Appendix to A nonparametric R^2 test for the presence of relevant variables

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Appendix

Below we list three lemmas that are used frequently and then we provide the proof of Theorems 1-5 and sketch of proof of Remarks 1 and 2. Throughout the proof, c will represent an inconsequential and arbitrary constant that may take different values in different contexts.

Lemma 1 *For second order U-statistic* $U_n = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_n(Z_t,Z_i)$, *where* $\phi_n(Z_t,Z_i)$ *is a symmetric function of* Z_t *and* Z_i *that could depend on* n, *and* $\{Z_t\}_{t=1}^n$ *is a sequence of IID random variables. Define* $\hat{U}_n = \frac{1}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i) - \frac{1}{2}E\phi_n(Z_t,Z_i)$, *where* $P(Z_i)$ *is the distribution function of* Z_i.
Then we have $U_n - \hat{U}_n = O_p(n^{-1}(E\phi_n^2(Z_t,Z_i))^{\frac{1}{2}})$.

Note: from Lemma 1 we obtain the following convenient expression.

$$\begin{aligned}
&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\phi_n(Z_t,Z_i) = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_n(Z_t,Z_i) + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t>i}}^{n}\phi_n(Z_t,Z_i) = \frac{2}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_n(Z_t,Z_i)\\
=\ &\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i) - E\phi_n(Z_t,Z_i) + O_p(n^{-1}(E\phi_n^2(Z_t,Z_i))^{\frac{1}{2}})
\end{aligned}$$

Lemma 1 is similar in spirit to Lemma 8.4 in Newey and McFadden (1994). Thus the proof is omitted.

Lemma 2 *Define* $S_{n,j}(z_0) = \frac{1}{n}\sum_{i=1}^n K_h(Z_i - z_0)\left(\frac{Z_i - z_0}{h}\right)^j g(U_i)w(Z_i - z_0; z_0), |j| = 0, \cdots, J,$
where Z_i, U_i *are iid,* $Z_i \in R^{l_c}, K_h(\cdot) = \frac{1}{h^{l_c}}K(\frac{\cdot}{h})$, *and* $K(\cdot)$ *is a kernel function defined on* R^{l_c}. *Here* $j = (j_1, j_2, \cdots, j_{l_c})$, $Z_i = (Z_{i,1}, Z_{i,2}, \cdots, Z_{i,l_c})$, $z_0 = (z_{0,1}, z_{0,2}, \cdots, z_{0,l_c})$, *and* $\left(\frac{Z_i - z_0}{h}\right)^j = \left(\frac{Z_{i,1}-z_{0,1}}{h}\right)^{j_1} \times \cdots \times \left(\frac{Z_{i,l_c}-z_{0,l_c}}{h}\right)^{j_{l_c}}$. *Assume*

L_1. $K(.)$ *is bounded with compact support and for Euclidean norm* $||.||$,

$$|u^j K(u) - v^j K(v)| \leq c_K||u - v||, \text{ for } 0 \leq |j| \leq J.$$

L_2. $g(U_i)$ *is a measurable function of* U_i *and* $E|g(U_i)|^s < \infty$ *for* $s > 2$.
L_3. *For* G *a compact subset of* $\Re^{l_c}$, *define the joint density of* Z_i *and* U_i *at* (z_0, u) *as* $f(z_0, u)$, *conditional density of* Z_i *and* U_i *given* U_i *at* $Z_i = z_0$ *and* $U_i = u$ *as* $f_{z|u}(z_0)$. *Assume*
$\sup_{z_0\in G}\int|g(u)|^s f_{z,u}(z_0,u)du < \infty$, $f_{z|u}(z_0) < \infty$, *and* $f_{z,u}(z_0, u)$ *is continuous around* z_0.
L_4. $w(Z_i - z_0; z_0)$ *is a function of* $Z_i - z_0$ *and* z_0. $|w(Z_i - z_0; z_0)| < c < \infty$, $|w(Z_i - z_0; z_0) - w(Z_i - z_k; z_k)| \leq c||z_0 - z_k||$ *almost everywhere.*
L_5. $\frac{nh^{l_c}}{ln(n)} \to \infty$.

$$\textit{Then for } z_0 \in G,\ \sup_{z_0\in G}|S_{n,j}(z_0) - E(S_{n,j}(z_0))| = O_{a.s.}\left(\left(\frac{nh^{l_c}}{ln(n)}\right)^{-\frac{1}{2}}\right).$$

Lemma 2 is similar in spirit to Lemma 1 in Yao and Zhang (2010). Thus the proof is omitted.
Comment: If $ES_{n,j}(z) = 0$, then L_3 could be replaced with
$L3'$: For G a compact subset of $\Re^{l_c}$, define the joint density of Z_i and U_i at (z_0, u) as $f(z_0, u)$, conditional density of Z_i and U_i given U_i at $Z_i = z_0$ and $U_i = u$ as $f_{z|u}(z_0)$. Assume $f_{z|u}(z_0) < \infty$.

Lemma 3 *For* $\{Z_t\}_{t=1}^n$ *an IID sequence of random variables, we define a third order U-statistic by*

$$U_n = \begin{pmatrix} n \\ 3 \end{pmatrix}^{-1}\sum_{(n,3)}\psi_n(Z_{i_1}, Z_{i_2}, Z_{i_3}),$$

where the sum $\sum_{(n,3)}$ *is taken over all subsets* $1 \leq i_1 < i_2 < i_3 \leq n$ *of* $\{1, 2, \cdots, n\}$ *and the function* $\psi_n(\cdot)$ *is symmetric in* Z_{i_1}, Z_{i_2}, *and* Z_{i_3}. *Define the conditional expectations* $\psi_{1n}(z_1) = E(\psi_n(Z_1, Z_2, Z_3)|Z_1 = z_1)$, $\psi_{2n}(z_1, z_2) = E(\psi_n(Z_1, Z_2, Z_3)|Z_1 = z_1, Z_2 = z_2)$ *and* $\psi_{3n}(z_1, z_2, z_3) = \psi_n(z_1, z_2, z_3)$. $\sigma_{1n}^2 = Var(\psi_{1n}(Z_1)), \sigma_{2n}^2 = Var(\psi_{2n}(Z_1, Z_2)), \sigma_{3n}^2 = Var(\psi_{3n}(Z_1, Z_2, Z_3))$. *Let* $\theta_n = E\psi_n(Z_1, Z_2, Z_3)$. *Then we have the following H-decomposition*

$$
\begin{aligned}
U_n &= \theta_n + \sum_{j=1}^{3} \binom{3}{j} H_n^{(j)} \\
&= \theta_n + \binom{n}{3}^{-1} \binom{n-1}{2} \left[\sum_{t=1}^{n} \psi_{1n}(Z_t) - n\theta_n\right] \\
&+ \binom{n}{3}^{-1} \binom{n-2}{1} \left[\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n} \psi_{2n}(Z_t, Z_i) - (n-1)\sum_{t=1}^{n}\psi_{1n}(Z_t) + \frac{n(n-1)}{2}\theta_n\right] \\
&+ \binom{n}{3}^{-1} \left\{ \sum_{(n,3)} \psi_n(Z_{i_1}, Z_{i_2}, Z_{i_3}) - \frac{(n-1)(n-2)}{2}\sum_{t=1}^{n}[\psi_{1n}(Z_t) - \theta_n] \right. \\
&\left. -(n-2)\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\psi_{2n}(Z_t, Z_i) - \psi_{1n}(Z_t) - \psi_{1n}(Z_i) + \theta_n] - \binom{n}{3}\theta_n \right\}
\end{aligned}
$$

Furthermore, we have $V(H_n^{(3)}) = \binom{n}{3}^{-1} \sum_{c=1}^{3}(-1)^{3-c}\binom{3}{c}\sigma_{cn}^2 = O(n^{-3}(\sigma_{1n}^2+\sigma_{2n}^2+\sigma_{3n}^2)) = O(n^{-3}\sigma_{3n}^2)$. $V(H_n^{(2)}) = O(n^{-2}\sigma_{2n}^2)$ *and* $V(H_n^{(1)}) = O(n^{-1}\sigma_{1n}^2)$.

Lemma 3: *Proof.* The lemma follows from Theorem 1-3 in section 1.6 of Lee (1990).

We find it easy to provide proof of Theorem 3 first as we base the proof of other Theorems on it.
Theorem 3: *Proof.*
Under the alternative hypothesis, $m(x) \neq \mu$. From the definition in equation (2), we define

$$
\tilde{R}^2 = 1 - \frac{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2}{\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2}, \text{ so } \hat{R}^2 = \tilde{R}^2 I\left(\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 \geq \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2\right).
$$

Since $\frac{\hat{a}}{\hat{b}} - \frac{a}{b} = \frac{1}{\hat{b}}\left[(\hat{a} - a) - (\hat{b} - b)\frac{a}{b}\right]$, we have

$$
\begin{aligned}
\tilde{R}^2 - R^2 &= -\frac{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2}{\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2} + \frac{E(y_t - m(x_t))^2}{V(y_t)} \\
= \quad & -\frac{1}{\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2}\left[\underbrace{\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E(y_t - m(x_t))^2}_{B_1} - \underbrace{\left(\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y_t)\right)}_{B_2}\frac{E(y_t - m(x_t))^2}{V(y_t)}\right].
\end{aligned}
$$

(1) We show $B_2\frac{E(y_t - m(x_t))^2}{V(y_t)} = \frac{1}{n}\sum_{t=1}^{n}\left\{\frac{E\sigma^2(x_t)}{V(y_t)}(y_t^2 - Ey_t^2) - 2(y_t - Em(x_t))\frac{E\sigma^2(x_t)}{V(y_t)}Em(x_t) + O_p(n^{-1})\right\}$.

$$
\begin{aligned}
\text{Note } B_2\frac{E(y_t - m(x_t))^2}{V(y_t)} &= \left(\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y_t)\right)\frac{E\sigma^2(x_t)}{V(y_t)} = \left(\frac{1}{n}\sum_{t=1}^{n}y_t^2 - \bar{y}^2 - V(y_t)\right)\frac{E\sigma^2(x_t)}{V(y_t)} \\
&= \frac{1}{n}\sum_{t=1}^{n}\left\{(y_t^2 - Ey_t^2)\frac{E\sigma^2(x_t)}{V(y_t)} - (\bar{y}^2 - (Ey_t)^2)\frac{E\sigma^2(x_t)}{V(y_t)}\right\}.
\end{aligned}
$$

So if $\bar{y}^2 = \frac{2}{n}\sum_{t=1}^{n}[y_t(Em(x_t)) - \frac{1}{2}(Em(x_t))^2] + O_p(n^{-1})$ then

$\bar{y}^2 - (Ey_t)^2 = \frac{2}{n}\sum_{t=1}^{n}y_tEm(x_t) - 2(Em(x_t))^2 + O_p(n^{-1}) = \frac{2Em(x_t)}{n}\sum_{t=1}^{n}(y_t - Em(x_t)) + O_p(n^{-1})$, then we have the claim in (1).

So we only need to show $\bar{y}^2 = \frac{2}{n}\sum_{t=1}^{n}[y_tEm(x_t) - \frac{1}{2}(Em(x_t))^2] + O_p(n^{-1})$.

$$\bar{y}^2 = \left[\frac{1}{n}\sum_{t=1}^{n} y_t\right]^2 = \frac{1}{n^2}\sum_{t=1}^{n} y_t^2 + \frac{1}{n^2}\sum_{\substack{t=1 \\ t\neq i}}^{n}\sum_{i=1}^{n} \underbrace{y_t y_i}_{\phi(y_t,y_i)} = B_{21} + B_{22}.$$

First, we note $B_{21} = O_p(n^{-1})$ since $\frac{1}{n}\sum_{t=1}^{n} y_t^2 \xrightarrow{p} Ey_t^2 < \infty$ by A2.

Second, we note in B_{22}, $\phi(y_t, y_i)$ is symmetric. We apply Lemma 1 on B_{22}. $E\phi^2(y_t, y_i) = (Ey_t^2)^2 < \infty$ since $t \neq i$ and by A2. $\int \phi(y_t, y_i)dP(y_i) = y_t E(y_i)$, so

$$\begin{aligned} B_{22} &= \frac{2}{n}\sum_{t=1}^{n} \int \phi(y_t, y_i)dP(y_i) - E\phi(y_t, y_i) + O_p(n^{-1}) = \frac{2}{n}\sum_{t=1}^{n} y_t E(y_i) - (Ey_t)^2 + O_p(n^{-1}) \\ &= \frac{2}{n}\sum_{t=1}^{n} [y_t(Em(x_t)) - \tfrac{1}{2}(Em(x_t))^2] + O_p(n^{-1}). \end{aligned}$$

From B_{21} and B_{22} we have the claim that $\bar{y}^2 = \frac{2}{n}\sum_{t=1}^{n} [y_t Em(x_t) - \frac{1}{2}(Em(x_t))^2] + O_p(n^{-1})$.

(2) $\frac{1}{n}\sum_{t=1}^{n} (y_t - \hat{m}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n} [\epsilon_t^2 + 2(m(x_t) - \hat{m}(x_t))\epsilon_t + (m(x_t) - \hat{m}(x_t))^2] = I_1 + I_2 + I_3$.

Note: $\hat{m}(x_t) - m(x_t) = \frac{1}{nh_n f(x_t)}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})\epsilon_i + \frac{1}{2nh_n f(x_t)}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n}) m^{(2)}(x_{it})(x_i - x_t)^2 + w_n(x_t)$, where $x_{it} = \lambda x_t + (1-\lambda)x_i$ for some $\lambda \in (0,1)$, and $w_n(x_t) = \hat{m}(x_t) - m(x_t) - \frac{1}{nh_n f(x_t)}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n}) y_{i*}$, where $y_{i*} = y_i - m(x_t) - m^{(1)}(x_t)(x_i - x_t) = \epsilon_i + \frac{1}{2}m^{(2)}(x_{it})(x_i - x_t)^2$.

(i) We show that $w_n(x_t) = O_p(R_{n,2}(x_t))$ uniformly for $x_t \in G$, where
$R_{n,2}(x_t) = |\frac{1}{n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n}) y_{i*}| + |\frac{1}{n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n}) y_{i*}|$.

Define $S_n(x_t) = \begin{pmatrix} S_{0n}(x_t) & S_{1n}(x_t) \\ S_{1n}(x_t) & S_{2n}(x_t) \end{pmatrix}$ with $S_{jn}(x_t) = \frac{1}{nh_n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n})^j$, and
$S(x_t) = \begin{pmatrix} f(x_t) & 0 \\ 0 & f(x_t)\sigma_K^2 \end{pmatrix} = \begin{pmatrix} S_0(x_t) & S_1(x_t) \\ S_1(x_t) & S_2(x_t) \end{pmatrix}$.

Then $\hat{m}(x_t) - m(x_t) = \frac{1}{nh_n}\sum_{i=1}^{n} W_n(\frac{x_i - x_t}{h_n}, x_t) y_{i*}$, where $W_n(z, x) = (1,0)S_n^{-1}(x)(1,z)' K(z)$. So

$$\begin{aligned} |w_n(x_t)| &= \frac{1}{nh_n}\left|\sum_{i=1}^{n} (W_n(\tfrac{x_i - x_t}{h_n}, x_t) - \tfrac{1}{f(x_t)}K(\tfrac{x_i - x_t}{h_n})) y_{i*}\right| \\ &= \frac{1}{nh_n}\left|(1,0)(S_n^{-1}(x_t) - S^{-1}(x_t))\begin{pmatrix} \sum_{t=1}^{n} K(\frac{x_i - x_t}{h_n}) y_{i*} \\ \sum_{t=1}^{n} K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n}) y_{i*} \end{pmatrix}\right| \\ &\leq \frac{1}{h_n}((1,0)(S_n^{-1}(x_t) - S^{-1}(x_t))^2(1,0)')^{\frac{1}{2}}\frac{1}{n}\left(\left|\sum_{t=1}^{n} K(\tfrac{x_i - x_t}{h_n}) y_{i*}\right| + \left|\sum_{t=1}^{n} K(\tfrac{x_i - x_t}{h_n})(\tfrac{x_i - x_t}{h_n}) y_{i*}\right|\right). \end{aligned}$$

We use Lemma 2 with A4, and follow Lemma 2 of Martins-Filho and Yao (2007) to obtain $w_n(x_t) = O_{a.s.}(R_{n,2}(x_t))$ uniformly for $x_t \in G$. With assumption A6-A8, we apply Lemma 2 to obtain
$\frac{1}{n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n}) y_{i*} = \frac{1}{n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})\epsilon_i + \frac{1}{2n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})(x_i - x_t)^2 m^{(2)}(x_{it}) = O_{a.s.}(h_n \left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}}) + O_{a.s.}(h_n^3)$ uniformly in $x_t \in G$. In a similar fashion we obtain $\frac{1}{n}\sum_{i=1}^{n} K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n}) y_{i*} = O_{a.s.}(h_n \left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}}) + O_{a.s.}(h_n^3)$ uniformly in $x_t \in G$. Thus, $\sup_{x_t \in G} |w_n(x_t)| = O_{a.s.}(h_n \left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}}) + O_{a.s.}(h_n^3)$.

(ii) Since $f(x_t) \geq \underline{B}_f$ in A4(1), results in (i) give us
$\sup_{x_t \in G} |\hat{m}(x_t) - m(x_t)| = O_{a.s.}(\left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}}) + O_{a.s.}(h_n^2)$.
$\sup_{x_t \in G} |\hat{m}(x_t) - m(x_t)|^2 = O_{a.s.}(\left(\frac{nh_n}{lnn}\right)^{-1}) + O_{a.s.}(h_n^4) + O_{a.s.}(h_n^2 \left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}})$.
So $\sqrt{n}\sup_{x_t \in G} |\hat{m}(x_t) - m(x_t)|^2 = O_{a.s.}(\frac{lnn}{\sqrt{nh_n^2}}) + O_{a.s.}(\sqrt{nh_n^8}) + O_{a.s.}((nh_n^8)^{\frac{1}{4}}(\frac{lnn}{\sqrt{nh_n^2}})^{\frac{1}{2}}) = o_{a.s.}(1)$ by A6.
We conclude $I_3 = o_p(n^{-\frac{1}{2}})$.

