{X_l - X_j}{h_n}\right) (m(X_l) - m(X_j) \right.$$
$$\left. - \quad m^{(1)}(X_j)(X_l - X_j)) \right\}$$

and the term inside the curly brackets $\{\cdot\}$ is $O_p(h_n^2)$ uniformly in G from Theorem 2. Hence,

$$\begin{aligned} |B_{n,41}(x)| &\leq O_p(h_n^2) \frac{1}{n g_n \bar{f}_n(x)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|v_{ij}|}{|v_{ii}|} K\left(\frac{X_i - x}{g_n}\right) \\ &\leq O_p(h_n^2) \frac{1}{n g_n \bar{f}_n(x)} \sum_{i=1}^{n} K\left(\frac{X_i - x}{g_n}\right) sup_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|v_{ij}|}{|v_{ii}|} \\ &\leq O_p(h_n^2) O(1) \frac{1}{n g_n \bar{f}_n(x)} \sum_{i=1}^{n} K\left(\frac{X_i - x}{g_n}\right) \end{aligned}$$

where $sup_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{|v_{ij}|}{|v_{ii}|} = O(1)$ by assumption. Furthermore, from Theorem 1 $\frac{1}{n g_n} \sum_{i=1}^{n} K\left(\frac{X_i - x}{g_n}\right) = O_p(1)$ and by assumption A1 $\bar{f}_n(x) \to \bar{f}(x)$. Hence, $sup_{x\in G}|B_{n,41}(x)| = O_p(h_n^2)$. Using similar arguments and Theorem 2 we have $sup_{x\in G}|B_{n,43}(x)| = O_p(h_n^3) + O_p\left(\left(\frac{nh_n}{ln(n)}\right)^{-1/2} h_n\right)$.

$$\begin{aligned} B_{n,42}(x) &= \frac{1}{n\bar{f}_n(x)} \sum_{l=1}^{n} U_l \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{n g_n h_n \bar{f}_n(X_j)} \sum_{i=1}^{n} \frac{v_{ij}}{v_{ii}} K\left(\frac{X_i - x}{g_n}\right) K\left(\frac{X_l - X_j}{h_n}\right) \\ &= \frac{1}{n\bar{f}_n(x)} \sum_{l=1}^{n} U_l \lambda_{ln}(x). \end{aligned}$$

Note that $E(B_{n,42}(x)) = 0$ and

$$\begin{aligned} V((n g_n)^{1/2} B_{n,42}(x)) &= \frac{g_n}{n \bar{f}_n(x)^2} \sum_{l=1}^{n} \sum_{k=1}^{n} E(U_l U_k \lambda_{ln}(x) \lambda_{kn}(x)) \\ &\leq \frac{g_n}{n \bar{f}_n(x)^2} \sum_{l=1}^{n} \sum_{k=1}^{n} |\omega_{lk}(\theta_0)||E(\lambda_{ln}(x)\lambda_{kn}(x))| \end{aligned}$$

We denote $a_{ij} = \frac{v_{ij}}{v_{ii}}$, $K_i = K\left(\frac{X_i - x}{g_n}\right)$, $K_{lj} = K\left(\frac{X_l - X_j}{h_n}\right)$ and examine

$$
\begin{aligned}
|E(\lambda_{ln}(x)\lambda_{kn}(x))| &= E\left|\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{n}\sum_{\substack{o=1\\o\neq m}}^{n}\frac{1}{n^2 g_n^2 h_n^2 \bar{f}_n(X_j)\bar{f}_n(X_o)}a_{ij}a_{mo}K_iK_mK_{lj}K_{ko}\right| \\
&\leq \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{n}\sum_{\substack{o=1\\o\neq m}}^{n}\frac{1}{n^2 g_n^2 h_n^2}|a_{ij}||a_{mo}|E\left(\frac{K_iK_mK_{lj}K_{ko}}{\bar{f}_n(X_j)\bar{f}_n(X_o)}\right).
\end{aligned}
$$

Since $inf_{x\in G}|\bar{f}_n(x)| > 0$ we have

$$
\begin{aligned}
V((ng_n)^{1/2}B_{n,42}(x)) &\leq \frac{Cg_n}{n\bar{f}_n(x)^2}\sum_{l=1}^{n}\sum_{k=1}^{n}|\omega_{lk}(\theta_0)|\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{n}\sum_{\substack{o=1\\o\neq m}}^{n}\frac{|a_{ij}||a_{mo}|}{n^2 g_n^2 h_n^2}E\left(K_iK_mK_{lj}K_{ko}\right) \\
&= \frac{Cg_n}{n\bar{f}_n(x)^2}\sum_{l=1}^{n}|\omega_{ll}(\theta_0)|\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{n}\sum_{\substack{o=1\\o\neq m}}^{n}\frac{|a_{ij}||a_{mo}|}{n^2 g_n^2 h_n^2}E\left(K_iK_mK_{lj}K_{lo}\right) \\
&+ \frac{Cg_n}{n\bar{f}_n(x)^2}\sum_{l=1}^{n}\sum_{\substack{k=1\\k\neq l}}^{n}|\omega_{lk}(\theta_0)|\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{m=1}^{n}\sum_{\substack{o=1\\o\neq m}}^{n}\frac{|a_{ij}||a_{mo}|}{n^2 g_n^2 h_n^2}E\left(K_iK_mK_{lj}K_{ko}\right) \\
&= T_{1n} + T_{2n}
\end{aligned}
$$

