Appendix to: Efficient semiparametric instrumental variable estimation under conditional heteroskedasticity

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August, 2011

For reviewing convenience only, not intended for publication.

Keywords and Phrases. Instrumental variables, semiparametric regression, efficient estimation.

JEL Classifications. C14, C21

Appendix 2

We first state Lemmas, which is used repeated in the proof below.

Lemma 1 *Define*

$$S_{n,j}(\mathbf{z_0}) = \frac{1}{n}\sum_{i=1}^{n} K_h(\mathbf{Z_i^c} - \mathbf{z_0^c})\left(\frac{\mathbf{Z_i^c} - \mathbf{z_0^c}}{h}\right)^j I(\mathbf{Z_i^d} = \mathbf{z_0^d}) g(U_i) w(\mathbf{Z_i^c} - \mathbf{z_0^c}; \mathbf{z_0}), |j| = 0, 1, \cdots, J,$$

where $\mathbf{Z_i}, U_i$ *are iid,* $\mathbf{Z_i^c} \in R^{l_c}, \mathbf{Z_i^d} \in R^{l_d}, K_h(\cdot) = \frac{1}{h^{l_c}}K(\frac{\cdot}{h})$*, and* $K(\cdot)$ *is a kernel function defined on* R^{l_c}*. Here similar to notations in A2(1),* $j = (j_1, j_2, \cdots, j_{l_c})$*,* $\mathbf{Z_i^c} = (\mathbf{Z_{i,1}^c}, \mathbf{Z_{i,2}^c}, \cdots, \mathbf{Z_{i,l_c}^c})$*,* $\mathbf{z_0^c} = (\mathbf{z_{0,1}^c}, \mathbf{z_{0,2}^c}, \cdots, \mathbf{z_{0,l_c}^c})$*, and* $\left(\frac{\mathbf{Z_i^c} - \mathbf{z_0^c}}{h}\right)^j = \left(\frac{\mathbf{Z_{i,1}^c} - \mathbf{z_{0,1}^c}}{h}\right)^{j_1} \times \cdots \times \left(\frac{\mathbf{Z_{i,l_c}^c} - \mathbf{z_{0,l_c}^c}}{h}\right)^{j_{l_c}}$*. Let* C *denote an arbitrary real number and* $C < \infty$*. Assume that*

L_1*.* $K(\cdot)$ *is bounded with compact support and for Euclidean norm* $||.||$*,*

$$|u^j K(u) - v^j K(v)| \leq c_K ||u - v||, \text{ for } 0 \leq |j| \leq J.$$

L_2*.* $g(U_i)$ *is a measurable function of* U_i *and* $E|g(U_i)|^s < C$ *for* $s > 2$*.*

L_3*. Let* $G = G^c \times G^d$*, a compact subset of* $\Re^{l_c + l_d}$*, where* G^c *and* G^d *are compact subsets of* R^{l_c} *and* R^{l_d} *respectively. Define the joint density of* $\mathbf{Z_i}$ *and* U_i *at* $(\mathbf{z_0}, u)$ *as* $f(\mathbf{z_0}, u)$*, conditional density of* $\mathbf{Z_i}$ *and* U_i *given* U_i *at* $\mathbf{Z_i} = \mathbf{z_0}$ *and* $U_i = u$ *as* $f_{z|u}(\mathbf{z_0})$*. Assume that* $\sup_{\mathbf{z_0} \in G} \int |g(u)|^s f_{z,u}(\mathbf{z_0}, u) du < \infty$*,* $f_{z|u}(\mathbf{z_0}) < C$ *for all* $\mathbf{z_0}$ *, and* $f_{z,u}(\mathbf{z_0}, u)$ *is continuous at* $\mathbf{z_0^c}$ *for all* $\mathbf{z_0^c}$*.*

L_4*.* $w(\mathbf{Z_i^c} - \mathbf{z_0^c}; \mathbf{z_0})$ *is a function of* $\mathbf{Z_i^c} - \mathbf{z_0^c}$ *and* $\mathbf{z_0}$*.* $|w(\mathbf{Z_i^c} - \mathbf{z_0^c}; \mathbf{z_0})| \leq C$*,* $|w(\mathbf{Z_i^c} - \mathbf{z_0^c}; \mathbf{z_0^c}, \mathbf{z_0^d}) - w(\mathbf{Z_i^c} - \mathbf{z_k^c}; \mathbf{z_k^c}, \mathbf{z_0^d})| \leq C||\mathbf{z_0^c} - \mathbf{z_k^c}||$ *almost everywhere.*

L_5*.* $nh^{l_c} \to \infty$*.*

Then for $\mathbf{z_0} = (\mathbf{z_0^c}, \mathbf{z_0^d}) \in G$*,*

$$\sup_{\mathbf{z_0} \in G} |S_{n,j}(\mathbf{z_0}) - E(S_{n,j}(\mathbf{z_0}))| = O_p\left(\left(\frac{nh^{l_c}}{ln(n)}\right)^{-\frac{1}{2}}\right).$$

See Yao and Zhang (2010) Lemma 1.