(iii) $I_2 = -\frac{2}{n}\sum_{t=1}^{n}(\hat{m}(x_t) - m(x_t))\epsilon_t$

$= \quad -\frac{2}{n^2 h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}\frac{1}{f(x_t)}K(\frac{x_i - x_t}{h_n})\epsilon_i\epsilon_t - \frac{1}{n^2 h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})(x_i - x_t)^2 m^{(2)}(x_{it})\epsilon_t - \frac{2}{n}\sum_{t=1}^{n}w_n(x_t)\epsilon_t$

$= \quad I_{21} + I_{22} + I_{23}$

$$I_{21} = -\frac{1}{n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\underbrace{[\frac{1}{h_n f(x_t)}K(\frac{x_i - x_t}{h_n})\epsilon_i\epsilon_t + \frac{1}{h_n f(x_i)}K(\frac{x_t - x_i}{h_n})\epsilon_t\epsilon_i]}_{\phi_n(Z_t, Z_i) \text{ where } Z_t = (x_t, \epsilon_t).}$$

$$= -\frac{2}{n^2 h_n}\sum_{t=1}^{n}\frac{K(0)\epsilon_t^2}{f(x_t)} - \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\phi_n(Z_t, Z_i) = -I_{211} - I_{212}.$$

First note: $nh_n E|I_{211}| = 2EK(0)\frac{\sigma^2(x_t)}{f(x_t)}$, so $I_{211} = O_p((nh_n)^{-1}) = o_p(n^{-\frac{1}{2}})$ with A6.
Second, we apply Lemma 1 to obtain

$$I_{212} - [\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i) - E\phi_n(Z_t, Z_i)] = O_p(n^{-1}(E\phi_n^2(Z_t, Z_i))^{\frac{1}{2}}).$$

We observe $\int\phi_n(Z_t, Z_i)dP(Z_i) = E\phi_n(Z_t, Z_i) = 0$ since $E(\epsilon_t|x_t) = 0$ in A3 and
$E\phi_n^2(Z_t, Z_i) \leq c[\frac{1}{h_n^2}E\frac{1}{f^2(x_t)}K^2(\frac{x_i - x_t}{h_n})\epsilon_i^2\epsilon_t^2 + \frac{1}{h_n^2}E\frac{1}{f^2(x_i)}K^2(\frac{x_t - x_i}{h_n})\epsilon_t^2\epsilon_i^2]$.
Since $\frac{1}{h_n}E\frac{1}{f^2(x_t)}K^2(\frac{x_i - x_t}{h_n})\epsilon_i^2\epsilon_t^2 = \int\frac{1}{f^2(x_t)}K^2(\psi)\sigma^2(x_t + h_n\psi)\sigma^2(x_t)f(x_t)f(x_t + h\psi)dx_t d\psi$
$\to \int K^2(\psi)d\psi\int(\sigma^2(x_t))^2 dx_t$, we have $(E\phi_n^2(Z_t, Z_i))^{\frac{1}{2}} = O(h_n^{-\frac{1}{2}})$.
So we conclude $I_{212} = O_p((n^2 h_n)^{-\frac{1}{2}}) = o_p(n^{-\frac{1}{2}})$ and $I_{21} = o_p(n^{-\frac{1}{2}})$.

$$I_{22} = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}[\frac{-1}{2h_n f(x_t)}K(\frac{x_i - x_t}{h_n})(x_i - x_t)^2 m^{(2)}(x_{it})\epsilon_t - \frac{1}{2h_n f(x_i)}K(\frac{x_t - x_i}{h_n})(x_t - x_i)^2 m^{(2)}(x_{ti})\epsilon_i]$$

$$= \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}[\psi_n(Z_t, Z_i) + \psi_n(Z_i, Z_t)] = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\phi_n(Z_t, Z_i).$$

Since $\phi_n(Z_t, Z_i)$ is symmetric, we apply Lemma 1 again to obtain
$I_{22} - [\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i) - E\phi_n(Z_t, Z_i)] = O_p(n^{-1}(E\phi_n^2(Z_t, Z_i))^{\frac{1}{2}})$.
Note $E\phi_n(Z_t, Z_i) = 0$ by A3.
$\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i) = \frac{-1}{n}\sum_{t=1}^{n}\frac{\epsilon_t}{h_n f(x_t)}E(K(\frac{x_i - x_t}{h_n})(x_i - x_t)^2 m^{(2)}(x_{it})|Z_t)$.
Since $\frac{1}{h_n}E(K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n})^2 m^{(2)}(x_{it})|Z_t) = \int K(\psi)\psi^2 m^{(2)}(x_t + \lambda h\psi)f(x_t + h\psi)d\psi$
$\to \sigma_K^2 m^{(2)}(x_t)f(x_t)$ for $\lambda \in (0, 1)$ by A4 and A5,
$E\frac{\sqrt{n}}{h_n^2}\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i) = 0$, and
$V(\frac{\sqrt{n}}{h_n^2}\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i)) = E\frac{\epsilon_t^2}{f^2(x_t)}[\frac{1}{h_n}E(K(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n})^2 m^{(2)}(x_{it})|Z_t)]^2$
$\to E(\sigma_K^2 m^{(2)}(x_t))^2\sigma^2(x_t) < \infty$, so $\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t, Z_i)dP(Z_i) = O_p(n^{-\frac{1}{2}}h_n^2) = o_p(n^{-\frac{1}{2}})$.

On the other hand, $E\phi_n^2(Z_t, Z_i) \leq c[E\psi_n^2(Z_t, Z_i) + E\psi_n^2(Z_i, Z_t)]$.
$\frac{1}{h_n^3}E\psi_n^2(Z_t, Z_i) = \frac{1}{h_n}E\frac{1}{4f^2(x_t)}K^2(\frac{x_i - x_t}{h_n})(\frac{x_i - x_t}{h_n})^4(m^{(2)}(x_{it}))^2\sigma^2(x_t)$
$\to \frac{1}{4}\int K^2(\psi)\psi^4 d\psi\int(m^{(2)}(x_t))^2\sigma^2(x_t)dx_t < \infty$. So $n^{-1}(E\phi_n^2(Z_t, Z_i))^{\frac{1}{2}} = O_p(n^{-1}h_n^{\frac{3}{2}}) = o_p(n^{-\frac{1}{2}})$.

With the results above we have $I_{22} = o_p(n^{-\frac{1}{2}})$.

$|I_{23}| \leq \sup_{x_t\in G}|w_n(x_t)|\frac{2}{n}\sum_{t=1}^{n}|\epsilon_t| = o_p(n^{-\frac{1}{2}})$, since $\sup_{x_t\in G}|w_n(x_t)| = O_p(h_n\left(\frac{nh_n}{lnn}\right)^{-\frac{1}{2}}) + O_p(h_n^3) = o_p(n^{-\frac{1}{2}})$
by A6 and $E|\epsilon_t| < \infty$ by A3. So in all we have $I_2 = o_p(n^{-\frac{1}{2}})$.

With (i)-(iii), we conclude that $\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 + o_p(n^{-\frac{1}{2}})$.

So $B_1 = \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E(y_t - m(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 - E\sigma^2(x_t) + o_p(n^{-\frac{1}{2}})$.

(3) It is easy to see that $\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 \xrightarrow{p} V(y_t) > 0$, so with results in (1) and (2),

$$\begin{aligned}
&\tilde{R}^2 - R^2 \\
= \quad & -\{[\tfrac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2]^{-1} - (V(y_t))^{-1} + (V(y_t))^{-1}\}\{\tfrac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 - E\sigma^2(x_t) + o_p(n^{-\frac{1}{2}}) \\
& -\tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{E\sigma^2(x_t)}{V(y_t)}(y_t^2 - Ey_t^2) - 2(y_t - Em(x_t))\tfrac{E\sigma^2(x_t)}{V(y_t)}Em(x_t)] + O_p(n^{-1})\} \\
= \quad & -[(V(y_t))^{-1} + o_p(1)]\{\tfrac{1}{n}\sum_{t=1}^{n}[\epsilon_t^2 - E\sigma^2(x_t) - \tfrac{E\sigma^2(x_t)}{V(y_t)}(y_t^2 - Ey_t^2) \\
& +2(y_t - Em(x_t))\tfrac{E\sigma^2(x_t)}{V(y_t)}Em(x_t)] + o_p(n^{-\frac{1}{2}})\} \\
= \quad & -[(V(y_t))^{-1}]\{\tfrac{1}{n}\sum_{t=1}^{n}[\epsilon_t^2 - \tfrac{E\sigma^2(x_t)}{V(y_t)}(y_t - E(y_t))^2]\} + o_p(n^{-\frac{1}{2}})
\end{aligned}$$

Let $W_t = \epsilon_t^2 - E\sigma^2(x) - \frac{E\sigma^2(x)}{V(y)}(y_t^2 - Ey^2) + 2\frac{E\sigma^2(x)}{V(y)}Em(x)(y_t - Em(x)) = \epsilon_t^2 - \frac{E\sigma^2(x_t)}{V(y_t)}(y_t - E(y_t))^2$. Since W_t is IID, with zero mean and variance EW_t^2, we apply the Central Limit Theorem to obtain

$$\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}W_t \xrightarrow{d} N(0, EW_t^2), \text{ and thus } \sqrt{n}(\tilde{R}^2 - R^2) \xrightarrow{d} N(0, (V(y_t))^{-2}EW_t^2).$$

(4) We show that $I(\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \leq \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2) - 1 = o_{a.s.}(1)$.

Note by Kolmogorov's Theorem and with assumptions A1 and A2, we have $\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 = \frac{1}{n}\sum_{t=1}^{n}y_t^2 - \bar{y}^2 \xrightarrow{a.s.} V(y_t)$, since $\frac{1}{n}\sum_{t=1}^{n}y_t^2 \xrightarrow{a.s.} Ey_t^2$ and $\bar{y}^2 \xrightarrow{a.s.} (Ey_t)^2$.

From result (2), $\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}[\epsilon_t^2 + 2(m(x_t) - \hat{m}(x_t))\epsilon_t + (m(x_t) - \hat{m}(x_t))^2]$.

By Kolmogorov's Theorem again, we have $\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 \xrightarrow{a.s.} E\epsilon_t^2 = E\sigma^2(x_t)$. As shown in result (2)(ii) above, $\sup_{x_t \in G}|\hat{m}(x_t) - m(x_t)| = o_{a.s.}(1)$. So

$|\frac{1}{n}\sum_{t=1}^{n}2(m(x_t) - \hat{m}(x_t))\epsilon_t| \leq o_{a.s.}(1)\frac{1}{n}\sum_{t=1}^{n}|\epsilon_t| = o_{a.s.}(1)$, since $\frac{1}{n}\sum_{t=1}^{n}|\epsilon_t| \xrightarrow{a.s.} E|\epsilon_t| < (E\epsilon_t^2)^{\frac{1}{2}} < \infty$.

Similarly, we obtain $\frac{1}{n}\sum_{t=1}^{n}(m(x_t) - \hat{m}(x_t))^2 = o_{a.s.}(1)$. So we have $\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \xrightarrow{a.s.} E\sigma^2(x_t)$.

Combine results above, we have $\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \xrightarrow{a.s.} V(y_t) - E\sigma^2(x_t) = V(m(x_t)) > 0$ under H_1. So we obtain the claimed result that $I(\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \leq \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2) - 1 = o_{a.s.}(1)$.

(5) Claim: $\sqrt{n}(I(\cdot) - 1)\tilde{R}^2 = o_p(1)$, where $I(\cdot) = I(\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \leq \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2)$.

Since $\tilde{R}^2 \xrightarrow{p} R^2 > 0$, we only need to show $\sqrt{n}(I(\cdot) - 1) = o_p(1)$.

$$\begin{aligned}
\text{By definition} \quad & I(\tfrac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 \leq \tfrac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2) \\
= \quad & I(\tfrac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E\sigma^2(x) + E\sigma^2(x) \leq \tfrac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y) + V(y)).
\end{aligned}$$

Define the event $D = \{w : |I(\cdot) - 1| \neq 0\}$, $A = \{w : |\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E\sigma^2(x)| > \delta_1\}$, and $B = \{w : |\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y)| > \delta_2\}$ for some $\delta_1, \delta_2 > 0$. Since $V(y) > E\sigma^2(x)$ under the alternative, we have $D \subset A \cup B$. From (1) and (2) above, we have $\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E\sigma^2(x) = O_p(n^{-1/2})$, and $\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y) = O_p(n^{-1/2})$. So when $I(\cdot) \neq 1$,

$|\sqrt{n}(I(\cdot)-1)| = \sqrt{n}I(D) \leq \sqrt{n}I(D)(I(A)+I(B))$.

With event A, $\sqrt{n}I(A) = \sqrt{n}1 < \frac{1}{\delta_1}|\sqrt{n}(\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 - E\sigma^2(x))| = O_p(1)$.

With event B, $\sqrt{n}I(B) = \sqrt{n}1 < \frac{1}{\delta_2}|\sqrt{n}(\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 - V(y))| = O_p(1)$.

Since $I(D) = o_p(1)$, we have the claimed result that $\sqrt{n}(I(\cdot)-1) = o_p(1)$.

By results (3)-(5) above, we have

$$\sqrt{n}(\hat{R}^2 - R^2) = \sqrt{n}(\tilde{R}^2 I(\cdot) - \tilde{R}^2 + \tilde{R}^2 - R^2) \xrightarrow{d} N(0, (V(y_t))^{-2}EW_t^2).$$

Theorem 1: *Proof.*

(a) Under H_0, $P(E(y_t|x_t) = \mu) = 1$, so we have $m(x_t) = \mu$, and thus $y_t = \mu + \epsilon_t$, $V(y_t) = E\sigma^2(x_t)$ and $R^2 = 0$. Recall the definition of $\tilde{R}^2 = \frac{\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2 - \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2}{\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2}$. We have from proof of Theorem 3 that $\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2 \xrightarrow{p} V(y_t) = E\sigma^2(x_t)$ under H_1.

(1) As in result (1) of Theorem 3's proof,

$\bar{y}^2 = \frac{2}{n}\sum_{t=1}^{n} y_t Em(x_t) - (Em(x_t))^2 + O_p(n^{-1}) = \mu^2 + \frac{2}{n}\sum_{t=1}^{n}\mu\epsilon_t + O_p(n^{-1})$, so

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2 = \frac{1}{n}\sum_{t=1}^{n}y_t^2 - \bar{y}^2 = \frac{1}{n}\sum_{t=1}^{n}(\mu+\epsilon_t)^2 - \bar{y}^2 \\
= \;& \mu^2 + \frac{2}{n}\sum_{t=1}^{n}\mu\epsilon_t + \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 - (\mu^2 + \frac{2}{n}\sum_{t=1}^{n}\mu\epsilon_t + O_p(n^{-1})) = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 + O_p(n^{-1}).
\end{aligned}$$

(2) $\frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 + \frac{2}{n}\sum_{t=1}^{n}(\mu - \hat{m}(x_t))\epsilon_t + \frac{1}{n}\sum_{t=1}^{n}(\mu - \hat{m}(x_t))^2$.

Define $e' = (1,0)$, and we follow result (2)(i) in Theorem 3 to write

$$\hat{m}(x_t) - \mu = \frac{1}{nh_n}\sum_{i=1}^{n}W_n(\frac{x_i - x_t}{h_n}, x_t)\epsilon_i = e'S_n^{-1}(x_t)\begin{bmatrix} \frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\epsilon_i \\ \frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\frac{x_i-x_t}{h_n}\epsilon_i \end{bmatrix}$$

(i) So $\frac{2}{n}\sum_{t=1}^{n}(\mu - \hat{m}(x_t))\epsilon_t = -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}e'S_n^{-1}(x_t)K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t\begin{bmatrix} 1 \\ \frac{x_i-x_t}{h_n}\end{bmatrix}$.

Recall the result in (2)(i) in Theorem 3 that $S_n(x_t) = \begin{bmatrix} S_{0n}(x_t) & S_{1n}(x_t) \\ S_{1n}(x_t) & S_{2n}(x_t)\end{bmatrix}$. Define $S_{jn}(x_t) = \frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})(\frac{x_i-x_t}{h_n})^j$, $j = 0,1,2$. We apply Lemma 2 to obtain $\sup_{x_t\in G}|S_{jn}(x_t) - ES_{jn}(x_t)| = O_{a.s.}((\frac{nh_n}{lnn})^{-\frac{1}{2}})$ with assumption A6 and A7. Furthermore, $ES_{jn}(x_t) = \int K(\psi)\psi^j f(x_t + h_n\psi)d\psi \to f(x_t)\int K(\psi)\psi^j d\psi$ uniformly over $x_t \in G$ with assumption A4. So $\sup_{x_t\in G}|S_{jn}(x_t) - S_j(x_t)| = o_{a.s.}(1)$. With similar arguments, $\sup_{x_t\in G}|S_{jn}^{-1}(x_t) - S_j^{-1}(x_t)| = o_{a.s.}(1)$. So we obtain

$$\begin{aligned}
&\frac{2}{n}\sum_{t=1}^{n}(\mu - \hat{m}(x_t))\epsilon_t = -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}e'[S_n^{-1}(x_t) - S^{-1}(x_t) + S^{-1}(x_t)]K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t\begin{bmatrix} 1 \\ \frac{x_i-x_t}{h_n}\end{bmatrix} \\
= \;& -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}e'S^{-1}(x_t)K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t\begin{bmatrix} 1 \\ \frac{x_i-x_t}{h_n}\end{bmatrix}(1 + o_{a.s.}(1)) \\
= \;& -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t(1 + o_{a.s.}(1)) \\
= \;& [-\frac{2}{n^2h_n}\sum_{t=1}^{n}\frac{1}{f(x_t)}K(0)\epsilon_t^2 - \frac{2}{n^2h_n}\sum_{\substack{t=1 \\ t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t](1 + o_{a.s.}(1)) \\
= \;& [A_{1n} + A_{2n}](1 + o_{a.s.}(1))
\end{aligned}$$

where $o_{a.s.}(1)$ above indicates the term of smaller magnitude almost surely.

(ii) $\frac{1}{n}\sum_{t=1}^{n}(\mu-\hat{m}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}\left[e' S_n^{-1}(x_t)\frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\epsilon_i \left[\begin{array}{c} 1 \\ \frac{x_i-x_t}{h_n}\end{array}\right]\right]^2$

since $\sup_{x_t\in G}|S_{jn}^{-1}(x_t)-S_j^{-1}(x_t)| = o_{a.s.}(1)$

$$= \underbrace{\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\epsilon_i\epsilon_j}_{I}(1+o_{a.s.}(1))$$

(a) When $t=i=j$, $I=\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\frac{1}{f^2(x_t)}K^2(0)\epsilon_t^2$. Since $|E\frac{1}{n}\sum_{t=1}^{n}\frac{1}{f^2(x_t)}\epsilon_t^2| = E\frac{\sigma^2(x_t)}{f^2(x_t)} < \infty$ with assumptions A3 and A4, $I = O_p((nh_n)^{-2}) = o_p(n^{-1})$.