We need to show that $T_{1n}, T_{2n} = o(1)$. The strategy we use is to establish the order of the partial sums that emerge from considering all possible combinations of the indexes l, k, i, j, m, o in T_{1n}, T_{2n}.[2] Each of these partial sums are shown to be $o_p(1)$ by first establishing the order of $\pi_n = \frac{1}{h_n^2 g_n^2}E(K_iK_mK_{lj}K_{lo})$ and $\rho_n = \frac{1}{h_n^2 g_n^2}E(K_iK_mK_{lj}K_{ko})$. Here we show the cases in which l and k are distinct from the indexes in the four inner sums, i.e., i, j, m, o.[3] We need to consider seven cases, and given A1 we have from calculating the expectations the following bounds: Case 1 ($i = m$ and $j = o$): $\pi_n \leq \frac{C}{g_n h_n}$, $\rho_n \leq \frac{C}{g_n}$; Case 2 ($i = o$ and $j = m$) $\pi_n \leq \frac{C}{h_n}$, $\rho_n \leq C$; Case 3 ($i = m$): $\pi_n \leq \frac{C}{g_n}$, $\rho_n \leq \frac{C}{g_n}$; Case 4 ($i = o$), Case 5 ($j = m$), Case 7 ($i \neq j \neq m \neq o$): $\pi_n \leq C$, $\rho_n \leq C$; Case 6 ($j = o$): $\pi_n \leq \frac{C}{h_n}$, $\rho_n \leq C$. We now denote the partial sums associated with $V((ng_n)^{1/2}B_{n,42}(x))$ in each of these cases by s_i, $i = 1, ..., 7$. Hence, we have the following inequalities, where the first term refers to the partial sums in T_{1n} and the second term refers to the partial sums in T_{2n} for each case.

$$
s_1 \leq \frac{C\bar{\omega}_n}{h_n\bar{f}_n^2(x)}\left(n^{-2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}|a_{ij}|^2\right) + \frac{C}{ng_n\bar{f}_n^2(x)}\sum_{l=1}^{n}g_n\sum_{\substack{k=1\\k\neq l}}^{n}|\omega_{lk}(\theta_0)|\left(n^{-2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}|a_{ij}|^2\right)
$$

[2] See the note on indexes in the end of this appendix.

[3] Bounds for all other cases described in appendix 1 are available from the authors upon request.

$$s_2 \leq \frac{C\bar{\omega}_n}{h_n \bar{f}_n^2(x)} g_n \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}||a_{ij}| \right) + \frac{C}{n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}||a_{ij}| \right)$$

$$s_3 \leq \frac{C\bar{\omega}_n}{\bar{f}_n^2(x)} \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{o=1 \\ o\neq i\neq j}}^{n} |a_{io}| \right) + \frac{C}{ng_n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{o=1 \\ o\neq i\neq j}}^{n} |a_{io}| \right)$$

$$s_4 \leq \frac{C\bar{\omega}_n g_n}{\bar{f}_n^2(x)} \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{m=1 \\ m\neq i\neq j}}^{n} |a_{mi}| \right) + \frac{C}{n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{m=1 \\ m\neq i\neq j}}^{n} |a_{mi}| \right)$$

$$s_5 \leq \frac{C\bar{\omega}_n g_n}{\bar{f}_n^2(x)} \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{o=1 \\ o\neq i\neq j}}^{n} |a_{jo}| \right) + \frac{C}{n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{o=1 \\ o\neq i\neq j}}^{n} |a_{jo}| \right)$$

$$s_6 \leq \frac{C\bar{\omega}_n g_n}{nh_n\bar{f}_n^2(x)} \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{m=1 \\ m\neq i\neq j}}^{n} |a_{mj}| \right) + \frac{C}{n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \sum_{\substack{m=1 \\ m\neq i\neq j}}^{n} |a_{mj}| \right)$$

$$s_7 \leq \frac{C\bar{\omega}_n g_n}{\bar{f}_n^2(x)} \left(n^{-2} \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} |a_{ij}| \right)^2 \right) + \frac{C}{n\bar{f}_n^2(x)} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k\neq l}}^{n} |\omega_{lk}(\theta_0)| \left(n^{-2} \left(\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}} |a_{ij}| \right)^2 \right)$$

By assumptions A1.6 and A3 we have that $\frac{1}{n}\sum_{l=1}^{n}\omega_{ll}(\theta_0) \to \bar{\omega}(\theta_0)$ and $inf_{x\in G}|\bar{f}_n(x)| > 0$. Furthermore, we note that from Theorem 1 $g_n \sum_{k=1,l\neq k}^{n} |\omega_{lk}(\theta_0)| = o(1)$ and consequently, provided that $sup_i \sum_{j=1,j\neq i} \frac{|v_{ij}|}{|v_{ii}|} = O(1)$ and $sup_i \sum_{j=1,j\neq i} \frac{|v_{ji}|}{|v_{jj}|} = O(1)$ the first term and second terms in each case are $o(1)$.

Therefore, $B_{n,42}(x) = o_p((ng_n)^{-1/2})$ and $B_{n,4}(x) = O_p(h_n^2) + o_p((ng_n)^{-1/2}) + O_p\left(\left(\frac{h_n}{n} ln(n)\right)^{1/2}\right)$. Now, provided that $\frac{h_n}{g_n} \to 0$ and $\frac{ng_n^3}{ln(n)} \to \infty$ we have that the last term is $o(g_n^2)$ and we obtain $B_{n,4}(x) = O_p(h_n^2) + o_p((ng_n)^{-1/2}) + o_p(g_n^2)$. Now, if $g_n = O(n^{-1/5})$ then $(ng_n)^{1/2}B_{n,3} = o_p(1)$ and consequently we have,

$$\sqrt{ng_n}\left(B_n(x) - \left(\sigma_K^2 \frac{m^{(2)}(x)}{2} g_n^2 + o_p(g_n^2)\right)\right) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}^2(x)} \int K^2(\phi)d\phi\right). \tag{23}$$

Lastly, it follows from arguments similar to those in the proof of Theorem 2 that

$$\sqrt{ng_n}\left(\hat{m}(x) - m(x) - \left(\sigma_K^2 \frac{m^{(2)}(x)}{2} g_n^2 + o_p(g_n^2)\right)\right) \xrightarrow{d} N\left(0, \frac{\bar{\omega}_f(x,\theta_0)}{\bar{f}^2(x)} \int K^2(\phi)d\phi\right) \tag{24}$$

which proves the theorem.

Theorem 4: *Proof* $\sqrt{ng_n}(\hat{m}(x) - \dot{m}(x)) = e'S_n^{-1}\begin{pmatrix} \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n K\left(\frac{X_i-x}{g_n}\right)q_i \\ \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n K\left(\frac{X_i-x}{g_n}\right)\left(\frac{X_i-x}{g_n}\right)q_i \end{pmatrix}$ where $q_i = \sum_{j=1,j\neq i}^n (a_{ij}(\dot{\theta}) - a_{ij}(\theta_0))(\check{m}(X_j) - m(X_j) - U_j)$ and since $S_n^{-1}(x) = O_p(1)$ and K has compact support, it suffices to show that $\frac{1}{\sqrt{ng_n}}\sum_{i=1}^n K\left(\frac{X_i-x}{g_n}\right)q_i = o_p(1)$. Hence, we must show that,

$$\alpha_n = \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{j=1,j\neq i}^n K\left(\frac{X_i-x}{g_n}\right)(a_{ij}(\dot{\theta}) - a_{ij}(\theta_0))U_j = o_p(1) \tag{25}$$

and

$$\beta_n = \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{j=1,j\neq i}^n K\left(\frac{X_i-x}{g_n}\right)(a_{ij}(\dot{\theta}) - a_{ij}(\theta_0))(\check{m}(X_j) - m(X_j)) = o_p(1) \tag{26}$$

Let $g_0(\theta) = 0$ and $I_{iwn} = \{j = 1, 2, ..., n : a_{ij}(\theta) = g_{wn}(\theta)\}$. Then,

$$\begin{aligned}
\alpha_n &= \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \left(\sum_{w=1}^W \sum_{\substack{j\in I_{iwn}\\ j\neq i}}^n K\left(\frac{X_i-x}{g_n}\right)(a_{ij}(\dot{\theta}) - a_{ij}(\theta_0))U_j\right. \\
&+ \left.\sum_{j\notin \cup_{w=1}^W I_{iwn}, j\neq i}^n K\left(\frac{X_i-x}{g_n}\right)(a_{ij}(\dot{\theta}) - a_{ij}(\theta_0))U_j\right) \\
&= \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{w=1}^W \sum_{\substack{j\in I_{iwn}\\ j\neq i}}^n K\left(\frac{X_i-x}{g_n}\right)(g_{wn}(\dot{\theta}) - g_{wn}(\theta_0))U_j \\
&+ \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{j\notin \cup_{w=1}^W I_{iwn}, j\neq i}^n K\left(\frac{X_i-x}{g_n}\right)(g_0(\dot{\theta}) - g_0(\theta_0))U_j \\
&= \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{w=1}^W \sum_{\substack{j\in I_{iwn}\\ j\neq i}}^n K\left(\frac{X_i-x}{g_n}\right)(g_{wn}(\dot{\theta}) - g_{wn}(\theta_0))U_j \\
&= \sum_{w=1}^W (g_{wn}(\dot{\theta}) - g_{wn}(\theta_0))\frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{\substack{j\in I_{iwn}\\ j\neq i}}^n K\left(\frac{X_i-x}{g_n}\right)U_j
\end{aligned}$$