(b) When $t=i$ (or $t=j$), $I = \frac{2}{n^3h_n^{\frac{3}{2}}}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\frac{1}{2\sqrt{h_n}f^2(x_t)}K(0)K(\frac{x_j-x_t}{h_n})\epsilon_t\epsilon_j = \frac{2}{n^3h_n^{\frac{3}{2}}}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\psi_n(Z_t,Z_j)$

where $Z_t=(x_t,\epsilon_t)$. We apply Lemma 1 on

$$nh_n^{\frac{3}{2}}I = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}(\psi_n(Z_t,Z_j)+\psi_n(Z_j,Z_t)) = \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\phi_n(Z_t,Z_j).$$

Note $\int\phi_n(Z_t,Z_j)dP(Z_j)=0$ and $E\phi_n(Z_t,Z_j)=0$. So $nh_n^{\frac{3}{2}}I = O_p(n^{-1}(E\phi_n^2(Z_t,Z_j))^{\frac{1}{2}})$.
$E\phi_n^2(Z_t,Z_j) \leq cE\psi_n^2(Z_t,Z_j) = \frac{c}{4h_n}K^2(0)E\frac{1}{f^4(x_t)}K^2(\frac{x_j-x_t}{h_n})\epsilon_t^2\epsilon_j^2$
$\to \frac{c}{4}K^2(0)E\frac{1}{f^3(x_t)}\sigma^4(x_t)\int K^2(\psi)d\psi < \infty$ with assumption A3 and A9.
So we conclude $nh_n^{\frac{3}{2}}I = O_p(n^{-1})$ and $I=o_p(n^{-1})$.

(c) When $i=j$ but $t\neq i$, $I = \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{f^2(x_t)}K^2(\frac{x_i-x_t}{h_n})\epsilon_i^2 = A_n$.

(d) When indices t,i and j are distinct, we let $\psi_n(Z_t,Z_i,Z_j) = \frac{1}{h_n^2}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\epsilon_i\epsilon_j$,

$$I = \frac{1}{n^3}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\t\neq i\neq j}}^{n}\psi_n(Z_t,Z_i,Z_j) = \frac{1}{3n^3}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\t\neq i\neq j}}^{n}\underbrace{(\psi_n(Z_t,Z_i,Z_j)+\psi_n(Z_i,Z_t,Z_j)+\psi_n(Z_j,Z_i,Z_t))}_{\phi_n(Z_t,Z_i,Z_j)}$$
$$= \frac{1}{n^3}\sum\sum\sum_{1=t<i<j=n}2\phi_n(Z_t,Z_i,Z_j),\ \phi_n(Z_t,Z_i,Z_j) \text{ is symmetric.}$$
$$= \frac{1}{3}\left[\frac{6}{n^3}-\binom{n}{3}^{-1}+\binom{n}{3}^{-1}\right]\sum\sum\sum_{1=t<i<j=n}\phi_n(Z_t,Z_i,Z_j)$$

Since $\frac{6}{n^3}-\binom{n}{3}^{-1} = O(n^{-4})$ and we note $U_n = \binom{n}{3}^{-1}\sum\sum\sum_{1=t<i<j=n}\phi_n(Z_t,Z_i,Z_j)$ is a third order U-statistic. We use the H-decomposition in Lemma 3. $E\phi_n(Z_t,Z_i,Z_j)=0$, $E(\phi_n(Z_t,Z_i,Z_j)|Z_t)=0$, and

$$E(\phi_n(Z_t,Z_i,Z_j)|Z_t,Z_i) = E(\psi_n(Z_t,Z_i,Z_j)+\psi_n(Z_i,Z_t,Z_j)+\psi_n(Z_j,Z_i,Z_t)|Z_t,Z_i)$$
$$= \frac{\epsilon_i\epsilon_t}{h_n}E(\frac{1}{h_nf^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i) = \phi_{2n}(Z_t,Z_i),$$

So $U_n = \frac{6}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i)+O_p(H_n^{(3)})$. $Var(H_n^{(3)}) = O(n^{-3}\sigma_{3n}^2)$. $\sigma_{3n}^2 \leq 3cE\psi_n^2(Z_t,Z_i,Z_j)$.

$h_n^2E\psi_n^2(Z_t,Z_i,Z_j) = \frac{1}{h_n^2}E\frac{1}{f^4(x_t)}K^2(\frac{x_i-x_t}{h_n})K^2(\frac{x_j-x_t}{h_n})\sigma^2(x_i)\sigma^2(x_j) \to (\int K^2(\psi)d\psi)^2E\frac{\sigma^4(x_t)}{f^2(x_t)} < \infty$ with assumptions A4 and A9. So $\sigma_{3n}^2 = O(h_n^{-2})$. $Var(H_n^{(3)}) = O(n^{-3}h_n^{-2}) = o(n^{-2})$, and thus
$U_n = \frac{6}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i)+o(n^{-1})$ and $I = \frac{1}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)]+o(n^{-1})$.

We conclude that

$$\frac{1}{n}\sum_{t=1}^{n}(\mu-\hat{m}(x_t))^2 - A_n(1+o_{a.s.}(1)) = \frac{1}{n(n-1)}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)] + o(n^{-1}))(1+o_{a.s.}(1))$$

$$= \frac{1}{n(n-1)}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\frac{\epsilon_i\epsilon_t}{h_n}E(\frac{1}{h_nf^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i)$$

$$+\frac{\epsilon_t\epsilon_i}{h_n}E(\frac{1}{h_nf^2(x_j)}K(\frac{x_t-x_j}{h_n})K(\frac{x_i-x_j}{h_n})|Z_i,Z_t)](1+o_{a.s.}(1)) + o_p(n^{-1})$$

$$= A_{3n}(1+o_{a.s.}(1)) + o_p(n^{-1})$$

We summarize the results obtained in (2) as

$$\frac{1}{n}\sum_{t=1}^{n}(y_t-\hat{m}(x_t))^2 - (A_n+A_{1n})(1+o_{a.s.}(1)) = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2 + (A_{2n}+A_{3n})(1+o_{a.s.}(1)) + o_p(n^{-1}).$$

A_n and A_{1n} are the bias terms. We focus on A_{2n} and A_{3n} to determine the asymptotic distribution.

(3) $A_{2n} = \frac{1}{n^2h_n}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[(-2)\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t - 2\frac{1}{f(x_i)}K(\frac{x_t-x_i}{h_n})\epsilon_t\epsilon_i]$.

$$A_{3n} = [\frac{1}{n(n-1)} - \frac{1}{n^2} + \frac{1}{n^2}]\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)]$$

$$= [O(n^{-3}) + \frac{1}{n^2}]\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)] = \frac{1}{n^2}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)] + o_p(n^{-1}),$$

since $A_{31n} = \frac{1}{n^2}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)]$ is a second order U-statistic in Lemma 1, we easily obtain $A_{31n} = O_p(n^{-1}(E\phi_{2n}^2(Z_t,Z_i))^{\frac{1}{2}})$ as ϵ_t has conditional mean zero as in Assumption A3.
Furthermore, $h_nE\phi_{2n}^2(Z_t,Z_i) = \frac{1}{h_n}E\sigma^2(x_i)\sigma^2(x_t)[E\frac{1}{h_nf^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i)]^2$
$= \int\sigma^2(x_t)\sigma^2(x_t-h_n\psi_1)[\int K(\psi)K(\psi_1+\psi)\frac{1}{f(x_t-h(\psi+\psi_1))}d\psi]^2f(x_t)f(x_t-h_n\psi_1)d\psi_1dx_t$
$\to \int\kappa^2(\psi_1)d\psi_1E\frac{\sigma^4(x_t)}{f(x_t)} < \infty$, so $(E\phi_{2n}^2(Z_t,Z_i))^{\frac{1}{2}} = O(h_n^{-\frac{1}{2}})$ and $A_{31n} = O_p((n^2h_n)^{-\frac{1}{2}})$.

$$A_{2n}+A_{31n} = \frac{1}{n^2h_n}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}\Big[(-2)\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i\epsilon_t - 2\frac{1}{f(x_i)}K(\frac{x_t-x_i}{h_n})\epsilon_t\epsilon_i$$

$$+\epsilon_i\epsilon_tE(\frac{1}{h_nf^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i) + \epsilon_t\epsilon_iE(\frac{1}{h_nf^2(x_j)}K(\frac{x_t-x_j}{h_n})K(\frac{x_i-x_j}{h_n})|Z_i,Z_t)\Big]$$

$$= \frac{1}{n^2h_n}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}[\psi_n(Z_t,Z_i)+\psi_n(Z_i,Z_t)+\psi_n'(Z_t,Z_i)+\psi_n'(Z_i,Z_t)]$$

$$= \frac{1}{n^2h_n}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}\phi_n(Z_t,Z_i).$$

Since $\phi_n(Z_t,Z_i)$ is symmetric and $E(\phi_n(Z_t,Z_i)|Z_t) = 0$, $A_{2n}+A_{31n}$ is a degenerated second order U-statistic. In the following we show

(i) $\frac{1}{h_n}E\phi_n^2(Z_t,Z_i) \to 2[2E\frac{\sigma^4(x_t)}{f(x_t)}(4\int K^2(\psi)d\psi + \int\kappa^2(\psi)d\psi - 4\int K(\psi)\kappa(\psi)d\psi)] = 2\sigma_\phi^2$.

(ii) $E\phi_n^4(Z_t,Z_i) = O(h_n)$.

(iii) For $G_n(Z_1,Z_2) = E(\phi_n(Z_t,Z_1)\phi_n(Z_t,Z_2)|Z_1,Z_2)$, we have $EG_n^2(Z_1,Z_2) = O(h_n^3)$.

From (i)-(iii), we have $\frac{EG_n^2(Z_1,Z_2)}{(E\phi_n^2(Z_t,Z_i))^2} = \frac{O(h_n^3)}{O(h_n^2)} \to 0$, and $\frac{\frac{1}{n}E\phi_n^4(Z_t,Z_i)}{(E\phi_n^2(Z_t,Z_i))^2} = \frac{\frac{1}{n}O(h_n)}{O(h_n^2)} \to 0$, which is condition (2.1) in Hall (1984). So we apply Hall's Central Limit Theorem to have

$$nh_n^{\frac{1}{2}}(A_{2n}+A_{31n}) \xrightarrow{d} N(0,\sigma_\phi^2).$$

$$\begin{aligned}
\text{(i) } E\phi_n^2(Z_t,Z_i) = {} & E\psi_n^2(Z_t,Z_i) + E\psi_n^2(Z_i,Z_t) + E\psi_n'^2(Z_t,Z_i) + E\psi_n'^2(Z_i,Z_t) \\
+ {} & 2E\psi_n(Z_t,Z_i)\psi_n(Z_i,Z_t) + 2E\psi_n(Z_t,Z_i)\psi_n'(Z_t,Z_i) + 2E\psi_n(Z_t,Z_i)\psi_n'(Z_i,Z_t) \\
+ {} & 2E\psi_n(Z_i,Z_t)\psi_n'(Z_t,Z_i) + 2E\psi_n(Z_i,Z_t)\psi_n'(Z_i,Z_t) + 2E\psi_n'(Z_t,Z_i)\psi_n'(Z_i,Z_t).
\end{aligned}$$

With assumptions A2, A3, A7 and A9,

$$\frac{1}{h_n}E\psi_n^2(Z_t,Z_i) = \frac{1}{h_n}E\psi_n^2(Z_i,Z_t)$$

$$= \frac{4}{h_n}E\frac{1}{f^2(x_t)}K^2(\frac{x_i-x_t}{h_n})\sigma^2(x_i)\sigma^2(x_t) \to 4\int K^2(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)} < \infty.$$

$\frac{1}{h_n}E\psi_n'^2(Z_t,Z_i) = \frac{1}{h_n}E\psi_n'^2(Z_i,Z_t)$
$= \frac{1}{h_n}E\sigma^2(x_i)\sigma^2(x_t)[E\frac{1}{h_n f^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i)]^2 \to \int \kappa^2(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)} < \infty.$

With the assumption A7 that $K(\cdot)$ is a symmetric function, we obtain

$\frac{2}{h_n}E\psi_n(Z_t,Z_i)\psi_n(Z_i,Z_t) \to 8\int K^2(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$, $\frac{2}{h_n}E\psi_n(Z_t,Z_i)\psi_n'(Z_t,Z_i) \to -4\int K(\psi)\kappa(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$,
$\frac{2}{h_n}E\psi_n(Z_i,Z_t)\psi_n'(Z_i,Z_t) \to -4\int K(\psi)\kappa(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$, $\frac{2}{h_n}E\psi_n'(Z_t,Z_i)\psi_n'(Z_i,Z_t) \to 2\int \kappa^2(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$,
$\frac{2}{h_n}E\psi_n(Z_t,Z_i)\psi_n'(Z_i,Z_t) \to -4\int K(\psi)\kappa(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$,
$\frac{2}{h_n}E\psi_n(Z_i,Z_t)\psi_n'(Z_t,Z_i) \to -4\int K(\psi)\kappa(\psi)d\psi E\frac{\sigma^4(x_t)}{f(x_t)}$,
so $\frac{1}{h_n}E\phi_n^2(Z_t,Z_i) \to 2\sigma_\phi^2$.

(ii) $E\phi_n^4(Z_t,Z_i) \le c[E\psi_n^4(Z_t,Z_i) + E\psi_n^4(Z_i,Z_t) + E\psi_n'^4(Z_t,Z_i) + E\psi_n'^4(Z_t,Z_i)]$ by c_r inequality.

$$\begin{aligned}\frac{1}{h_n}E\psi_n^4(Z_t,Z_i) &= \frac{1}{h_n}E\psi_n^4(Z_i,Z_t) = \frac{16}{h_n}E\frac{1}{f^4(x_t)}K^4(\frac{x_i-x_t}{h_n})E(\epsilon_i^4|x_i)E(\epsilon_t^4|x_t)\\ &< \frac{16c}{h_n}E\frac{1}{f^4(x_t)}K^4(\frac{x_i-x_t}{h_n}) \text{ by assumption A9,}\\ &\to 16c\int K^4(\psi)d\psi\int\frac{1}{f^2(x_t)}dx_t.\\ \frac{1}{h_n}E\psi_n'^4(Z_t,Z_i) &= \frac{1}{h_n}E\psi_n'^4(Z_i,Z_t)\\ &= \frac{1}{h_n}E[E(\epsilon_i^4|x_i)E(\epsilon_t^4|x_t)[E\frac{1}{h_n f^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i)]^4]\\ &< \frac{c}{h_n}E[[E\frac{1}{h_n f^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|Z_t,Z_i)]^4] \text{ by assumption A9,}\\ &\to c\int\kappa^4(\psi)d\psi\int\frac{1}{f^2(x_t)}dx_t.\end{aligned}$$

So we have $E\phi_n^4(Z_t,Z_i) = O(h_n)$.

(iii) Since
$$\begin{aligned}G_n(Z_1,Z_2) &= E(\phi_n(Z_t,Z_1)\phi_n(Z_t,Z_2)|Z_1,Z_2)\\ &= E\{[\psi_n(Z_t,Z_1)+\psi_n(Z_1,Z_t)+\psi_n'(Z_t,Z_1)+\psi_n'(Z_1,Z_t)]\\ &\quad\times[\psi_n(Z_t,Z_2)+\psi_n(Z_2,Z_t)+\psi_n'(Z_t,Z_2)+\psi_n'(Z_2,Z_t)]|Z_1,Z_2\}\\ &= G_1+G_2+\cdots+G_{16}.\end{aligned}$$

$EG_n^2(Z_1,Z_2) = E[G_1+\cdots+G_{16}]^2 \le c\{EG_1^2+EG_2^2+\cdots+EG_{16}^2\}$. Let's consider

$$\begin{aligned}\frac{1}{h_n^3}EG_1^2 &= \frac{1}{h_n^3}E\{E[\psi_n(Z_t,Z_1)\psi_n(Z_t,Z_2)|Z_1,Z_2]\}^2\\ &= \frac{16}{h_n}E\{\sigma^2(x_1)\sigma^2(x_2)[\frac{1}{h_n}E(\frac{1}{f^2(x_t)}K(\frac{x_1-x_t}{h_n})K(\frac{x_2-x_t}{h_n})\sigma^2(x_t)|x_1,x_2)]^2\}\\ &\to 16\int\sigma^8(x_1)dx_1\int\kappa^2(\psi_1)d\psi_1.\end{aligned}$$

We follow similar arguments to show the other terms in $EG_n^2(Z_1,Z_2) = O(h_n^3)$. So we have the claim in (iii).

(4) Combining results in (1)-(3), we conclude

$$nh_n^{\frac{1}{2}}\{\frac{1}{n}\sum_{t=1}^n(y_t-\bar{y})^2 - \frac{1}{n}\sum_{t=1}^n(y_t-\hat{m}(x_t))^2 + (A_n+A_{1n})(1+o_{a.s.}(1))\} \xrightarrow{d} N(0,\sigma_\phi^2).$$

Since $\frac{1}{n}\sum_{t=1}^n(y_t-\bar{y})^2 \xrightarrow{p} E\sigma^2(x_t) > 0$, we have

$$nh_n^{\frac{1}{2}}\{\tilde{R}^2 + [\frac{1}{n}\sum_{t=1}^n(y_t-\bar{y})^2]^{-1}((A_n+A_{1n})(1+o_{a.s.}(1)))\} \xrightarrow{d} N(0,(E\sigma^2(x_t))^{-2}\sigma_\phi^2).$$

We follow the proof in Theorem 3 to show $I(\cdot) = I(\frac{1}{n}\sum_{t=1}^n(y_t-\bar{y})^2 \ge \frac{1}{n}\sum_{t=1}^n(y_t-\hat{m}(x_t))^2) \xrightarrow{p} 1$. So $nh_n^{\frac{1}{2}}\{\hat{R}^2 + I(\cdot)[\frac{1}{n}\sum_{t=1}^n(y_t-\bar{y})^2]^{-1}((A_n+A_{1n})(1+o_{a.s.}(1)))\} \xrightarrow{d} N(0,(E\sigma^2(x_t))^{-2}\sigma_\phi^2)$, and $T_n \xrightarrow{d} N(0,1)$ as desired.

For Theorem 1(b), it is sufficient to prove the following results:

(i) $\hat{A}_n - A_n = o_p((nh_n^{\frac{1}{2}})^{-1})$. (ii) $\hat{A}_{1n} - A_{1n} = o_p((nh_n^{\frac{1}{2}})^{-1})$. (iii) $\hat{V}_T - V_T = o_p(1)$.