But given TA 4.1, the consistency of $\dot{\theta}$ and the fact that W is finite and does not depend on n, it suffices to show that $\alpha_{n1} = \frac{1}{\sqrt{ng_n}}\sum_{i=1}^n \sum_{j\in I_{iwn}, j\neq i}^n K\left(\frac{X_i-x}{g_n}\right)U_j = O_p(1)$ for arbitrary w. Given the independence of $\{X_i\}$ and $\{U_i\}$ and taking expectation of the square yields,

$$\begin{aligned}
E(\alpha_{n1}^2) &= \frac{1}{ng_n}\sum_{i=1}^n E\left(K^2\left(\frac{X_i-x}{g_n}\right)\right)E\left(\left(\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}} U_\tau\right)^2\right) \\
&+ \frac{1}{ng_n}\sum_{i=1}^n \sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}} \sum_{\substack{j=1\\ j\neq i}}^n \sum_{\substack{t\in I_{jwn}\\ t\neq j}} E\left(K\left(\frac{X_i-x}{g_n}\right)K\left(\frac{X_j-x}{g_n}\right)\right)E(U_t U_\tau)
\end{aligned}$$

$$\leq \frac{C}{n}\sum_{i=1}^{n}E\left(\left(\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}}U_\tau\right)^2\right)+\frac{Cg_n}{n}\sum_{i=1}^{n}\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{t\in I_{jwn}\\ t\neq j}}E\left(U_tU_\tau\right)$$

$$\leq \frac{C}{n}\sum_{i=1}^{n}\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}}\sum_{\substack{t\in I_{iwn}\\ t\neq i}}|\omega_{t\tau}|+\frac{Cg_n}{n}\sum_{i=1}^{n}\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{t\in I_{jwn}\\ t\neq j}}|\omega_{t\tau}|$$

By TA 4.2 τ belongs to at most $\aleph$ different index sets I_{iwn} (the same for t) hence given that $|\omega_{t\tau}|$ is bounded the first term on the right hand side of the last inequality is bounded by $C\aleph^2$. For the second term, note that $\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{t\in I_{jwn}\\ t\neq j}}|\omega_{t\tau}|\leq\aleph\sum_{t=1}^{n}|\omega_{t\tau}|\leq C\aleph$ by assumptions TA 4.3, hence

$$\frac{Cg_n}{n}\sum_{i=1}^{n}\sum_{\substack{\tau\in I_{iwn}\\ \tau\neq i}}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{t\in I_{jwn}\\ t\neq j}}|\omega_{t\tau}|\leq g_nC\aleph^2=o(1).$$

The same manipulations used above show that

$$\beta_n=\sum_{w=1}^{W}(g_{wn}(\dot{\theta})-g_{wn}(\theta_0))\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}K\left(\frac{X_i-x}{g_n}\right)(\check{m}(X_j)-m(X_j))$$

and therefore we need only show that $\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}K\left(\frac{X_i-x}{g_n}\right)(\check{m}(X_j)-m(X_j))=O_p(1)$. Let K_i and K_{lj} be as defined in the proof of Theorem 3, then we can write

$$\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}K\left(\frac{X_i-x}{g_n}\right)(\check{m}(X_j)-m(X_j)) = \beta_{1n}(x)+\beta_{2n}(x)+\beta_{3n}(x),$$

where

$$\beta_{1n}(x)=\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}\sum_{l=1}^{n}\frac{K_iK_{lj}}{nh_n\bar{f}_n(X_j)}(m(X_l)-m(X_j)-m^{(1)}(X_j)(X_l-X_j),$$

$$\beta_{2n}(x)=\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}\sum_{l=1}^{n}\frac{K_iK_{lj}}{nh_n\bar{f}_n(X_j)}U_l,$$

$$\beta_{3n}(x)=\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}K_i\left(O_p(h_n^3)+O_p\left(h_n\left(\frac{nh_n}{ln(n)}\right)^{-1/2}\right)\right).$$

We show that $\beta_{in}(x)=O_p(1)$ for $i=1,2,3$. From Theorem 2,

$$|\beta_{1n}(x)| \leq h_n^2O_p(1)\frac{1}{\sqrt{ng_n}}\sum_{i=1}^{n}\sum_{\substack{j\in I_{iwn}\\ j\neq i}}K_i$$

$$\leq \aleph h_n^2 O_p(1)(ng_n)^{1/2}\frac{1}{ng_n}\sum_{i=1}^n K_i \leq \aleph(ng_n)^{1/2}h_n^2 O_p(1) \text{ since } \tfrac{1}{ng_n}\textstyle\sum_{i=1}^n K_i = O_p(1).$$
$$= O_p(1) \text{ provided } g_n = O(n^{-1/5}),\ h_n = O(n^{-1/5}).$$

Also,

$$|\beta_{3n}(x)| \leq \aleph h_n^3(ng_n)^{1/2}O_p(1)\frac{1}{ng_n}\sum_{i=1}^n K_i + \aleph\left(\frac{nh_n}{ln(n)}\right)^{-1/2} h_n(ng_n)^{1/2}O_p(1)\frac{1}{ng_n}\sum_{i=1}^n K_i$$
$$\leq \aleph h_n^3(ng_n)^{1/2}O_p(1) + \aleph\left(\frac{nh_n}{ln(n)}\right)^{-1/2} h_n(ng_n)^{1/2}O_p(1) = \left((nh_n^6 g_n)^{1/2} + (g_n h_n ln(n))^{1/2}\right)\aleph O_p(1)$$
$$= O_p(1) \text{ provided } g_n = O(n^{-1/5}),\ h_n = O(n^{-1/5}).$$

We now examine $\beta_{2n}(x)$. We write,

$$\beta_{2n}(x) = \sqrt{ng_n}\frac{1}{n}\sum_{l=1}^n U_l \frac{1}{nh_n g_n}\sum_{i=1}^n \sum_{\substack{j\in I_{iwn}\\ j\neq i}} \frac{K_i K_{lj}}{\bar{f}_n(X_j)}$$
$$= \sqrt{ng_n}\frac{1}{n}\sum_{l=1}^n U_l c_{nl} \text{ where } c_{nl} = \tfrac{1}{nh_n g_n}\textstyle\sum_{i=1}^n \sum_{\substack{j\in I_{iwn}\\ j\neq i}} \tfrac{K_i K_{lj}}{\bar{f}_n(X_j)}.$$

Since $\{X_i\}$ and $\{U_i\}$ are independent it is easy to verify $E(\beta_{2n}(x)) = 0$ and

$$V(\beta_{2n}(x)) = ng_n\frac{1}{n^2}\sum_{l=1}^n\sum_{k=1}^n E(U_l U_k)E(c_{nl}c_{nk})$$
$$\leq \frac{g_n}{n}\sum_{l=1}^n\sum_{k=1}^n |\omega_{lk}(\theta_0)||E(c_{nl}c_{nk})| \text{ and since } inf_{x\in G}|\bar{f}_n(x)| > 0,$$
$$\leq C\frac{g_n}{n}\sum_{l=1}^n\sum_{k=1}^n |\omega_{lk}(\theta_0)|\frac{1}{n^2 h_n^2 g_n^2}\sum_{i=1}^n\sum_{\substack{j\in I_{iwn}\\ j\neq i}}\sum_{m=1}^n\sum_{\substack{o\in I_{mwn}\\ o\neq m}} E(K_i K_{lj} K_m K_{ko})$$
$$= C\frac{g_n}{n}\sum_{l=1}^n \omega_{ll}(\theta_0)\frac{1}{n^2}\sum_{i=1}^n\sum_{\substack{j\in I_{iwn}\\ j\neq i}}\sum_{m=1}^n\sum_{\substack{o\in I_{mwn}\\ o\neq m}} E(K_i K_{lj} K_m K_{lo})\frac{1}{h_n^2 g_n^2}$$
$$+ C\frac{g_n}{n}\sum_{l=1}^n\sum_{\substack{k=1\\ k\neq l}}^n |\omega_{lk}(\theta_0)|\frac{1}{n^2}\sum_{i=1}^n\sum_{\substack{j\in I_{iwn}\\ j\neq i}}\sum_{m=1}^n\sum_{\substack{o\in I_{mwn}\\ o\neq m}} E(K_i K_{lj} K_m K_{ko})\frac{1}{h_n^2 g_n^2}$$
$$= T_{1n} + T_{2n}.$$

We need to show that $T_{1n}, T_{2n} = O(1)$. We adopt the same strategy used in Theorem 3, i.e., establish the order of partial sums that emerge from considering all possible combinations of the indexes l, k, i, j, m, o in $T_{n1}, T_{n,2}$. Each of these partial sums is bounded by establishing the order $\pi_n = E(K_i K_{lj} K_m K_{lo})\frac{1}{h_n^2 g_n^2}$ and $\rho_n = E(K_i K_{lj} K_m K_{ko})\frac{1}{h_n^2 g_n^2}$.