(i) Note under H_0, $y_t = \mu + \epsilon_t$. We estimate ϵ_t by $\tilde{\epsilon}_t = y_t - \bar{y}$. So $\tilde{\epsilon}_t - \epsilon_t = (y_t-\bar{y}) - (y_t-\mu) = u - \bar{y} = O_p(n^{-\frac{1}{2}})$. Also note: $\sup_{x_t\in G}|\hat{f}(x_t) - f(x_t)| = \sup_{x_t\in G}|\hat{f}(x_t) - E\hat{f}(x_t) + E\hat{f}(x_t) - f(x_t)|$. Since by Lemma 2, $\sup_{x_t\in G}|\hat{f}(x_t) - E\hat{f}(x_t)| = O_{a.s.}((\frac{lnn}{nh_n})^{\frac{1}{2}})$. From Theorem 3, we have $E\hat{f}(x_t) - f(x_t) = O(h_n)$ uniformly over $x_t \in G$. Let $L_n = (\frac{lnn}{nh_n})^{\frac{1}{2}} + h_n$. Then $\sup_{x_t\in G}|\hat{f}(x_t) - f(x_t)| = O_p(L_n) = o_p(h_n^{\frac{1}{2}})$.

Furthermore, $\sup_{x_t\in G}|\hat{f}^{-1}(x_t) - f^{-1}(x_t)| \le \frac{\sup_{x_t\in G}|\hat{f}(x_t)-f(x_t)|}{\inf_{x_t\in G}\hat{f}(x_t)\inf_{x_t\in G}f(x_t)}$. With assumption A4(1), $\inf_{x_t\in G}\hat{f}(x_t) \ge$

$\inf_{x_t\in G}(\hat{f}(x_t)-f(x_t))+\inf_{x_t\in G}f(x_t)>0$, since $\inf_{x_t\in G}(\hat{f}(x_t)-f(x_t))\leq\sup_{x_t\in G}|\hat{f}(x_t)-f(x_t)|=o_p(1)$. So we also have $\sup_{x_t\in G}|\hat{f}^{-1}(x_t)-f^{-1}(x_t)|=O_p(L_n)$.

$$\begin{aligned}\hat{A}_n-A_n= & \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})[\tfrac{1}{\hat{f}^2(x_t)}-\tfrac{1}{f^2(x_t)}][\tilde{\epsilon}_i^2-\epsilon_i^2+\epsilon_i^2]+\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})\tfrac{1}{f^2(x_t)}[\tilde{\epsilon}_i^2-\epsilon_i^2]\\ = & I_1+I_2\end{aligned}$$

$$I_1\leq O_p(L_n)\{\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})\epsilon_i^2+\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i+\epsilon_i)(\tilde{\epsilon}_i-\epsilon_i)|\}.$$

With Lemma 1, we easily obtain $\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\frac{x_i-x_t}{h_n})\epsilon_i^2=O_p((nh_n)^{-1})$.

$$\begin{aligned}&\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i-\epsilon_i+2\epsilon_i)(\tilde{\epsilon}_i-\epsilon_i)|\\ \leq\ & \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})(\tilde{\epsilon}_i-\epsilon_i)^2+\frac{2}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i-\epsilon_i)\epsilon_i|\\ =\ & I_{11}+I_{12}=O_p((n^{\frac{3}{2}}h_n)^{-1}),\end{aligned}$$

since $I_{11}=O_p(n^{-1})O_p((nh_n)^{-1})=O_p((n^2h_n)^{-1})$ and $I_{12}=O_p(n^{-\frac{1}{2}})O_p((nh_n)^{-1})=O_p((n^{\frac{3}{2}}h_n)^{-1})$. Given the note above, we have $I_1\leq O_p(L_n)O_p((nh_n)^{-1})=o_p((nh_n^{\frac{1}{2}})^{-1})$.

$$\begin{aligned}I_2= & \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})\tfrac{1}{f^2(x_t)}(\tilde{\epsilon}_i-\epsilon_i)(\tilde{\epsilon}_i-\epsilon_i+2\epsilon_i)\\ = & \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})\tfrac{1}{f^2(x_t)}(\tilde{\epsilon}_i-\epsilon_i)^2+\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K^2(\tfrac{x_i-x_t}{h_n})\tfrac{1}{f^2(x_t)}(\tilde{\epsilon}_i-\epsilon_i)2\epsilon_i\\ = & I_{21}+I_{22}=O_p((n^{\frac{3}{2}}h_n)^{-1})=o_p((nh_n^{\frac{1}{2}})^{-1}),\end{aligned}$$

since $I_{21}=O_p(n^{-1})O_p((nh_n)^{-1})=O_p((n^2h_n)^{-1})$ and $I_{22}=O_p(n^{-\frac{1}{2}})O_p((nh_n)^{-1})=O_p((n^{\frac{3}{2}}h_n)^{-1})$. So we conclude $\hat{A}_n-A_n=o_p((nh_n^{\frac{1}{2}})^{-1})$.

$$\begin{aligned}\text{(ii) }\hat{A}_{1n}-A_{1n}= & -\frac{2}{n^2h_n}\sum_{t=1}^{n}[\tfrac{1}{\hat{f}(x_t)}-\tfrac{1}{f(x_t)}]K(0)(\tilde{\epsilon}_t^2-\epsilon_t^2+\epsilon_t^2)-\frac{2}{n^2h_n}\sum_{t=1}^{n}\tfrac{K(0)}{f(x_t)}(\tilde{\epsilon}_t^2-\epsilon_t^2)\\ = & o_p((nh_n^{\frac{1}{2}})^{-1})\text{ with similar arguments.}\end{aligned}$$

(iii) Given assumption A2 that $V(y)=E\sigma^2(x)>0$, we easily obtain $\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2\xrightarrow{p}E\sigma^2(x)$. Thus $[\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2]^{-2}\xrightarrow{p}(E\sigma^2(x))^{-2}$. So to prove $\hat{V}_T-V_T=o_p(1)$, we only need to show

(1) $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)}\xrightarrow{p}E\frac{\sigma^4(x_t)}{f(x_t)}=\int\sigma^4(x_t)dx_t$ and

(2) $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2}{h_n\hat{f}^2(x_t)}-\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)}=o_p(1)$.

$$\begin{aligned}\text{(1)}\quad & \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\tfrac{x_i-x_t}{h_n})\tfrac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)}=\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\psi_n(Z_t,Z_i)=\frac{1}{2n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}(\psi_n(Z_t,Z_i)+\psi_n(Z_i,Z_t))\\ = & \frac{1}{2n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\phi_n(Z_t,Z_i),\text{ where }Z_t=(x_t,\epsilon_t)\text{ and }\phi_n(Z_t,Z_i)\text{ is symmetric.}\\ = & \frac{1}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i)-\tfrac{1}{2}E\phi_n(Z_t,Z_i)+O_p(n^{-1}(E\phi_n^2(Z_t,Z_i))^{\frac{1}{2}})\text{ by Lemma 1.}\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}E\phi_n(Z_t,Z_i)= & E\psi_n(Z_t,Z_i)=\frac{1}{h_n}\int K(\tfrac{x_i-x_t}{h_n})\sigma^2(x_i)\sigma^2(x_t)\tfrac{f(x_i)f(x_t)}{f^2(x_t)}dx_idx_t\\ \rightarrow & \int\sigma^4(x_t)dx_t.\end{aligned}$$

$$h_n E\phi_n^2(Z_t,Z_i) \leq 4h_nE\psi_n^2(Z_t,Z_i) = \frac{4}{h_n}\int K^2(\frac{x_i-x_t}{h_n})E(\epsilon_i^4|x_i)E(\epsilon_t^4|x_t)\frac{f(x_t)f(x_i)}{f^4(x_t)}dx_idx_t$$
$$\rightarrow 4\int K^2(\psi)d\psi E(\frac{E^2(\epsilon_t^4|x_t)}{f^3(x_t)}) < \infty \text{ by assumption A9.}$$

Since $E(\frac{1}{n}\sum_{t=1}^{n}\int \phi_n(Z_t,Z_i)dP(Z_i)) = E(\phi_n(Z_t,Z_i)) = 2E\psi_n(Z_t,Z_i) \rightarrow \int 2\sigma^4(x_t)dx_t$,

$$V(\frac{1}{n}\sum_{t=1}^{n}\int \phi_n(Z_t,Z_i)dP(Z_i)) = \frac{1}{n}V(\int \psi_n(Z_t,Z_i)dP(Z_i)+\int\psi_n(Z_i,Z_t)dP(Z_i))$$
$$\leq \frac{C}{n}E\psi_n^2(Z_t,Z_i) = O(\frac{1}{nh_n}), \text{ so } V(\frac{1}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i)) \rightarrow 0.$$

Since $O_p(n^{-1}(E\phi_n^2(Z_t,Z_i))^{\frac{1}{2}}) = O_p(n^{-1}h_n^{-\frac{1}{2}})$, so $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)} \xrightarrow{p} \int\sigma^4(x_t)dx_t$.

$$(2)\quad \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\tilde\epsilon_i^2\tilde\epsilon_t^2}{h_n\hat f^2(x_t)} - \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)}$$
$$= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}[\frac{1}{\hat f^2(x_t)}-\frac{1}{f^2(x_t)}]K(\frac{x_i-x_t}{h_n})\tilde\epsilon_i^2\tilde\epsilon_t^2 + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde\epsilon_i^2\tilde\epsilon_t^2-\epsilon_i^2\epsilon_t^2)$$
$$= o_p(1).$$

Given assumption A8, we have $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})|\epsilon_i|^l|\epsilon_t|^m = O_p(1)$ with the application of Lemma 1 for $l,m=\{0,1,2\}$. Furthermore, $\sup_{x_t\in G}|\frac{1}{\hat f(x_t)}-\frac{1}{f(x_t)}| = O_p(L_n)$.

$$\tilde\epsilon_i^2\tilde\epsilon_t^2-\epsilon_i^2\epsilon_t^2$$
$$= (\tilde\epsilon_i-\epsilon_i+\epsilon_i)^2(\tilde\epsilon_t-\epsilon_t+\epsilon_t)^2-\epsilon_i^2\epsilon_t^2$$
$$= (\tilde\epsilon_i-\epsilon_i)^2(\tilde\epsilon_t-\epsilon_t)^2+(\tilde\epsilon_i-\epsilon_i)^2\epsilon_t^2+2\epsilon_t(\tilde\epsilon_i-\epsilon_i)^2(\tilde\epsilon_t-\epsilon_t)+\epsilon_i^2(\tilde\epsilon_t-\epsilon_t)^2+2\epsilon_t(\tilde\epsilon_t-\epsilon_t)\epsilon_i^2$$
$$+2\epsilon_i(\tilde\epsilon_i-\epsilon_i)(\tilde\epsilon_t-\epsilon_t)^2+2\epsilon_i\epsilon_t^2(\tilde\epsilon_i-\epsilon_i)+4\epsilon_i\epsilon_t(\tilde\epsilon_i-\epsilon_i)(\tilde\epsilon_t-\epsilon_t).$$

Since $\tilde\epsilon_i-\epsilon_i = O_p(n^{-\frac{1}{2}})$, we obtain the claim in (2).

Theorem 2: *Proof.*

1. The asymptotic local power of the test is examined with the sequence of Pitman local alternative: $H_1(l_n): m(x) = \mu + l_nD(x)$. Since

$$\frac{1}{n}\sum_{t=1}^{n}(y_t-\hat m(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)+\epsilon_t-\hat m(x_t))^2$$
$$= \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2+\frac{2}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat m(x_t))\epsilon_t+\frac{1}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat m(x_t))^2,$$

we establish the following results under $H_1(l_n)$ with $l_n = \frac{1}{\sqrt{nh_n^{\frac{1}{2}}}}$:

(1) $\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar y)^2 = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2+\frac{l_n^2}{n}\sum_{t=1}^{n}D(x_t)(D(x_t)-E(D(x_t)))+o_p((nh_n^{\frac{1}{2}})^{-1})$.

(2) $\frac{2}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat m(x_t))\epsilon_t = o_p((nh_n^{\frac{1}{2}})^{-1})+\frac{2}{n}\sum_{t=1}^{n}(\mu-\hat m(x_t))\epsilon_t$.

(3) $\frac{1}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat m(x_t))^2 = o_p((nh_n^{\frac{1}{2}})^{-1})+\frac{1}{n}\sum_{t=1}^{n}(\mu-\hat m(x_t))^2$.

Given the results in Theorem 1(a), and $\frac{1}{n}\sum_{t=1}^{n}D(x_t)(D(x_t)-E(D(x_t))) \xrightarrow{p} V(D(x_t))$, we obtain the claim in part 1 of Theorem 2.

(1) With result in (1) Theorem 3, we have $\bar y^2 = \frac{2}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)+\epsilon_t)(\mu+l_nED(x_t))-(\mu+l_nED(x_t))^2+O_p(n^{-1})$. So

$$\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar y)^2 = \frac{1}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)+\epsilon_t)^2-\bar y^2$$
$$= \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2+\frac{2l_n}{n}\sum_{t=1}^{n}\epsilon_t(D(x_t)-E(D(x_t))+\frac{l_n}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t))(D(x_t)-E(D(x_t))$$
$$+(\mu+l_nED(x_t))\frac{l_n}{n}\sum_{t=1}^{n}(ED(x_t)-D(x_t))+O_p(n^{-1}).$$

Since $E\frac{1}{n}\sum_{t=1}^{n}\epsilon_t(D(x_t)-E(D(x_t))=0$ and $nV(\frac{1}{n}\sum_{t=1}^{n}\epsilon_t(D(x_t)-E(D(x_t)))) = E(\sigma^2(x_t)(D(x_t)-E(D(x_t)))^2) <$

∞ by assumption A10, $\frac{2l_n}{n}\sum_{t=1}^{n}\epsilon_t(D(x_t)-E(D(x_t)))) = O_p((nh_n^{\frac{1}{2}})^{-\frac{1}{2}} * n^{-\frac{1}{2}}) = o_p((nh_n^{\frac{1}{2}})^{-1})$.

Similarly, $(\mu + l_nE(D(x_t)))\frac{l_n}{n}\sum_{t=1}^{n}(ED(x_t)-D(x_t)) = o_p((nh_n^{\frac{1}{2}})^{-1})$. So we have the claim in (1).

(2) Since $m(x_t) = \mu + l_nD(x_t)$, we follow Theorem 1(a) to have

$$\begin{aligned}\hat{m}(x_t)-m(x_t) = & \frac{1}{nh_n}\sum_{i=1}^{n}W_n(\tfrac{x_i-x_t}{h_n},x_t)(y_i-\mu-l_nD(x_t)) \\ = & e'S_n^{-1}(x_t)\left[\begin{array}{c}\frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})((D(x_i)-D(x_t))l_n+\epsilon_i) \\ \frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})(\frac{x_i-x_t}{h_n})((D(x_i)-D(x_t))l_n+\epsilon_i)\end{array}\right].\end{aligned}$$

$$\begin{aligned}& \frac{2}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat{m}(x_t))\epsilon_t \\ = & -\frac{2l_n}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}e'S_n^{-1}(x_t)K(\tfrac{x_i-x_t}{h_n})(D(x_i)-D(x_t))\epsilon_t\left[\begin{array}{c}1\\ \frac{x_i-x_t}{h_n}\end{array}\right] \\ & -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}e'S_n^{-1}(x_t)K(\tfrac{x_i-x_t}{h_n})\epsilon_i\epsilon_t\left[\begin{array}{c}1\\ \frac{x_i-x_t}{h_n}\end{array}\right] \\ = & I_2+\frac{2}{n}\sum_{t=1}^{n}(\mu-\hat{m}(x_t))\epsilon_t.\end{aligned}$$

To obtain the claim in (2), we only need to show $I_2 = o_p((nh_n^{\frac{1}{2}})^{-1})$.

Since $\sup_{x_t\in G}|S_{jn}^{-1}(x_t)-S_j^{-1}(x_t)| = o_{a.s.}(1)$, we follow Theorem 1 (a) (2)(i) to have

$$I_2 = -\frac{2l_n}{n^2h_n}\sum_{t=1}^{n}\sum_{i=1}^{n}\frac{1}{f(x_t)}K(\tfrac{x_i-x_t}{h_n})(D(x_i)-D(x_t))\epsilon_t(1+o_{a.s.}(1)) = I_{21}(1+o_{a.s.}(1)).$$

When $t=i$, $I_{21}=0$. So we have

$$\begin{aligned}\frac{I_{21}}{l_n} = & -\frac{2}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n}\frac{1}{h_nf(x_t)}K(\tfrac{x_i-x_t}{h_n})(D(x_i)-D(x_t))\epsilon_t = -\frac{2}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n}\psi_n(Z_t,Z_i) \\ = & -\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n}(\psi_n(Z_t,Z_i)+\psi_n(Z_i,Z_t)) = -\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n}\phi_n(Z_t,Z_i) \\ = & -\frac{2}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i)+O_p(n^{-1}(E\phi_n^2(Z_t,Z_i))^{\frac{1}{2}})\end{aligned}$$

by Lemma 1, since $E\phi_n(Z_t,Z_i)=0$.

$\int\phi_n(Z_t,Z_i)dP(Z_i) = \frac{\epsilon_t}{f(x_t)}\frac{1}{h_n}\int K(\frac{x_i-x_t}{h_n})(D(x_i)-D(x_t))f(x_i)dx_i$,

$E(\frac{1}{n}\sum_{t=1}^{n}\phi_n(Z_t,Z_i)dP(Z_i)) = 0$,

$V(\frac{1}{n}\sum_{t=1}^{n}\int\phi_n(Z_t,Z_i)dP(Z_i)) = \frac{1}{n}\int\frac{\sigma^2(x_t)}{f(x_t)}[\int K(\psi)(D(x_t+h_n\psi)-D(x_t))f(x_t+h_n\psi)d\psi]^2dx_t = o(n^{-1})$.

$h_nE\phi_n^2(Z_t,Z_i) \le 4h_nE\psi_n^2(Z_t,Z_i) = \int\frac{K^2(\psi)}{f^2(x_t)}(D(x_t+h_n\psi)-D(x_t))^2\sigma^2(x_t)f(x_t)f(x_t+h_n\psi)d\psi dx_t \to 0$.