We need to show that $T_{1n}, T_{2n} = o(1)$. The strategy we use is to establish the order of the partial sums that emerge from considering all possible combinations of the indexes l, k, i, j, m, o in T_{1n}, T_{2n}.[4] Each of these partial sums are shown to be $o_p(1)$ by first establishing the order of $\pi_n = \frac{1}{h_n^2 g_n^2} E(K_i K_m K_{lj} K_{lo})$ and $\rho_n = \frac{1}{h_n^2 g_n^2} E(K_i K_m K_{lj} K_{ko})$. Here we show the cases in which l and k are distinct from the indexes in the four inner sums, i.e., i, j, m, o.[5] We need to consider seven cases, and given A1 we have from calculating the expectations the following bounds: Case 1 ($i = m$ and $j = o$): $\pi_n \leq \frac{C}{g_n h_n}$, $\rho_n \leq \frac{C}{g_n}$; Case 2 ($i = o$ and $j = m$) $\pi_n \leq \frac{C}{h_n}$, $\rho_n \leq C$; Case 3 ($i = m$): $\pi_n \leq \frac{C}{g_n}$, $\rho_n \leq \frac{C}{g_n}$; Case 4 ($i = o$), Case 5 ($j = m$), Case 7 ($i \neq j \neq m \neq o$): $\pi_n \leq C$, $\rho_n \leq C$; Case 6 ($j = o$): $\pi_n \leq \frac{C}{h_n}$, $\rho_n \leq C$. We now denote the partial sums associated with $V((ng_n)^{1/2} B_{n,42}(x))$ in each of these cases by s_i, $i = 1, ..., 7$. Hence, we have the following inequalities, where the first term refers to the partial sums in T_{1n} and the second term refers to the partial sums in T_{2n} for each case.

$$s_1 \leq \frac{C}{nh_n}\bar{\omega}_n + \frac{\aleph C}{n^2 g_n} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)|, s_2 \leq \frac{C g_n}{nh_n}\bar{\omega}_n + \frac{C}{n^2} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)|$$

$$s_3 \leq \frac{C\aleph^2}{n}\bar{\omega}_n + \frac{C\aleph^2}{n^2 g_n} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)|, s_4 \leq \frac{C\aleph^2 g_n}{n}\bar{\omega}_n + \frac{C\aleph^2}{n^2} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)|.$$

Case 5 is identical to Case 4 and

$$s_6 \leq \frac{C\aleph^2 g_n}{nh_n}\bar{\omega}_n + \frac{C\aleph^2}{n^2} \sum_{l=1}^{n} g_n \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)|, s_7 \leq C\bar{\omega}_n g_n \aleph^2 C + \frac{C g_n}{n} \sum_{l=1}^{n} \sum_{\substack{k=1 \\ k \neq l}}^{n} |\omega_{lk}(\theta_0)| \aleph^2.$$

Hence, given A1 and the fact that from Theorem 1 $g_n \sum_{k=1, l \neq k}^{n} |\omega_{lk}(\theta_0)| = o(1)$ we conclude that in each case the first and second terms are O(1).

Note on Indexes: To construct the set of all index combinations for the six-fold sums we first note that for the four inner sums we need to consider seven different possible cases for i, j, m, o: Case 1 ($i = m$ and $j = o$, $i \neq j$); Case 2 ($i = o$ and $j = m$, $i \neq j$); Case 3 ($i = m$, but i, j, o distinct); Case 4 ($i = o$, but i, j, m distinct); Case 5 ($j = m$, but i, m, o distinct); Case 6 ($j = o$, but i, j, m distinct); Case 7 ($i \neq j \neq m \neq o$). In each of these cases we must then investigate all possible subcases where l and k are equal or distinct from the indexes considered in T_{1n} and T_{2n}.

[4]See the note on indexes in the end of this appendix.
[5]Bounds for all other cases described in the appendix 1 are available from the authors upon request.

Case 1: For the term T_{1n} there are 3 subcases: 1.1) l, i, j distinct; 1.2) $l = i$ and i, j distinct; 1.3) $l = j$ and i, j distinct. For the term T_{2n} there are 7 subcases: 1.1) l, k, i, j distinct; 1.2) $k = i$, l, k, j distinct; 1.3) $k = j$, l, k, i distinct; 1.4) $l = i$, l, k, j distinct; 1.5) $l = j$, l, k, i distinct; 1.6) $l = i$, $k = j$, l, k distinct; 1.7) $l = j$, $k = i$, l, k distinct.

Case 2: The subcases are identical to those in Case 1.