So $I_{21} = o_p(n^{-\frac{1}{2}}\frac{1}{\sqrt{nh_n^{\frac{1}{2}}}}) = o_p((nh_n^{\frac{1}{2}})^{-1})$ and $I_2 = o_p((nh_n^{\frac{1}{2}})^{-1})$.

$$\begin{aligned}(3)\quad & \frac{1}{n}\sum_{t=1}^{n}(\mu+l_nD(x_t)-\hat{m}(x_t))^2 \\ = & \frac{1}{n}\sum_{t=1}^{n}\left[e'S_n^{-1}(x_t)\frac{1}{nh_n}\sum_{i=1}^{n}K(\tfrac{x_i-x_t}{h_n})(l_n(D(x_i)-D(x_t))+\epsilon_i)\left(\begin{array}{c}1\\ \frac{x_i-x_t}{h_n}\end{array}\right)\right]^2 \\ = & \frac{1}{n}\sum_{t=1}^{n}(\mu-\hat{m}(x_t))^2+\{\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{f^2(x_t)}K(\tfrac{x_i-x_t}{h_n})K(\tfrac{x_j-x_t}{h_n})[l_n^2(D(x_i)-D(x_t))(D(x_j)-D(x_t))] \\ & +\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{f^2(x_t)}K(\tfrac{x_i-x_t}{h_n})K(\tfrac{x_j-x_t}{h_n})l_n(D(x_i)-D(x_t))\epsilon_j \\ & +\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{f^2(x_t)}K(\tfrac{x_i-x_t}{h_n})K(\tfrac{x_j-x_t}{h_n})l_n(D(x_j)-D(x_t))\epsilon_i\}(1+o_{a.s.}(1)) \\ = & \frac{1}{n}\sum_{t=1}^{n}(\mu-\hat{m}(x_t))^2+I_{31}+I_{32}+I_{33}.\end{aligned}$$

Since I_{33} is similar to I_{32}, we show (i) $I_{31} = o_p((nh_n^{\frac{1}{2}})^{-1})$, and (ii) $I_{32} = o_p((nh_n^{\frac{1}{2}})^{-1})$, which are sufficient for the claim in (3).

(i) We note when $t = i = j$, $I_{31} = 0$. Similarly, when $t = i$ or $t = j$, $I_{31} = 0$. So we consider the case when $i = j$, but $t \neq i$. Since $\frac{1}{l_n^2} = \frac{1}{nh_n^{\frac{1}{2}}}$, we only need to show $\frac{I_{31}}{l_n^2} = o_p(1)$.

$$\frac{I_{31}}{l_n^2} = \frac{1}{nh_n}\frac{1}{n^2h_n}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{f^2(x_t)}K^2(\tfrac{x_i-x_t}{h_n})(D(x_i)-D(x_t))^2 = \frac{1}{nh_n}I_{311}.$$

$$I_{311} = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\underbrace{\frac{1}{h_nf^2(x_t)}K^2(\frac{x_i-x_t}{h_n})(D(x_i)-D(x_t))^2}_{\psi_n(x_i,x_t)}$$

$$= \frac{1}{2n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}(\psi_n(x_t,x_i)+\psi_n(x_i,x_t)) = \frac{1}{2n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\phi_n(x_t,x_i).$$

Since $\int\phi_n(x_t,x_i)dP(x_i) = \int\frac{1}{f^2(x_t)}K^2(\psi)(D(x_t+h_n\psi)-D(x_t))^2f(x_t+h_n\psi)d\psi+\int\frac{1}{f(x_t+h_n\psi)}K^2(\psi)(D(x_t)-D(x_t+h_n\psi))^2d\psi$, $E\int\phi_n(x_t,x_i)dP(x_i) = E\phi_n(x_t,x_i)\to 0$.

$h_nE\phi_n^2(x_t,x_i)\leq 4h_nE\psi_n^2(x_t,x_i) = 4\frac{1}{h_n}E\frac{1}{f^4(x_t)}K^4(\frac{x_i-x_t}{h_n})(D(x_i)-D(x_t))^4\to 0$, $O_p(n^{-1}(E\phi_n^2(x_t,x_i))^{\frac{1}{2}}) = o_p((nh_n^{\frac{1}{2}})^{-1})$, and $V(\phi_n(x_t,x_i) = o(h_n^{-1})$, so by Lemma 1, we have

$$I_{311} = \frac{1}{n}\sum_{t=1}^{n}\int\phi_n(x_t,x_i)dP(x_i) - \frac{1}{2}E\phi_n(x_t,x_i) + o_p((nh_n^{\frac{1}{2}})^{-1}) = o_p(1).$$

So we have the claimed result that $I_{31} = o_p((nh_n^{\frac{1}{2}})^{-1})$.

When t, i, j are three distinct indices,

$$\frac{I_{31}}{l_n^2} = \frac{1}{n^3}\sum_{\substack{t=1\\t\neq i\neq j}}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{\frac{1}{h_n^2f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})(D(x_i)-D(x_t))(D(x_j)-D(x_t))}_{\psi_n(x_t,x_i,x_j)}$$

$$= \frac{1}{3n^3}\sum_{\substack{t=1\\t\neq i\neq j}}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{[\psi_n(x_t,x_i,x_j)+\psi_n(x_i,x_t,x_j)+\psi_n(x_j,x_i,x_t)]}_{\phi_n(x_t,x_i,x_j)} = \frac{1}{n^3}\sum\sum\sum_{1=t<i<j=n}2\phi_n(x_t,x_i,x_j)$$

$$= \frac{1}{6}\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1}\sum\sum\sum_{1=t<i<j=n}2\phi_n(x_t,x_i,x_j)(1+O(n^{-4})) = \frac{1}{6}I_{311}(1+O(n^{-4})).$$

I_{311} is a U-statistic. We consider the H-decomposition in Lemma 3 and write

$$I_{311} = \frac{3}{n}\sum_{t=1}^{n}\int 2\phi_n(x_t,x_i,x_j)dP(x_i)dP(x_j) - 2\theta_n + O_p(H_n^{(2)}+H_n^{(3)}).$$

$\theta_n = 6E\psi_n(x_t,x_i,x_j) = 6\int\frac{1}{f(x_t)}K(\psi_1)K(\psi_2)(D(x_t+h_n\psi_1)-D(x_t))(D(x_t+h_n\psi_2)-D(x_t))f(x_t+h_n\psi_1)f(x_t+h_n\psi_2)dx_td\psi_1d\psi_2\to 0$.

$E[\int 2\phi_n(x_t,x_i,x_j)dP(x_i)dP(x_j)] = \theta_n\to 0$. Furthermore,

$$\begin{aligned} & h_n^2E(\textstyle\int 2\phi_n(x_t,x_i,x_j)dP(x_i)dP(x_j))^2 \leq 4h_n^2E(\phi_n^2(x_t,x_i,x_j)) \leq 4ch_n^2E(\psi_n^2(x_t,x_i,x_j)) \\ = & 4c\textstyle\int\frac{1}{f^4(x_t)}K^2(\psi_1)K^2(\psi_2)(D(x_t+h_n\psi_1)-D(x_t))^2(D(x_t+h_n\psi_2)-D(x_t))^2 \\ & \times f(x_t)f(x_t+h_n\psi_1)f(x_t+h_n\psi_2)dx_td\psi_1d\psi_2 \\ \to & 0. \end{aligned}$$

$V(\int 2\phi_n(x_t,x_i,x_j)dP(x_i)dP(x_j)) = o(h_n^{-2})$. Since $nh_n^2\to\infty$, we have

$$\frac{3}{n}\sum_{t=1}^{n}\int 2\phi_n(x_t,x_i,x_j)dP(x_i)dP(x_j) - 2\theta_n = o_p(1).$$

$V(H_n^{(2)}) = O(n^{-2}\sigma_{2n}^2)$, $V(H_n^{(3)}) = O(n^{-3}\sigma_{3n}^2)$. $\sigma_{2n}^2 = V(\int 2\phi_n(x_t,x_i,x_j)dP(x_j))\leq cE\phi_n^2(x_t,x_i,x_j)$, $\sigma_{3n}^2\leq cE\phi_n^2(x_t,x_i,x_j) = o(h_n^{-2})$, $V(H_n^{(2)}) = o(n^{-2}h_n^{-2})$, and $V(H_n^{(3)}) = o(n^{-3}h_n^{-2})$, so $O_p(H_n^{(2)}+H_n^{(3)}) = o_p((nh_n)^{-1})$. So we conclude $I_{311} = o_p(1)$ and $I_{31} = o_p((nh_n^{\frac{1}{2}})^{-1})$.

(ii) With similar arguments as (i), we apply Lemma 1 and 3 to obtain $I_{32} = o_p((nh_n^{\frac{1}{2}})^{-1})$.

With results in (i) and (ii) above, we obtain the claim in (3) and the claim in part 1 of Theorem 2.

2. We follow the proof in Theorem 1(b) closely. It is sufficient to prove the following results:

(i) $\hat{A}_n - A_n = o_p((nh_n^{\frac{1}{2}})^{-1})$. (ii) $\hat{A}_{1n} - A_{1n} = o_p((nh_n^{\frac{1}{2}})^{-1})$. (iii) $\hat{V}_T - V_T = o_p(1)$.

Note under $H_1(l_n): y_t = \mu + l_n D(x_t) + \epsilon_t$, so

$$\begin{aligned}\tilde{\epsilon}_t - \epsilon_t = & \ (y_t - \bar{y}) - (y_t - \mu - l_n D(x_t)) \\ = & \ \mu + l_n ED(x_t) - \bar{y} + l_n(D(x_t) - ED(x_t)) = O_p(n^{-\frac{1}{2}}) + l_n(D(x_t) - ED(x_t)).\end{aligned}$$

(i) As show in Theorem 1(b),

$$\begin{aligned}\hat{A}_n - A_n = & \ \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})[\tfrac{1}{\hat{f}^2(x_t)} - \tfrac{1}{f^2(x_t)}][\tilde{\epsilon}_i^2 - \epsilon_i^2 + \epsilon_i^2] + \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})\tfrac{1}{f^2(x_t)}[\tilde{\epsilon}_i^2 - \epsilon_i^2] \\ & = I_1 + I_2\end{aligned}$$

$$I_1 \leq \ O_p(L_n)\{\frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})\epsilon_i^2 + \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i + \epsilon_i)(\tilde{\epsilon}_i - \epsilon_i)|\}.$$

With Lemma 1, we easily obtain $\frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\frac{x_i-x_t}{h_n})\epsilon_i^2 = O_p((nh_n)^{-1})$.

$$\begin{aligned} & \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i - \epsilon_i + 2\epsilon_i)(\tilde{\epsilon}_i - \epsilon_i)| \\ \leq & \ \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})(\tilde{\epsilon}_i - \epsilon_i)^2 + \frac{2}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(\tilde{\epsilon}_i - \epsilon_i)\epsilon_i| \\ = & \ I_{11} + I_{12}. \end{aligned}$$

$$\begin{aligned} I_{11} = & \ O_p(n^{-1})\frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n}) + \frac{l_n^2}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})(D(x_i) - ED(x_i))^2 \\ & + O_p(n^{-\frac{1}{2}})\frac{2l_n}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})(D(x_i) - ED(x_i)) \\ = & \ I_{111} + I_{112} + I_{113} \end{aligned}$$

We can show that $I_{111} = O_p(n^{-2}h_n^{-1})$, $I_{112} = O_p(n^{-2}h_n^{-\frac{3}{2}})$, and $I_{113} = O_p(n^{-2}h_n^{-\frac{5}{4}})$. So $I_{11} = O_p(n^{-2}h_n^{-\frac{5}{4}}) = o_p((nh_n^{\frac{1}{2}})^{-1})$.

$$\begin{aligned} I_{12} = & \ O_p(n^{-\frac{1}{2}})\frac{2}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})|\epsilon_i| + \frac{2l_n}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K^2(\tfrac{x_i-x_t}{h_n})|(D(x_i) - ED(x_i))\epsilon_i| \\ = & \ I_{121} + I_{122}. \end{aligned}$$

We can show that $I_{121} = O_p(n^{-\frac{3}{2}}h_n^{-1})$ and $I_{122} = O_p(n^{-\frac{3}{2}}h_n^{-\frac{5}{4}})$. So $I_{12} = O_p(n^{-\frac{3}{2}}h_n^{-\frac{5}{4}})$.
So $I_1 = O_p(L_n)O_p((nh_n)^{-1}) = o_p((nh_n^{\frac{1}{2}})^{-1})$.

With similar arguments, we have $I_2 = o_p((nh_n^{\frac{1}{2}})^{-1})$. Thus, $\hat{A}_n - A_n = o_p((nh_n^{\frac{1}{2}})^{-1})$.

The proof of (ii) and (iii) follows from Theorem 1(b) in an analogous fashion.

Theorem 4: *Proof.* Recall the definition of $\hat{T}_n$ in section 2,

$$\hat{T}_n = \frac{nh_n^{\frac{1}{2}}\{\hat{R}^2 + I(\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2 \geq \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2)\frac{\hat{A}_{1n}+\hat{A}_n}{\frac{1}{n}\sum_{t=1}^{n}(y_t-\bar{y})^2}\}}{\sqrt{\hat{V}_T}}.$$

From Theorem 3, $\hat{R}^2 \xrightarrow{p} R^2 > 0$ under H_1. From Theorem 3 proof (4), we know $I(\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 \geq \frac{1}{n}\sum_{t=1}^{n}(y_t - \hat{m}(x_t))^2) \xrightarrow{p} 1$. It is also shown that $\frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2 \xrightarrow{p} E\sigma^2(x_t) > 0$.

Under H_1, $y_t = m(x_t) + \epsilon_t$, so

$$\begin{aligned}\tilde{\epsilon}_t - \epsilon_t = & \ (y_t - \bar{y}) - (y_t - m(x_t)) = m(x_t) - E(m(x_t)) + (E(m(x_t)) - \bar{y}) \\ = & \ m(x_t) - E(m(x_t)) + O_p(n^{-\frac{1}{2}}).\end{aligned}$$

Following the proof in Theorem 1(b), it is easy to see that $\hat{A}_n = \frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{\hat{f}^2(x_t)}K^2(\frac{x_i-x_t}{h_n})\tilde{\epsilon}_i^2 = O_p((nh_n)^{-1}))$, $\hat{A}_{1n} = -\frac{2}{n^2h_n}\sum_{t=1}^{n}\frac{K(0)}{\hat{f}(x_t)}\tilde{\epsilon}_t^2 = O_p((nh_n)^{-1}))$. So to prove Theorem 4, we only need to show $\hat{V}_T$'s probability limit is positive, since

$\hat{T}_n = \{nh_n^{\frac{1}{2}}[R^2 + O_p((nh_n)^{-1})]\}O_p(1) \to \infty$ at rate $nh_n^{\frac{1}{2}}$, as $R^2 > 0$ under H_1. So $P(\hat{T}_n > c_n) \to 1$ for any positive constant $c_n = o(nh_n^{\frac{1}{2}})$. Thus the $\hat{T}_n$ test is consistent.

It is sufficient to show

$$\begin{aligned}&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n\hat{f}^2(x_t)}K(\frac{x_i-x_t}{h_n})\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2\\ \xrightarrow{p}\ & E\left(\frac{\sigma^4(x_t)}{f(x_t)}\right) + E\left(\frac{(m(x_t)-E(m(x_t)))^4}{f(x_t)}\right) + 2E\left(\frac{(m(x_t)-E(m(x_t))^2\sigma^2(x_t)}{f(x_t)}\right),\ \text{which is positive.}\end{aligned}$$

Theorem 1(b) (iii)(1) shows $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^2\epsilon_t^2 \xrightarrow{p} E\left(\frac{\sigma^4(x_t)}{f(x_t)}\right)$. So we only need to show

$$\begin{aligned}&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n\hat{f}^2(x_t)}K(\frac{x_i-x_t}{h_n})\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2 - \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^2\epsilon_t^2\\ =\ & o_p(1) + E\left(\frac{(m(x_t)-E(m(x_t)))^4}{f(x_t)}\right) + 2E\left(\frac{(m(x_t)-E(m(x_t)))^2\sigma^2(x_t)}{f(x_t)}\right).\end{aligned}$$

$$\begin{aligned}&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2}{h_n\hat{f}^2(x_t)} - \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}K(\frac{x_i-x_t}{h_n})\frac{\epsilon_i^2\epsilon_t^2}{h_nf^2(x_t)}\\ =\ & \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}[\frac{1}{\hat{f}^2(x_t)} - \frac{1}{f^2(x_t)}]K(\frac{x_i-x_t}{h_n})\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2 + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2 - \epsilon_i^2\epsilon_t^2).\end{aligned}$$

Note $\sup_{x_t\in G}|\frac{1}{\hat{f}(x_t)} - \frac{1}{f(x_t)}| = O_p(L_n)$, which implies $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}[\frac{1}{\hat{f}^2(x_t)} - \frac{1}{f^2(x_t)}]K(\frac{x_i-x_t}{h_n})\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2 = o_p(1)$. Following Theorem 1(b)(iii)(2), we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde{\epsilon}_i^2\tilde{\epsilon}_t^2 - \epsilon_i^2\epsilon_t^2)\\ =\ & \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})[(\tilde{\epsilon}_i-\epsilon_i)^2(\tilde{\epsilon}_t-\epsilon_t)^2 + (\tilde{\epsilon}_i-\epsilon_i)^2\epsilon_t^2 + 2\epsilon_t(\tilde{\epsilon}_i-\epsilon_i)^2(\tilde{\epsilon}_t-\epsilon_t)\\ &+\epsilon_i^2(\tilde{\epsilon}_t-\epsilon_t)^2 + 2\epsilon_t(\tilde{\epsilon}_t-\epsilon_t)\epsilon_i^2 + 2\epsilon_i(\tilde{\epsilon}_i-\epsilon_i)(\tilde{\epsilon}_t-\epsilon_t)^2 + 2\epsilon_i\epsilon_t^2(\tilde{\epsilon}_i-\epsilon_i) + 4\epsilon_i\epsilon_t(\tilde{\epsilon}_i-\epsilon_i)(\tilde{\epsilon}_t-\epsilon_t)]\\ =\ & \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde{\epsilon}_i-\epsilon_i)^2(\tilde{\epsilon}_t-\epsilon_t)^2 + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_t^2(\tilde{\epsilon}_i-\epsilon_i)^2\\ &+\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^2(\tilde{\epsilon}_t-\epsilon_t)^2 + o_p(1),\end{aligned}$$

with Lemma 1 and $\tilde{\epsilon}_t - \epsilon_t = m(x_t) - E(m(x_t)) + O_p(n^{-\frac{1}{2}})$. Consider

$$\begin{aligned}&\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde{\epsilon}_i-\epsilon_i)^2(\tilde{\epsilon}_t-\epsilon_t)^2\\ =\ & O_p(n^{-\frac{1}{2}}) + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(m(x_i)-E(m(x_i)))^2(m(x_t)-E(m(x_t)))^2\\ =\ & O_p(n^{-\frac{1}{2}}) + \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\psi_n(x_t,x_i) = O_p(n^{-\frac{1}{2}}) + I.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}I =\ & \frac{1}{2n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}(\psi_n(x_t,x_i) + \psi_n(x_i,x_t)) = \frac{1}{2n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\phi_n(x_t,x_i)\\ =\ & \frac{1}{n}\sum_{t=1}^{n}\int\phi_n(x_t,x_i)dP(x_i) - \frac{1}{2}E\phi_n(x_t,x_i) + O_p(n^{-1}(E\phi_n^2(x_t,x_i))^{\frac{1}{2}})\end{aligned}$$

$E(\int\phi_n(x_t,x_i)dP(x_i)) = E(\phi_n(x_t,x_i)) \to 2E\left(\frac{(m(x_t)-E(m(x_t)))^4}{f(x_t)}\right)$,

$h_nE(\phi_n^2(x_t,x_i)) \leq 4h_nE(\psi_n^2(x_t,x_i)) \to 4\int K^2(\psi)d\psi E\left(\frac{(m(x_t)-E(m(x_t)))^8}{f^3(x_t)}\right) < \infty$, so

$O_p(n^{-1}(E\phi_n^2(x_t,x_i))^{\frac{1}{2}}) = O_p(n^{-1}h_n^{-\frac{1}{2}})$. Since $V(\frac{1}{n}\sum_{t=1}^{n}\int \phi_n(x_t,x_i)dP(x_i)) = O((nh_n)^{-1})$, we conclude $I \xrightarrow{p} E\left(\frac{(m(x_t)-E(m(x_t)))^4}{f(x_t)}\right)$. So

$$\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})(\tilde{\epsilon}_i-\epsilon_i)^2(\tilde{\epsilon}_t-\epsilon_t)^2 = o_p(1)+E\left(\frac{(m(x_t)-E(m(x_t)))^4}{f(x_t)}\right).$$

Similarly, we show $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_t^2(\tilde{\epsilon}_i-\epsilon_i)^2 = o_p(1)+E\left(\frac{(m(x_t)-E(m(x_t)))^2\sigma^2(x_t)}{f(x_t)}\right)$,

$$\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{h_n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^2(\tilde{\epsilon}_t-\epsilon_t)^2 = o_p(1)+E\left(\frac{(m(x_t)-E(m(x_t)))^2\sigma^2(x_t)}{f(x_t)}\right).$$

Combing above results, we obtain the claim in Theorem 4.

Theorem 5: *Proof.* Recall the definition of

$$\hat{T}_n^* = \frac{nh_n^{\frac{1}{2}}\{\hat{R}^{2*}-I(\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2\geq\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2)\frac{\hat{A}_{1n}^*+\hat{A}_n^*}{\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2}\}}{\sqrt{\hat{V}_T^*}}, \text{ for } \hat{R}^{2*} = \underbrace{\left[1-\frac{\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2}{\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2}\right]}_{\tilde{R}^{2*}} I(\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2\geq\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2).$$

To prove the Theorem, it is sufficient to show that given $W=\{x_i,y_i\}_{i=1}^n$,

(1) $\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2 = \frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*)^2+O_p(n^{-1})$.

Since $\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t^*-\hat{m}^*(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*)^2+\frac{2}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))\epsilon_t^*+\frac{1}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))^2$,

(2) we show $\frac{2}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))\epsilon_t^* = (A_{1n}^*+A_{2n}^*)(1+o_p(1))$, where $A_{1n}^* = -\frac{2}{n^2h_n}\sum_{t=1}^{n}\frac{K(0)}{f(x_t)}(\epsilon_t^*)^2$, and $A_{2n}^* = -\frac{2}{n^2h_n}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^*\epsilon_t^*$.

(3) $\frac{1}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))^2-A_n^*(1+o_p(1)) = A_{3n}^*(1+o_p(1))+o_p(n^{-1})$, $A_n^* = \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t\neq i}}^{n}\frac{1}{f^2(x_t)}K^2(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^2$,

$$\begin{aligned} A_{3n}^* = {} & \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\t\neq i\neq j}}^{n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\epsilon_i^*\epsilon_j^* = \frac{1}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\frac{\epsilon_i^*\epsilon_t^*}{h_n^2}E(K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})\frac{1}{f^2(x_j)}|x_t,x_i) \\ & +\frac{\epsilon_t^*\epsilon_i^*}{h_n^2}E(K(\frac{x_t-x_j}{h_n})K(\frac{x_i-x_j}{h_n})\frac{1}{f^2(x_j)}|x_i,x_t)]+o_p(n^{-1}). \end{aligned}$$

Results (2) and (3) above imply

$$\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2-(A_n^*+A_{1n}^*)(1+o_p(1)) = \frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*)^2+(A_{2n}^*+A_{3n}^*)(1+o_p(1))+o_p(n^{-1}).$$

(4) $(A_{2n}^*+A_{3n}^*)(S_n^*)^{-1}\xrightarrow{d}N(0,1)$, where

$(S_n^*)^2 = E((\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_n(Z_t^*,Z_i^*))^2|W) = n^{-2}h_n^{-1}\sigma_\phi^2+o_p(n^{-2}h_n^{-1})$, and

$$\begin{aligned} & \phi_n(Z_t^*,Z_i^*) \\ = {} & \frac{1}{n^2h_n}[-\frac{2}{f(x_t)}K(\frac{x_i-x_t}{h_n})\epsilon_i^*\epsilon_t^*-\frac{2}{f(x_i)}K(\frac{x_t-x_i}{h_n})\epsilon_t^*\epsilon_i^* \\ & +\epsilon_i^*\epsilon_t^*E(\frac{1}{h_nf^2(x_j)}K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})|x_t,x_i)+\epsilon_t^*\epsilon_i^*E(\frac{1}{h_nf^2(x_j)}K(\frac{x_t-x_j}{h_n})K(\frac{x_i-x_j}{h_n})|x_t,x_i)]. \end{aligned}$$

Combining results (1)-(4), we conclude

$$\frac{nh_n^{\frac{1}{2}}}{nh_n^{\frac{1}{2}}S_n^*}\{\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2-\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2+(A_n^*+A_{1n}^*)(1+o_p(1))\}\xrightarrow{d}N(0,1).$$

Since $\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2\xrightarrow{p}E\sigma^2(x_t)>0$, we have

$$\frac{nh_n^{\frac{1}{2}}}{nh_n^{\frac{1}{2}}\frac{S_n^*}{E(\sigma^2(x_t))}}\{\tilde{R}^{2*}+[\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2]^{-1}((A_n^*+A_{1n}^*)(1+o_p(1)))\}\xrightarrow{d}N(0,1).$$

We follow the proof in Theorem 1(a) to show $I(\cdot) = I(\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2\geq\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2)\xrightarrow{p}1$. So

$$\frac{nh_n^{\frac{1}{2}}}{nh_n^{\frac{1}{2}}\frac{S_n^*}{E(\sigma^2(x_t))}}\{\hat{R}^{2*}+I(\cdot)[\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2]^{-1}((A_n^*+A_{1n}^*)(1+o_p(1)))\}\xrightarrow{d}N(0,1).$$

(5) If we further have $A_n^*-\hat{A}_n^*=o_p((nh_n^{\frac{1}{2}})^{-1})$, $A_{1n}^*-\hat{A}_{1n}^*=o_p((nh_n^{\frac{1}{2}})^{-1})$, and $\hat{V}_T^*-[nh_n^{\frac{1}{2}}\frac{S_n^*}{E(\sigma^2(x_t))}]^2=o_p(1)$, the claim in Theorem 5 follows.

$$(1)\ \epsilon_{t,0}^* = y_t^*-\hat{\mu}^*=y_t^*-\frac{1}{n}\sum_{t=1}^{n}y_t^*=\bar{y}+\epsilon_t^*-\bar{y}-\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*=\epsilon_t^*-\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*,$$

So $\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2=\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*-\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*)^2=\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*)^2-(\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*)^2$. Since $E_{\hat{F}_t}(\epsilon_t^*)=0$, $E_{\hat{F}_t}(\epsilon_t^*)^2=\hat{\epsilon}_t^2$, so for given W, $\sqrt{n}\frac{\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*}{(\frac{1}{n}\sum_{t=1}^{n}\hat{\epsilon}_t^2)^{\frac{1}{2}}}\xrightarrow{d}N(0,1)$. In Theorem 3, we show that $\frac{1}{n}\sum_{t=1}^{n}\hat{\epsilon}_t^2=\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2+o_p(n^{-\frac{1}{2}})=O_p(1)$, so $\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*=O_p(n^{-\frac{1}{2}})$ and $(\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^*)^2=O_p(n^{-1})$. Thus, we conclude

$$\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,0}^*)^2=\frac{1}{n}\sum_{t=1}^{n}(\epsilon_t^*)^2+O_p(n^{-1}).$$

$$\begin{aligned}(2)\ \hat{m}^*(x)-\bar{y} &= \frac{1}{nh_n}\sum_{i=1}^{n}W_n(\tfrac{x_i-x}{h_n},x)(y_i^*-\bar{y})=\frac{1}{nh_n}\sum_{i=1}^{n}W_n(\tfrac{x_i-x}{h_n},x)\epsilon_i^*\\ &= e'S_n^{-1}(x)\begin{bmatrix}\frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x}{h_n})\epsilon_i^*\\ \frac{1}{nh_n}\sum_{i=1}^{n}K(\frac{x_i-x}{h_n})\frac{x_i-x}{h_n}\epsilon_i^*\end{bmatrix}\end{aligned}$$

We follow the proof in Theorem 1(a)(2)(i) to have $\frac{2}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))\epsilon_t^*=(A_{1n}^*+A_{2n}^*)(1+o_p(1))$.

(3) With similar arguments as in Theorem 1(a)(2)(ii), we obtain

$$\frac{1}{n}\sum_{t=1}^{n}(\bar{y}-\hat{m}^*(x_t))^2=\underbrace{\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\epsilon_i^*\epsilon_j^*}_{I}(1+o_p(1)).$$

(a) When $t=i=j$, $I=\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\frac{K^2(0)}{f^2(x_t)}(\epsilon_t^*)^2$. Since $E_{\hat{F}_t}(\epsilon_t^*)^2=\hat{\epsilon}_t^2$,

$$\begin{aligned}E(|I||W) &= \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\frac{K^2(0)}{f^2(x_t)}E_{\hat{F}_t}(\epsilon_t^*)^2=\frac{1}{n^3h_n^2}\sum_{t=1}^{n}\frac{K^2(0)}{f^2(x_t)}\hat{\epsilon}_t^2\\ &= \frac{1}{n^3h_n^2}\sum_{t=1}^{n}\frac{K^2(0)}{f^2(x_t)}[\epsilon_t^2+2(m(x_t)-\hat{m}(x_t))\epsilon_t+(m(x_t)-\hat{m}(x_t))^2]\\ &= O_p((n^2h_n^2)^{-1})=o_p(n^{-1}),\ \text{since}\ \sup_{x_t\in G}|m(x_t)-\hat{m}(x_t)|=o_p(1).\end{aligned}$$

(b) When $t=i$ (or $t=j$),

$$\begin{aligned}nh_n^{\frac{3}{2}}I &= \frac{2}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\frac{K(0)}{2\sqrt{h_n}f^2(x_t)}K(\tfrac{x_j-x_t}{h_n})\epsilon_t^*\epsilon_j^*=\frac{2}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\psi_n(Z_t,Z_j)\\ &= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}(\psi_n(Z_t,Z_j)+\psi_n(Z_j,Z_t))=\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}\phi_n(Z_t,Z_j)\end{aligned}$$

where $Z_t\equiv Z_t^*=(x_t,\epsilon_t^*)$. Since $E(\phi_n(Z_t,Z_j)|W,Z_t)=0$, we apply Lemma 1 to obtain $nh_n^{\frac{3}{2}}I=O_p((\frac{1}{n^4}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}E(\phi_n^2(Z_t,Z_j)|W))^{\frac{1}{2}})$.

$$\begin{aligned}&\frac{1}{n^4}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}E(\phi_n^2(Z_t,Z_j)|W)\\ =\ & \frac{1}{4n^4}\sum_{t=1}^{n}\sum_{\substack{j=1\\t\neq j}}^{n}[\tfrac{K^2(0)}{h_nf^4(x_t)}K^2(\tfrac{x_j-x_t}{h_n})\hat{\epsilon}_t^2\hat{\epsilon}_j^2+\tfrac{K^2(0)}{h_nf^4(x_j)}K^2(\tfrac{x_t-x_j}{h_n})\hat{\epsilon}_j^2\hat{\epsilon}_t^2+\tfrac{2K^2(0)}{h_nf^2(x_t)f^2(x_j)}K(\tfrac{x_j-x_t}{h_n})K(\tfrac{x_t-x_j}{h_n})\hat{\epsilon}_t^2\hat{\epsilon}_j^2]\\ =\ & I_1+I_2+I_3.\end{aligned}$$

Let's take a look at I_1.

$$\begin{aligned}I_1 = & \frac{1}{4n^4}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}\frac{K^2(0)}{h_n f^4(x_t)}K^2(\frac{x_j-x_t}{h_n})(m(x_t)-\hat{m}(x_t)+\epsilon_t)^2(m(x_j)-\hat{m}(x_j)+\epsilon_j)^2\\
= & \frac{1}{4n^4}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}\frac{K^2(0)}{h_n f^4(x_t)}K^2(\frac{x_j-x_t}{h_n})[(m(x_t)-\hat{m}(x_t))^2(m(x_j)-\hat{m}(x_j))^2+(m(x_t)-\hat{m}(x_t))^2\epsilon_j^2\\
& +2(m(x_t)-\hat{m}(x_t))^2(m(x_j)-\hat{m}(x_j))\epsilon_j+\epsilon_t^2(m(x_j)-\hat{m}(x_j))^2+\epsilon_t^2\epsilon_j^2+2\epsilon_t^2(m(x_j)-\hat{m}(x_j))\epsilon_j\\
& +2(m(x_t)-\hat{m}(x_t))\epsilon_t(m(x_j)-\hat{m}(x_j))^2+2(m(x_t)-\hat{m}(x_t))\epsilon_t\epsilon_j^2\\
& +4(m(x_t)-\hat{m}(x_t))(m(x_j)-\hat{m}(x_j))\epsilon_t\epsilon_j].\end{aligned}$$

Let's consider a generic term showing up repeatedly as

$$\begin{aligned}I_{110} = & \frac{1}{n^2}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}\frac{K^2(0)}{h_n f^4(x_t)}K^2(\frac{x_j-x_t}{h_n})|\epsilon_t|^k|\epsilon_j|^\tau = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}\psi_n(Z_t,Z_j)\\
= & \frac{1}{2n^2}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}(\psi_n(Z_t,Z_j)+\psi_n(Z_j,Z_t)) = \frac{1}{2n^2}\sum_{\substack{t=1\\t\neq j}}^{n}\sum_{j=1}^{n}\phi_n(Z_t,Z_j)\\
= & \frac{1}{n}\sum_{t=1}^{n}E(\phi_n(Z_t,Z_j)|Z_t)-\frac{1}{2}E\phi_n(Z_t,Z_j)+O_p(\frac{1}{n}(E\phi_n^2(Z_t,Z_j))^{\frac{1}{2}})\end{aligned}$$

where $Z_t=(x_t,\epsilon_t)$, $k,\tau=\{0,1,2\}$, and we apply Lemma 1 in the last line.
$E(\phi_n(Z_t,Z_j)|Z_t)=E(\psi_n(Z_t,Z_j)|Z_t)+E(\psi_n(Z_j,Z_t)|Z_t)$,

$E(\psi_n(Z_t,Z_j)|Z_t)=\int\frac{K^2(0)}{h_n f^4(x_t)}K^2(\frac{x_j-x_t}{h_n})|\epsilon_t|^k E(|\epsilon_j|^\tau|x_j)f(x_j)dx_j\to\frac{K^2(0)}{f^3(x_t)}|\epsilon_t|^k E(|\epsilon_t|^\tau|x_t)\int K^2(\psi)d\psi$,

$E(\psi_n(Z_j,Z_t)|Z_t)=\int\frac{K^2(0)}{h_n f^4(x_j)}K^2(\frac{x_t-x_j}{h_n})|\epsilon_t|^\tau E(|\epsilon_j|^k|x_j)f(x_j)dx_j\to\frac{K^2(0)}{f^3(x_t)}|\epsilon_t|^\tau E(|\epsilon_t|^k|x_t)\int K^2(\psi)d\psi$,

$E(\phi_n(Z_t,Z_j))\to 2K^2(0)\int K^2(\psi)d\psi E[E(|\epsilon_t|^k|x_t)E(|\epsilon_t|^\tau|x_t)\frac{1}{f^3(x_t)}]$, and

$h_nE\phi_n^2(Z_t,Z_j)\leq\ Ch_n^2E\psi_n^2(Z_t,Z_j)\to K^2(0)\int K^4(\psi)d\psi E[E(|\epsilon_t|^{2k}|x_t)E(|\epsilon_t|^{2\tau}|x_t)\frac{1}{f^7(x_t)}]<\infty$ with assumption A9 and A4(1).

So we conclude $E\phi_n^2(Z_t,Z_j)=O(\frac{1}{h_n})$, $\frac{1}{n}\sum_{t=1}^{n}E(\phi_n(Z_t,Z_j)|Z_t)=O_p(1)$ since $nh_n\to\infty$, $E(\phi_n(Z_t,Z_j))=O(1)$, $O_p(\frac{1}{n}(E\phi_n^2(Z_t,Z_j))^{\frac{1}{2}})=O_p((nh_n^{\frac{1}{2}})^{-1})$, and thus $I_{110}=O_p(1)$. Furthermore, since $\sup_{x_t\in G}|\hat{m}(x_t)-m(x_t)|=o_p(1)$, we have $I_1=O_p(n^{-2})$.

With similar arguments, we obtain $I_2=O_p(n^{-2})$ and $I_3=O_p(n^{-2})$. So in all, we conclude $nh_n^{\frac{3}{2}}I=O_p(n^{-1})$, and thus $I=O_p((n^2h_n^{\frac{3}{2}})^{-1})=o_p(n^{-1})$.

(c) When $i=j$ and $t\neq i$, $I=\frac{1}{n^3h_n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{f^2(x_t)}K^2(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^2=A_n^*$.