Case 3: For the term T_{1n} there are 4 subcases: 3.1) l, i, j, o distinct; 3.2) $l = i$ and i, j, o distinct; 3.3) $l = j$ and i, j, o distinct; 3.4) $l = o$ and i, j, o distinct. For the term T_{2n} there are 13 subcases: 3.1) l, k, i, j, o distinct; 3.2) $k = i$, l, k, j, o distinct; 3.3) $l = i$, i, k, j, o distinct; 3.4) $k = j$, i, l, j, o distinct; 3.5) $l = j$, l, k, i, o distinct; 3.6) $l = o$, l, k, i, j distinct; 3.7) $k = o$, l, i, j, k distinct; 3.8) $l = i$, $k = j$, l, k, o distinct; 3.9) $l = j$, $i = k$, l, k, o distinct; 3.10) $l = i$, $k = o$, l, k, j distinct; 3.11) $l = o$, $i = k$, l, k, j distinct; 3.12) $l = j$, $k = o$, i, l, k distinct; 3.13) $l = o$, $k = j$, l, k, i distinct

Case 4: For the term T_{1n} there are 4 subcases: 4.1) l, i, j, m distinct; 4.2) $l = m$ and i, j, l distinct; 4.3) $l = i$ and i, j, m distinct; 4.4) $l = j$ and i, j, m distinct. For the term T_{2n} there are 13 subcases: 4.1) l, k, i, j, m distinct; 4.2) $k = m$, l, k, j, i distinct; 4.3) $l = m$, l, i, k, j distinct; 4.4) $k = i$, l, k, j, m distinct; 4.5) $l = i$, l, k, j, m distinct; 4.6) $k = j$, l, k, i, m distinct; 4.7) $l = j$, m, i, j, k distinct; 4.8) $l = m$, $k = i$, l, k, j distinct; 4.9) $l = i$, $m = k$, l, k, j distinct; 4.10) $l = m$, $k = j$, l, k, i distinct; 4.11) $l = j$, $m = k$, l, k, i distinct; 4.12) $l = i$, $k = j$, m, l, k distinct; 4.13) $l = j$, $k = i$, l, k, m distinct.

Case 5: identical to Case 4 due to symmetry.

Case 6: For the term T_{1n} there are 4 subcases: 6.1) l, i, j, m distinct; 6.2) $l = i$ and l, m, j distinct; 6.3) $l = m$ and i, j, l distinct; 6.4) $l = j$ and i, l, m distinct. For the term T_{2n} there are 13 subcases: 6.1) l, k, i, j, m distinct; 6.2) $k = i$, l, k, j, m distinct; 6.3) $l = i$, l, k, m, j distinct; 6.4) $k = m$, l, k, j, i distinct; 6.5) $l = m$, l, k, i, j distinct; 6.6) $k = j$, l, k, i, m distinct; 6.7) $l = j$, m, i, l, k distinct; 6.8) $l = i$, $k = m$, l, k, j distinct; 6.9) $k = i$, $m = l$, l, k, j distinct; 6.10) $l = i$, $k = j$, l, k, m distinct; 6.11) $l = j$, $i = k$, l, k, m distinct; 6.12) $l = m$, $k = j$, i, l, k distinct; 6.13) $l = j$, $k = m$, l, k, i distinct.

Case 7: For the term T_{1n} there are 5 subcases: 7.1) $l \neq i \neq j \neq m \neq o$; 7.2) $l = i$ and l, j, m, o are distinct; 7.3) $l = j$ and l, i, m, o are distinct; 7.4) $l = m$ and i, j, l, o are distinct; 7.5) $l = o$ and i, j, m, l are distinct.

For the term T_{2n} there are 21 subcases: 7.1) $l \neq k \neq i \neq j \neq m \neq o$; 7.2) $l = i$, $j = k$ and l, j, m, o are distinct; 7.3) $l = k$,$j = l$ and i, j, m, o are distinct; 7.4) $l = i$,$k = m$ and i, j, m, o are distinct; 7.5) $i = k$, $l = m$ and i, j, m, o are distinct; 7.6) $l = i$, $k = o$ and i, j, m, o are distinct; 7.7) $i = k$, $l = o$ and i, j, m, o are distinct; 7.8) $l = j$, $k = m$ and i, j, m, o are distinct; 7.9) $j = k$, $l = m$ and i, j, m, o are distinct; 7.10) $l = j$, $k = o$ and i, j, m, o are distinct; 7.11) $j = k$, $l = o$ and i, j, m, o are distinct; 7.12) $l = m$, $k = o$ and i, j, m, o are distinct; 7.13) $m = k$, $l = o$ and i, j, m, o are distinct; 7.14) $i = k$, l, k, j, m, o are distinct; 7.15) $i = l$, l, k, j, m, o are distinct; 7.16) $j = k$, l, k, i, m, o are distinct; 7.17) $l = j$, l, k, i, m, o are distinct; 7.18) $m = k$, l, k, i, j, o are distinct; 7.19) $m = l$, l, k, i, j, o are distinct; 7.20) $o = k$, l, k, i, j, m are distinct; 7.21) $l = o$, l, k, i, j, m are distinct.

Appendix 2

Table 1 Average bias($\times 10^{-2}$)(B), Standard deviation(S) and Root Mean Squared Error(R) with panel data models and $J = 2$

$n = 100$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	.335	.336	.336	.392	.333	.335	1.078	.349	.356
HU1	-.709	.472	.474	.721	.467	.477	-10.294	.519	.569
HU2	.315	.338	.338	.175	.333	.335	.420	.352	.358
RWC	.322	.284	.285	.318	.281	.285	1.449	.294	.306
2SLL	.278	.277	.278	.268	.275	.278	1.042	.289	.298
FHU1	-.707	.463	.465	.755	.460	.470	-9.999	.506	.551
FHU2	.329	.337	.337	.163	.333	.335	.431	.351	.357
FRWC	.327	.285	.286	.320	.282	.286	1.451	.296	.308
F2SLL	.289	.280	.280	.271	.277	.280	1.056	.291	.300

$n = 150$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	-.020	.271	.272	-.118	.270	.274	1.371	.285	.295
HU1	-.496	.373	.375	-.416	.374	.385	-9.795	.423	.479
HU2	.093	.271	.272	.304	.273	.276	.906	.289	.297
RWC	-.047	.228	.230	-.121	.229	.236	1.694	.242	.257
2SLL	-.051	.223	.224	-.162	.225	.230	1.364	.238	.249
FHU1	-.502	.368	.370	-.409	.370	.381	-9.597	.419	.471
FHU2	.102	.271	.271	.297	.272	.276	.931	.289	.297
FRWC	-.048	.229	.231	-.120	.230	.236	1.689	.243	.257
F2SLL	-.054	.224	.225	-.158	.226	.231	1.365	.239	.250

$n = 200$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	-.348	.237	.237	-.638	.237	.240	.120	.249	.256
HU1	-.397	.330	.335	.203	.334	.348	-10.232	.376	.451
HU2	-.604	.237	.237	-.955	.239	.241	.062	.247	.253
RWC	-.372	.198	.199	-.705	.201	.207	.364	.209	.221
2SLL	-.387	.194	.194	-.652	.197	.201	.125	.204	.213
FHU1	-.393	.327	.331	.210	.331	.345	-10.050	.373	.443
FHU2	-.602	.236	.237	-.953	.238	.241	.061	.247	.253
FRWC	-.371	.199	.200	-.706	.202	.207	.365	.210	.221
F2SLL	-.383	.194	.195	-.652	.197	.201	.129	.205	.214

Table 2 Average bias(B), Standard deviation(S) and Root Mean Squared Error(R) with AR(2) model

$n = 100$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	.081	.227	.227	.149	.225	.229	.510	.245	.252
HU1	-.285	.207	.208	.415	.210	.213	-.623	.236	.241
HU2	.214	.221	.221	.419	.220	.223	.648	.239	.246
VFF	.071	.202	.203	.243	.203	.207	.567	.220	.228
2SLL	.089	.203	.203	.228	.203	.208	.554	.221	.228
FHU1	-.284	.212	.212	.357	.213	.216	-.838	.243	.248
FHU2	.198	.221	.222	.419	.220	.223	.662	.239	.246
FVFF	.069	.203	.204	.225	.204	.209	.576	.222	.230
F2SLL	.085	.204	.205	.212	.205	.209	.561	.222	.230

$n = 200$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	.384	.156	.157	.011	.162	.166	.452	.171	.179
HU1	.214	.146	.147	.273	.151	.155	-.649	.166	.171
HU2	.335	.153	.154	.038	.158	.162	.418	.170	.177
VFF	.420	.141	.142	-.018	.145	.149	.422	.154	.162
2SLL	.424	.142	.142	-.017	.146	.150	.419	.154	.162
FHU1	.230	.147	.148	.264	.152	.155	-.633	.169	.174
FHU2	.347	.153	.154	.015	.158	.162	.419	.170	.176
FVFF	.412	.141	.142	-.029	.145	.150	.435	.154	.162
F2SLL	.415	.142	.142	-.023	.146	.150	.435	.154	.163

$n = 400$	$m_1(x)$			$m_2(x)$			$m_3(x)$		
estimators	B	S	R	B	S	R	B	S	R
LLE	-.174	.111	.112	-.102	.114	.119	.332	.128	.135
HU1	-.484	.103	.104	.089	.108	.113	-.513	.125	.128
HU2	-.181	.108	.109	.000	.112	.117	.332	.126	.132
VFF	-.184	.099	.101	-.113	.102	.109	.297	.114	.121
2SLL	-.188	.099	.101	-.115	.102	.109	.290	.114	.121
FHU1	-.488	.104	.105	.063	.109	.113	-.515	.127	.130
FHU2	-.193	.108	.109	-.009	.112	.117	.327	.126	.132
FVFF	-.182	.099	.101	-.113	.103	.109	.295	.114	.122
F2SLL	-.188	.099	.101	-.114	.103	.109	.289	.114	.122

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