(d) When indices t,i, and j are distinct, we let $\psi_n(Z_t,Z_i,Z_j)=\frac{1}{h_n^2f^2(x_t)}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\epsilon_i^*\epsilon_j^*$, and we follow Theorem 1(a)(2)(ii)(d) to have

$$I=\frac{1}{3}\left[O(n^{-4})+\binom{n}{3}^{-1}\right]\sum\sum\sum_{1=t<i<j=n}\phi_n(Z_t,Z_i,Z_j),$$

where $\phi_n(Z_t,Z_i,Z_j)=\psi_n(Z_t,Z_i,Z_j)+\psi_n(Z_i,Z_t,Z_j)+\psi_n(Z_j,Z_i,Z_t)$ and $Z_t=(x_t,\epsilon_t^*)$. We focus on the third order U-statistic $U_n=\binom{n}{3}^{-1}\sum\sum\sum_{1=t<i<j=n}\phi_n(Z_t,Z_i,Z_j)$.

We note $\theta_n=E(\phi_n(Z_t,Z_i,Z_j)|W)=0$ and $\phi_{1n}(Z_t)=E(\phi_n(Z_t,Z_i,Z_j)|W,Z_t)=0$ because $E_{\hat{F}_t}\epsilon_t^*=0$. So we apply the H-decomposition in Lemma 3 to have

$$\begin{aligned}U_n = & \binom{n}{3}^{-1}\binom{n-2}{1}\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}\phi_{2n}(Z_t,Z_i)\\
& +\underbrace{\binom{n}{3}^{-1}\{\sum\sum\sum_{1=t<i<j=n}\phi_n(Z_t,Z_i,Z_j)-(n-2)\sum_{\substack{t=1\\t<i}}^{n}\sum_{i=1}^{n}\phi_{2n}(Z_t,Z_i)\}}_{U_{1n}},\end{aligned}$$

where $\phi_{2n}(Z_t,Z_i)=\frac{\epsilon_i^*\epsilon_t^*}{h_n^2}E(K(\frac{x_i-x_j}{h_n})K(\frac{x_t-x_j}{h_n})\frac{1}{f^2(x_j)}|x_t,x_i)$.

Since $(n-2)\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i) = \sum\sum\sum_{1=t<i<j=n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_t,Z_j)+\phi_{2n}(Z_i,Z_j)]$, we have

$$\begin{aligned}
U_{1n} = &\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1}\sum\sum\sum_{1=t<i<j=n}\{\phi_n(Z_t,Z_i,Z_j)-[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_t,Z_j)+\phi_{2n}(Z_i,Z_j)]\}\\
= &\begin{pmatrix} n \\ 3 \end{pmatrix}^{-1}\sum\sum\sum_{1=t<i<j=n}\Phi_n(Z_t,Z_i,Z_j).
\end{aligned}$$

By construction for given W, conditioning on Z_t, Z_i, Z_j, or $\{Z_t,Z_i\},\{Z_t,Z_j\},\{Z_i,Z_j\}$, the expectation of $\Phi_n(Z_t,Z_i,Z_j)$ is zero. So $E(U_{1n}|W)=0$. Furthermore, $E(\Phi_n(Z_t,Z_i,Z_j)\Phi_n(Z_{t'},Z_{i'},Z_{j'})|W)=0$ if $\{t,i,j\}\neq\{t',i',j'\}$. So

$$\begin{aligned}
V(U_{1n}|W) = &\begin{pmatrix} n \\ 3 \end{pmatrix}^{-2}\sum\sum\sum_{1=t<i<j=n}E(\Phi_n^2(Z_t,Z_i,Z_j)|W)\\
\leq\ & c\begin{pmatrix} n \\ 3 \end{pmatrix}^{-2}\sum\sum\sum_{1=t<i<j=n}E(\phi_n^2(Z_t,Z_i,Z_j)|W)\leq c\begin{pmatrix} n \\ 3 \end{pmatrix}^{-2}\sum\sum\sum_{1=t<i<j=n}E(\psi_n^2(Z_t,Z_i,Z_j)|W)\\
=\ & c\begin{pmatrix} n \\ 3 \end{pmatrix}^{-2}\sum\sum\sum_{1=t<i<j=n}\tfrac{1}{h_n^4f^4(x_t)}K^2(\tfrac{x_i-x_t}{h_n})K^2(\tfrac{x_j-x_t}{h_n})\hat{\epsilon}_i^2\hat{\epsilon}_j^2\\
=\ & c\begin{pmatrix} n \\ 3 \end{pmatrix}^{-2}\sum\sum\sum_{1=t<i<j=n}\tfrac{1}{h_n^4f^4(x_t)}K^2(\tfrac{x_i-x_t}{h_n})K^2(\tfrac{x_j-x_t}{h_n})[\epsilon_i^2\epsilon_j^2+\epsilon_i^2(m(x_j)-\hat{m}(x_j))^2\\
&+2\epsilon_i^2\epsilon_j(m(x_j)-\hat{m}(x_j))+(m(x_i)-\hat{m}(x_i))^2\epsilon_j^2+(m(x_i)-\hat{m}(x_i))^2(m(x_j)-\hat{m}(x_j))^2\\
&+2(m(x_i)-\hat{m}(x_i))^2\epsilon_j(m(x_j)-\hat{m}(x_j))+2(m(x_i)-\hat{m}(x_i))\epsilon_i\epsilon_j^2\\
&+2(m(x_i)-\hat{m}(x_i))\epsilon_i(m(x_j)-\hat{m}(x_j))^2+4(m(x_i)-\hat{m}(x_i))(m(x_j)-\hat{m}(x_j))\epsilon_i\epsilon_j]\\
=\ & I_1+I_2+\cdots+I_9
\end{aligned}$$

$I_1=O_p(n^{-3}h_n^{-2})=o_p(n^{-2})$ as we have show in Theorem 1(a)(2)(ii)(d) for term $H_n^{(3)}$. With assumption A9, and $\sup_{x_t\in G}|\hat{m}(x_t)-m(x_t)|=o_p(1)$, we have $I_i=o_p(n^{-3}h_n^{-2})=o_p(n^{-2})$ for $i=2,\cdots,9$ with similar arguments as in (b) above. So we conclude $U_{1n}=o_p(n^{-1})$. So

$$U_n = \begin{pmatrix} n \\ 3 \end{pmatrix}^{-1}\begin{pmatrix} n-2 \\ 1 \end{pmatrix}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i)+o_p(n^{-1})=\tfrac{6}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i)+o_p(n^{-1}).$$

$$\begin{aligned}
I =\ & \tfrac{2}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\phi_{2n}(Z_t,Z_i)+o_p(n^{-1})\\
=\ & \tfrac{1}{n(n-1)}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\tfrac{\epsilon_i^*\epsilon_t^*}{h_n^2}E(K(\tfrac{x_i-x_j}{h_n})K(\tfrac{x_t-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_t,x_i)+\tfrac{\epsilon_t^*\epsilon_i^*}{h_n^2}E(K(\tfrac{x_t-x_j}{h_n})K(\tfrac{x_i-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_i,x_t)]+o_p(n^{-1}).
\end{aligned}$$

Combining results in (a)-(d) above, we obtain the claim in (3).

(4) $$A_{3n}^* = [O(n^{-3})+\tfrac{1}{n^2}]\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)]=\underbrace{\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\phi_{2n}(Z_t,Z_i)+\phi_{2n}(Z_i,Z_t)]}_{A_{31n}^*}+o_p(n^{-1}),$$

$$\begin{aligned}
\text{where } A_{31n}^* =\ & \tfrac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}[\tfrac{\epsilon_i^*\epsilon_t^*}{h_n^2}E(K(\tfrac{x_i-x_j}{h_n})K(\tfrac{x_t-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_t,x_i)\\
&+\tfrac{\epsilon_t^*\epsilon_i^*}{h_n^2}E(K(\tfrac{x_t-x_j}{h_n})K(\tfrac{x_i-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_i,x_t)]+o_p(n^{-1}).
\end{aligned}$$

Since $E(A_{31n}^*|W)=0$,

$$\begin{aligned}
V(A_{31n}^*|W) =\ & \tfrac{1}{n^4}\sum_{t=1}^{n}\sum_{\substack{i=1\\t<i}}^{n}\{\tfrac{1}{h_n^4}[E(K(\tfrac{x_i-x_j}{h_n})K(\tfrac{x_t-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_t,x_i)]^2\hat{\epsilon}_i^2\hat{\epsilon}_t^2\\
&+\tfrac{1}{h_n^4}[E(K(\tfrac{x_t-x_j}{h_n})K(\tfrac{x_i-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_i,x_t)]^2\hat{\epsilon}_t^2\hat{\epsilon}_i^2\\
&+\tfrac{2}{h_n^4}[E(K(\tfrac{x_i-x_j}{h_n})K(\tfrac{x_t-x_j}{h_n})\tfrac{1}{f^2(x_j)}|x_t,x_i)]^2\hat{\epsilon}_i^2\hat{\epsilon}_t^2\}\\
=\ & O_p(n^{-2}h_n^{-1}),
\end{aligned}$$

we conclude $A_{31n}^*=O_p((nh_n^{\frac{1}{2}})^{-1})$. It is sufficient to show $(A_{2n}^*+A_{31n}^*)(S_n^*)^{-1}\xrightarrow{d}N(0,1)$.

$$
\begin{aligned}
& A^*_{2n} + A^*_{31n} \\
= & \sum_{\substack{t=1 \\ t<i}}^{n}\sum_{i=1}^{n} \frac{1}{n^2 h_n}[-\frac{2}{f(x_t)}K(\frac{x_i - x_t}{h_n})\epsilon^*_i\epsilon^*_t - \frac{2}{f(x_i)}K(\frac{x_t - x_i}{h_n})\epsilon^*_t\epsilon^*_i \\
& +\epsilon^*_i\epsilon^*_t E(\frac{1}{h_n}K(\frac{x_i - x_j}{h_n})K(\frac{x_t - x_j}{h_n})\frac{1}{f^2(x_j)}|x_t, x_i) + \epsilon^*_t\epsilon^*_i E(\frac{1}{h_n}K(\frac{x_t - x_j}{h_n})K(\frac{x_i - x_j}{h_n})\frac{1}{f^2(x_j)}|x_i, x_t)] \\
= & \sum_{\substack{t=1 \\ t<i}}^{n}\sum_{i=1}^{n}[\psi_n(Z^*_t, Z^*_i) + \psi_n(Z^*_i, Z^*_t) + \psi'_n(Z^*_t, Z^*_i) + \psi'_n(Z^*_i, Z^*_t)] \\
= & \sum_{\substack{t=1 \\ t<i}}^{n}\sum_{i=1}^{n}\phi_n(Z^*_t, Z^*_i), \text{ where } Z^*_t = (x_t, \epsilon^*_t).
\end{aligned}
$$

Since $\phi_n(Z^*_t, Z^*_i)$ is symmetric and $E(\phi_n(Z^*_t, Z^*_i)|W, Z^*_t) = 0$, $A^*_{2n} + A^*_{31n}$ is a degenerate second order U-statistic. We apply Proposition 3.2 of de Jong (1987) to have $(A^*_{2n} + A^*_{3n})(S^*_n)^{-1} \xrightarrow{d} N(0,1)$ if we have G_I, G_{II} and G_{IV} are of order $o_p((S^*_n)^4)$, where

$$
\begin{aligned}
(S^*_n)^2 = & \; E((\sum_{\substack{t=1 \\ t<i}}^{n}\sum_{i=1}^{n}\phi_n(Z^*_t, Z^*_i))^2|W), G_I = \sum_{\substack{t=1 \\ t<i}}^{n}\sum_{i=1}^{n}E(\phi^4_n(Z^*_t, Z^*_i)|W) \\
G_{II} = & \sum\sum\sum_{1=t<i<j=n}\{E(\phi^2_n(Z^*_t, Z^*_i)\phi^2_n(Z^*_t, Z^*_j)|W) + E(\phi^2_n(Z^*_i, Z^*_t)\phi^2_n(Z^*_i, Z^*_j)|W) \\
& +E(\phi^2_n(Z^*_j, Z^*_t)\phi^2_n(Z^*_j, Z^*_i)|W)\} = G_{II1} + G_{II2} + G_{II3}, \\
G_{IV} = & \sum\sum\sum\sum_{1=t<i<j<l=n}\{E(\phi_n(Z^*_t, Z^*_i)\phi_n(Z^*_t, Z^*_j)\phi_n(Z^*_l, Z^*_i)\phi_n(Z^*_l, Z^*_j)|W) \\
& +E(\phi_n(Z^*_t, Z^*_i)\phi_n(Z^*_t, Z^*_l)\phi_n(Z^*_j, Z^*_i)\phi_n(Z^*_j, Z^*_l)|W) \\
& +E(\phi_n(Z^*_t, Z^*_j)\phi_n(Z^*_t, Z^*_l)\phi_n(Z^*_i, Z^*_j)\phi_n(Z^*_i, Z^*_l)|W)\} = G_{IV1} + G_{IV2} + G_{IV3}.
\end{aligned}
$$

We can show that

$$
\begin{aligned}
(S^*_n)^2 = & \; n^{-2}h_n^{-1}2\int(\sigma^2(x_t))^2dx_t\int(2K(\psi) - \kappa(\psi))^2d\psi + o_p(n^{-2}h_n^{-1}) \\
= & \; n^{-2}h_n^{-1}\sigma^2_\phi + o_p(n^{-2}h_n^{-1}) = O_p(n^{-2}h_n^{-1}).
\end{aligned}
$$

and that G_I, G_{II} and G_{IV} are of order $o_p((S^*_n)^4)$, and details are omitted for brevity.

(5) (i) $A^*_n - \hat{A}^*_n = o_p((nh_n^{\frac{1}{2}})^{-1})$ conditioning on W.

As shown in (1), $\epsilon^*_{t,0} = \epsilon^*_t - \frac{1}{n}\sum_{t=1}^{n}\epsilon^*_t = \epsilon^*_t + O_p(n^{-\frac{1}{2}})$, and $E(|\epsilon^*_t|^0|W) = 1$, $E(|\epsilon^*_t|^1|W) = \frac{2}{\sqrt{5}}|\hat{\epsilon}_t|$, $E(|\epsilon^*_t|^2|W) = \hat{\epsilon}^2_t$, the claimed result follows with similar arguments as in Theorem 1(b).

(ii) $A^*_{1n} - \hat{A}^*_{1n} = o_p((nh_n^{\frac{1}{2}})^{-1})$ conditioning on W can be shown similarly.

(iii) We claim: $\hat{V}^*_T - \frac{(nh_n^{\frac{1}{2}})^2(S^*_n)^2}{(E(\sigma^2(x_t)))^2} = o_p(1)$ conditioning on W, where in (4) above, we have shown $(nh_n^{\frac{1}{2}})^2(S^*_n)^2 = \sigma^2_\phi + o_p(1)$ with $\sigma^2_\phi = 2\int\sigma^4(x_t)dx_t\int(2K(\psi) - \kappa(\psi))^2d\psi$, and by definition, $\hat{V}^*_T = \frac{\sigma^{2*}_\phi}{(\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_{t,0})^2)^2}$, with $\sigma^{2*}_\phi = [\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1 \\ t\neq i}}^{n}K(\frac{x_i - x_t}{h_n})\frac{(\epsilon^*_{i,0})^2(\epsilon^*_{t,0})^2}{h_n\hat{f}^2(x_t)}]2\int(2K(\psi) - \kappa(\psi))^2d\psi$.

Since $\frac{\hat{a}}{\hat{b}} - \frac{a}{b} = \frac{1}{\hat{b}}[\hat{a} - a - (\hat{b} - b)\frac{a}{b}]$, it is sufficient to prove the claim by showing

(a) $(\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_{t,0})^2)^2 \xrightarrow{p} (E(\sigma^2(x_t)))^2$ conditioning on W, where $(E(\sigma^2(x_t)))^2 > 0$, and

(b) $\frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1 \\ t\neq i}}^{n}K(\frac{x_i - x_t}{h_n})\frac{(\epsilon^*_{i,0})^2(\epsilon^*_{t,0})^2}{h_n\hat{f}^2(x_t)} \xrightarrow{p} \int\sigma^4(x_t)dx_t$.

(a) It is sufficient to show $\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_{t,0})^2 \xrightarrow{p} E(\sigma^2(x_t))$ conditioning on W.

Note $\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_{t,0})^2 = \frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_t)^2 + O_p(n^{-1})$. Furthermore, $E(\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_t)^2|W) = E(|\frac{1}{n}\sum_{t=1}^{n}(\epsilon^*_t)^2||W) = \frac{1}{n}\sum_{t=1}^{n}\hat{\epsilon}^2_t = \frac{1}{n}\sum_{t=1}^{n}\epsilon^2_t + o_p(1)$. Since $\frac{1}{n}\sum_{t=1}^{n}\epsilon^2_t \xrightarrow{p} E(\sigma^2(x_t))$, we have the claimed result in (a).

$$(b)\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\frac{(\epsilon_{i,0}^*)^2(\epsilon_{t,0}^*)^2}{h_n\hat{f}^2(x_t)} = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\frac{(\epsilon_i^*+O_p(n^{-\frac{1}{2}}))^2(\epsilon_t^*+O_p(n^{-\frac{1}{2}}))^2}{h_n}[\frac{1}{\hat{f}^2(x_t)}-\frac{1}{f^2(x_t)}]$$
$$+\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}K(\frac{x_i-x_t}{h_n})\frac{(\epsilon_i^*+O_p(n^{-\frac{1}{2}}))^2(\epsilon_t^*+O_p(n^{-\frac{1}{2}}))^2}{h_n}\frac{1}{f^2(x_t)}$$
$$= V_{11}+V_{12}.$$

$$V_{11} \leq \sup_{x_t\in G}|\frac{1}{\hat{f}^2(x_t)}-\frac{1}{f^2(x_t)}|\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{x_i-x_t}{h_n})[(\epsilon_i^*)^2+2\epsilon_i^*O_p(n^{-\frac{1}{2}})+O_p(n^{-1})]$$
$$\times[(\epsilon_t^*)^2+2\epsilon_t^*O_p(n^{-\frac{1}{2}})+O_p(n^{-1})]$$
$$= O_p(L_n)\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{x_i-x_t}{h_n})[(\epsilon_i^*)^2(\epsilon_t^*)^2+2(\epsilon_i^*)^2\epsilon_t^*O_p(n^{-\frac{1}{2}})+(\epsilon_i^*)^2O_p(n^{-1})$$
$$+2\epsilon_i^*(\epsilon_t^*)^2O_p(n^{-\frac{1}{2}})+4\epsilon_i^*\epsilon_t^*O_p(n^{-1})+2\epsilon_i^*O_p(n^{-\frac{3}{2}})+(\epsilon_t^*)^2O_p(n^{-1})+2\epsilon_t^*O_p(n^{-\frac{3}{2}})+O_p(n^{-2})].$$

Let's consider $V_{110}=\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^2(\epsilon_t^*)^2$.

$E(|V_{110}|W) = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{x_i-x_t}{h_n})(\hat{\epsilon}_i)^2(\hat{\epsilon}_t)^2 = O_p(1)$, following similar arguments as in term I_{110} in (3)(b) above. We can show similarly for $(k,\tau)=(2,1),(2,0),(1,2),(1,1),(1,0),(0,2),(0,1),(0,0)$ that $\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_n}K(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^k(\epsilon_t^*)^\tau=O_p(1)$, and thus, $V_{11}=O_p(L_n)=o_p(1)$.

$$V_{12} \leq \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})[(\epsilon_i^*)^2(\epsilon_t^*)^2+2(\epsilon_i^*)^2\epsilon_t^*O_p(n^{-\frac{1}{2}})+(\epsilon_i^*)^2O_p(n^{-1})$$
$$+2\epsilon_i^*(\epsilon_t^*)^2O_p(n^{-\frac{1}{2}})+4\epsilon_i^*\epsilon_t^*O_p(n^{-1})+2\epsilon_i^*O_p(n^{-\frac{3}{2}})+(\epsilon_t^*)^2O_p(n^{-1})+2\epsilon_t^*O_p(n^{-\frac{3}{2}})+O_p(n^{-2})].$$

Let's consider $V_{120}=\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^2(\epsilon_t^*)^2$.

$E(V_{120}|W) = E(|V_{120}||W) = \frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})(\hat{\epsilon}_i)^2(\hat{\epsilon}_t)^2 = \int\sigma^4(x_t)dx_t+o_p(1)$, as shown in Theorem 1(b)(iii). Following similar arguments, we can show similarly for (k,τ) considered above, $\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\frac{1}{h_nf^2(x_t)}K(\frac{x_i-x_t}{h_n})(\epsilon_i^*)^k(\epsilon_t^*)^\tau = O_p(1)$, and thus, $V_{12}=\int\sigma^4(x_t)dx_t+o_p(1)$. Results on V_{11} and V_{12} give the claimed result in (b).

Sketch of proof of Remark 1 in section 3: *Proof.*

Based on our Theorems 1-5, we now establish the asymptotic properties of the alternative tests in section 2. We cite the conditions 1-7 for $d=1$ in Doksum and Samarov (1995) which are used to establish the asymptotic distribution for the nonparametric R^2 estimators. To obtain the properties of the tests, we need the following additional assumptions.

R1. The weight function $w(x_i)$ in their condition 6 satisfies $\frac{1}{n}\sum_{i=1}^{n}w(x_i)=1$.

R2. $f(x)$, the marginal density of x in their condition 3 is a bounded function.

R3. The kernel function $K(\cdot)$ in their condition 5 satisfies the Lipschitz condition.

R4. For $\epsilon=y-m(x)$, $\sigma^2(x)=E(\epsilon^2|x)$ is a continuous function in $x\in S_x$, where S_x is the support of $f(x)$. $E(\epsilon^4|x)$ is a continuous function in $x\in S_x$ and $E(\epsilon^4|x)<\infty$.

R5 The derivative of $m(x)$ up to order k is a bounded function, and uniformly continuous on S_x.

For $\hat{\eta}_1^2=\frac{S_y^2-\frac{1}{n}\sum_{i=1}^{n}(y_i-\tilde{m}(x_i))^2w(x_i)}{S_y^2}$:

(1) $S_y^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 w(x_i) - \bar{y}_w^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 w(x_i) - 2\bar{y}_w \mu_{y,w} + (\mu_{y,w})^2 + O_p(n^{-1})$ by Theorem 3's proof (1).

$\frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{m}(x_i))^2 w(x_i) = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 w(x_i) - \frac{2}{n}\sum_{i=1}^{n}(\tilde{m}(x_i) - m(x_i))\epsilon_i w(x_i) + \frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i) - m(x_i))^2 w(x_i)$.

We observe that with the conditions, $\sup_{\bar{\Phi}} |\frac{1}{\tilde{f}(x_i)} - \frac{1}{f(x_i)}| = o_p(1)$ where Φ is defined in condition 1 as an open convex set in $\Re$ such that $\inf_\Phi f(x) \geq \delta$ for some $\delta > 0$ and $\bar{\Phi}$ denotes its bounded closure. Define $g(x) = m(x)f(x)$, then $\tilde{m}(x_i) - m(x_i) = \frac{\tilde{g}(x_i)}{\tilde{f}(x_i)} - \frac{g(x_i)}{f(x_i)} = \frac{\tilde{g}(x_i) - \tilde{f}(x_i)m(x_i)}{f(x_i)} + (\frac{1}{\tilde{f}(x_i)} - \frac{1}{f(x_i)})(\tilde{g}(x_i) - \tilde{f}(x_i)m(x_i))$ $= \frac{\tilde{g}(x_i) - \tilde{f}(x_i)m(x_i)}{f(x_i)}(1 + o_p(1))$ uniformly over $x_i \in \Phi$. Then

$$
\begin{aligned}
& -\frac{2}{n}\sum_{i=1}^{n}(\tilde{m}(x_i) - m(x_i))\epsilon_i w(x_i) = -2[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{f(x_i)}(\tilde{g}(x_i) - \tilde{f}(x_i)m(x_i))\epsilon_i w(x_i)](1 + o_p(1)) \\
= \; & -2[\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} K(\frac{x_j - x_i}{h_n})(m(x_j) - m(x_i))\frac{\epsilon_i w(x_i)}{h_n f(x_i)} + \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} K(\frac{x_j - x_i}{h_n})\frac{\epsilon_j \epsilon_i w(x_i)}{h_n f(x_i)}](1 + o_p(1)) \\
= \; & -2[T_{na} + T_{nb}]. \\
& \frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i) - m(x_i))^2 w(x_i) = [\frac{1}{n}\sum_{i=1}^{n}\frac{1}{f^2(x_i)}(\tilde{g}(x_i) - \tilde{f}(x_i)m(x_i))^2 w(x_i)](1 + o_p(1)) \\
= \; & \{\frac{1}{n(n-1)^2}\sum_{i=1}^{n}\sum_{j \neq i}\sum_{l \neq i}\frac{w(x_i)}{h_n^2 f^2(x_i)}K(\frac{x_j - x_i}{h_n})K(\frac{x_l - x_i}{h_n})[(m(x_j) - m(x_i))(m(x_l) - m(x_i)) \\
& +(m(x_j) - m(x_i))\epsilon_l + (m(x_l) - m(x_i))\epsilon_j + \epsilon_l \epsilon_j]\}(1 + o_p(1)) \\
= \; & \{T_{nc} + T_{nd} + T_{ne} + T_{nf}\}(1 + o_p(1)). \\
T_{nf} = \; & [\frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i<j}}^{n}\epsilon_i \epsilon_j E(\frac{w(x_l)}{h_n^2 f^2(x_l)}K(\frac{x_i - x_l}{h_n})K(\frac{x_j - x_l}{h_n})|x_i, x_j) \\
& +\frac{1}{n(n-1)^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} K^2(\frac{x_j - x_i}{h_n})\epsilon_j^2 \frac{w(x_i)}{h_n^2 f^2(x_i)}](1 + o_p(1)) = [T_{nf1} + T_{n0}](1 + o_p(1)).
\end{aligned}
$$

Under H_0, $T_{ni} = 0$ for $i = a, c, d, e$. So $\frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{m}(x_i))^2 w(x_i) = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 w(x_i) + [-2T_{nb} + T_{nf1} + T_{n0}](1 + o_p(1))$. H_0 implies $y_i = \mu + \epsilon_i$, so $S_y^2 = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 w(x_i) + O_p(n^{-1}) = \sigma_{y,w}^2 + o_p(1)$. The structure of $\hat{T}_{1n}$ is similar to $\hat{T}_n$ and we follow Theorem 1(a) to show $nh_n^{1/2}[2T_{nb} - T_{nf1}] \xrightarrow{d} N(0, \sigma_{\phi 1}^2)$ for $\sigma_{\phi 1}^2 = E[\frac{w^2(x)}{f(x)}\sigma^4(x)]2\int(2K(\psi) - \kappa(\psi))^2 d\psi$. It implies $nh_n^{1/2}(\hat{\eta}_1^2 + (S_y^2)^{-1}T_{n0}) \xrightarrow{d} N(0, (\sigma_{y,w}^4)^{-1}\sigma_{\phi 1}^2)$. Following Theorem 1(b), we show further that $\hat{T}_{n0} - T_{n0} = o_p((nh_n^{1/2})^{-1})$ and $\hat{\sigma}_{\phi 1}^2 \xrightarrow{p} \sigma_{\phi 1}^2$, which gives the claimed result in (1).

(2) Under H_1, $T_{ni} \neq 0$ for $i = a, c, d, e$. We use the additional assumption R5 and follow Theorem 3 to show that $\hat{\eta}_1^2 = (\sigma_{y,w}^2)^{-1}(\sigma_{y,w}^2 - E(\sigma^2(x)w(x))) = \eta_w^2 > 0$, and $\hat{T}_{n0} = O_p((nh_n)^{-1})$, so $\hat{T}_{1n} = nh_n^{1/2}\eta_w^2 + o_p((nh_n^{1/2}))$ and $P(\hat{T}_{1n} > c_n) \to 1$ for $c_n = o((nh_n^{1/2}))$.

(3) Since $y_i^* = \bar{y}_w + \epsilon_i^*$, $S_y^{*2} = \frac{1}{n}\sum_{i=1}^{n}(\epsilon_{i,0}^*)^2 w(x_i) = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^{*2} w(x_i) + O_p(n^{-1})$.

We note that $\tilde{m}^*(x_t) - \bar{y}_w = \frac{1}{(n-1)h_n f(x_t)}\sum_{i \neq t} K(\frac{x_i - x_t}{h_n})\epsilon_i^*(1 + o_p(1))$, then

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}(\epsilon_{t,b}^*)^2 w(x_t) = \; & \frac{1}{n}\sum_{t=1}^{n}(\bar{y}_w + \epsilon_t^* - \tilde{m}^*(x_t))^2 w(x_t) \\
= \; & \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^{*2} + \frac{2}{n}\sum_{t=1}^{n}(\bar{y}_w - \tilde{m}^*(x_t))\epsilon_t^* w(x_t) + \frac{1}{n}\sum_{t=1}^{n}(\bar{y}_w - \tilde{m}^*(x_t))^2 \\
= \; & \frac{1}{n}\sum_{t=1}^{n}\epsilon_t^{*2} - \frac{2}{n(n-1)}\sum_{t=1}^{n}\sum_{i \neq t} K(\frac{x_i - x_t}{h_n})\frac{w(x_t)}{h_n f(x_t)}\epsilon_i^* \epsilon_t^*(1 + o_p(1)) \\
& +\frac{1}{n(n-1)^2}\sum_{t=1}^{n}\sum_{i \neq t}\sum_{j \neq t} K(\frac{x_i - x_t}{h_n})K(\frac{x_j - x_t}{h_n})\frac{w(x_t)}{h_n^2 f^2(x_t)}\epsilon_i^* \epsilon_j^*(1 + o_p(1)).
\end{aligned}
$$

The structure is similar to that in $\hat{T}_n^*$ and we follow Theorem 5 to obtain the claimed result.

For $\hat{\eta}_2^2 = \frac{\frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i) - \bar{m})^2 w(x_i)}{S_y^2}$:

$$\frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-\bar{m})^2w(x_i) = \frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-m(x_i))^2w(x_i)+\frac{1}{n}\sum_{i=1}^{n}(m(x_i)-\bar{m})^2w(x_i) + \frac{2}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-m(x_i))(m(x_i)-\bar{m})^2w(x_i).$$

Given the similar structure with $\hat{\eta}_1^2$, we only observe that

$$\frac{1}{n}\sum_{i=1}^{n}(m(x_i)-\bar{m})^2w(x_i) = \begin{cases} O_p(n^{-1})+o_p((nh_n^{1/2})^{-1}) \text{ under } H_0. \\ \int m^2(x_i)f(x_i)w(x_i)dx_i - \mu_{y,w}^2 + o_p(1) \text{ under } H_1. \end{cases}$$

$$\frac{2}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-m(x_i))(m(x_i)-\bar{m})^2w(x_i) = \begin{cases} O_p(n^{-1})+o_p((nh_n^{1/2})^{-1}) \text{ under } H_0. \\ O_p(h_n^{k+1})+O_p((nh_n^{1/2})^{-1}h_n)+O_p(n^{-1/2}) \\ +2\mu_{y,w}[O_p(h_n^{k+1})+O_p((nh_n^{1/2})^{-1})+O_p(n^{1/2})] \text{ under } H_1. \end{cases}$$

We follow similar arguments in Theorems 1-4 to obtain the desired results in (1) and (2).

For the claim in (3) for the bootstrap test statistic $\hat{T}_{2n}^*$, note
$\tilde{m}^*(x_t)-\bar{y}_w = \frac{1}{(n-1)h_nf(x_t)}\sum_{i\neq t}K(\frac{x_i-x_t}{h_n})\epsilon_i^*(1+o_p(1))$ and
$\bar{m}^*-\bar{y}_w = \frac{1}{n(n-1)}\sum_{t=1}^{n}\sum_{i\neq t}K(\frac{x_i-x_t}{h_n})\frac{w(x_t)\epsilon_i^*}{h_nf(x_t)}(1+o_p(1))$. We can show that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-\bar{m}^*)^2w(x_i) \\ =\ & \frac{1}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-\bar{y}_w)^2w(x_i)+\frac{1}{n}\sum_{i=1}^{n}(\bar{y}_w-\bar{m}^*)^2w(x_i)+\frac{2}{n}\sum_{i=1}^{n}(\tilde{m}(x_i)-\bar{y}_w)(\bar{y}_w-\bar{m}^*)w(x_i) \\ =\ & [\frac{1}{n(n-1)^2}\sum_{t=1}^{n}\sum_{i\neq t}K^2(\frac{x_i-x_t}{h_n})\frac{w(x_t)\epsilon_i^*2}{h_n^2f^2(x_t)}+\frac{1}{n(n-1)^2}\sum_{t=1}^{n}\sum_{i\neq t}\sum_{j\neq t}K(\frac{x_i-x_t}{h_n})K(\frac{x_j-x_t}{h_n})\frac{w(x_t)\epsilon_i^*\epsilon_j^*}{h_n^2f^2(x_t)}](1+o_p(1)) \\ & +o_p((nh_n^{1/2})^{-1}). \end{aligned}$$

We follow Theorem 5 to obtain the claimed result in (3) for $\hat{T}_{2n}^*$.

Sketch of proof of Remark 2 in section 3: *Proof.*

Since the bootstrap test asymptotically has the null limiting distribution of $\hat{T}_{nG}$, we briefly look at the correction terms in $\hat{T}_{nG}$. Under homoskedasticity and bounded support for X denoted by G,

$$A_{nG} = \frac{1}{n^3h_n^{2d}}\sum_{\substack{t=1 \\ t\neq i}}^{n}\sum_{i=1}^{n}K_{it}^2\frac{\epsilon_i^2}{f^2(X_t)} = \frac{1}{nh_n^d}\sigma^2\int K^2(\psi)d\psi(1+o_p(1)),$$

$$A_{1nG} = -\frac{2}{n^2h_n^d}\sum_{t=1}^{n}K(0)\frac{\epsilon_t^2}{f(X_t)} = -\frac{2}{nh_n^d}\sigma^2K(0)\int_G dX(1+o_p(1)),$$

$\frac{1}{n}\sum_{t=1}^{n}(y_t-\hat{r}(x_{1t}))^2 \xrightarrow{p} \sigma^2$, and $\hat{V}_{TG} \xrightarrow{p} \int_G dX2\int(2K(\psi)-\kappa(\psi))^2d\psi$, so we could focus on $\hat{R}_{nG}^2$ to construct the test, as the other correction terms are independent of the DGP characteristics under H_0.

The justification of the bootstrap procedure is based on the facts that when we use resample with replacement from the centered $\{\hat{\epsilon}_i\}_{i=1}^n$ to construct $\{\epsilon_i^*\}_{i=1}^n$, we have

$E(\epsilon_i^*|W) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\epsilon}_i-\bar{\hat{\epsilon}}) = 0$ for $\bar{\hat{\epsilon}} = \frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i$ and $E(\epsilon_i^{*2}|W) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\epsilon}_i-\bar{\hat{\epsilon}})^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2-\bar{\hat{\epsilon}}^2$.

Since $\bar{\hat{\epsilon}} = \frac{1}{n}\sum_{i=1}^{n}(y_i-\hat{m}(x_i)) = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i - \frac{1}{n}\sum_{i=1}^{n}(\hat{m}(x_i)-m(x_i)) = O_p(n^{1/2})+O_p(h_n^v)$, and we can show that $\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\hat{m}(x_i))^2 = \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2+o_p(n^{-1/2})$, we obtain $E(\epsilon_i^{*2}|W) = \sigma^2+O_p(n^{-1/2})$. These observations enable us to follow the proof in Theorem 6 to obtain the claimed result.

References

de Jong, P., 1987. A central limit theorem for generalized quadratic forms. Probability Theory and Related Filelds 75, 261–277.

Doksum, K., Samarov, A., 1995. Nonparametric estimation of global functionals and a meaure of the explanatory power of covariates in regression. Annals of Statistics 23, 1443–1473.

Hall, P., 1984. Central Limit Theorem for Integrated Square Error of Multivariate Nonparametric Density Estimators. Journal of Multivariate Analysis 14, 1–16.

Lee, A. J., 1990. U-statistics Theory and Pracice. Marcel Dekker, New York.

Martins-Filho, C., Yao, F., 2007. Nonparametric frontier estimation via local linear regression. Journal of Econometrics 141, 283–319.

Newey, W., McFadden, D., 1994. Large sample estimation and hypothesis testing. In: Engle, R. F., McFadden, D. L. (Eds.), Handbook of Econometrics. Vol. 4. Elsevier Science B.V., Amsterdam.

Yao, F., Zhang, J. S., 2010. Efficient semiparametric instrumental variable estimation. Working paper, Economics Department, West Virginia University.
URL `http://community.wvu.edu/ fy006/hp/yao-zhang2011.pdf`