Lemma 2 *Define* $\hat{g}_k(\mathbf{Z_t}) \equiv \hat{E}(X_{t,k}|\mathbf{Z_t})$*, and* $\hat{g}_{1,k}(Z_{1t}) \equiv \hat{E}(X_{t,k}|Z_{1t})$ *for* $k = 1, 2, \cdots, K$*. Let* $L_{1n} = (\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}} + h_1^{s+1}$*,* $L_{2n} = (\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}} + h_2^{s_1+1}$*, and* $L_n = L_{1n} + L_{2n}$*. With assumptions A2 and A3, we have*

(1)$\sup_{z_{10} \in G_1} |\hat{f}_1(z_{10}) - f_1(z_{10})| = O_p(L_{1n})$*.* $\inf_{z_{10} \in G_1} \hat{f}_1(z_{10}) > C > 0$*.*

(2)$\sup_{\mathbf{z_0} \in G} |\hat{f}(\mathbf{z_0}) - f_z(\mathbf{z_0})| = O_p(L_{2n})$*.* $\inf_{\mathbf{z_0} \in G} \hat{f}(\mathbf{z_0}) > C > 0$*.*

(3) $\sup_{z_{10} \in G_1} |\hat{g}_{1,j}(z_{10}) - g_{1,j}(z_{10})| = O_p(L_{1n})$*.*

(4) $\sup_{\mathbf{z_0} \in G} |\hat{g}_j(\mathbf{z_0}) - g_j(\mathbf{z_0})| = O_p(L_{2n})$*.*

(5) $\sup_{Z_{1t} \in G_1} |\hat{E}(m(Z_{1t})|Z_{1t}) - m(Z_{1t})| = O_p(h_1(\frac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}}) + O(h_1^{s+1})$*.*

(6) $\sup_{\mathbf{Z_t} \in G} |\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t})| = O_p(h_2(\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-\frac{1}{2}}) + O(h_2^{s_1+1})$*.*

$$\begin{aligned}&\sup_{\mathbf{Z_t} \in G} |\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) \\ &- \frac{1}{f_z(\mathbf{Z_t}) n h_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} K_2\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2}\right) I(\mathbf{Z_i^d} = \mathbf{Z_t^d})[m(Z_{1i}) - m(Z_{1t})]| \\ = \;& O_p(h_2(\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-1}) + O(h_2^{2s_1+1}) + O_p(h_2^{s_1+1}(\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-\frac{1}{2}}).\end{aligned}$$

(7) $\sup_{\mathbf{Z_t} \in G} |\hat{E}(\epsilon_t|\mathbf{Z_t})| = O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-\frac{1}{2}})$*.*

$$\begin{aligned}&\sup_{\mathbf{Z_t} \in G} |\hat{E}(\epsilon_t|\mathbf{Z_t}) - \frac{1}{f_z(\mathbf{Z_t}) n h_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} K_2\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2}\right) I(\mathbf{Z_i^d} = \mathbf{Z_t^d})\epsilon_i| \\ = \;& O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-1}) + O_p(h_2^{s_1+1}(\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-\frac{1}{2}}).\end{aligned}$$

(8) $\sup_{Z_{1t} \in G_1} |\hat{E}(\epsilon_t|Z_{1t})| = O_p((\frac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}})$*.*

See Theorem 1 in Yao and Zhang for proof.

Lemma 3 *With assumption A1-A5, A6(1)(ii), (3), we have*

$$\sup_{\mathbf{Z_t}\in G}\left|\hat{\sigma}^2(\mathbf{Z_t})-\sigma^2(\mathbf{Z_t})\right|=O_p(L_n)+O_p(n^{-\frac{1}{2}}).$$

Lemma 3: *Proof.*

(a)We first note since $\tilde{\beta}-\beta=O_p(n^{-\frac{1}{2}})$ from Theorem 1 in Yao and Zhang (2010),

$$\tilde{\epsilon}_t=m(Z_{1t})-\hat{E}(m(Z_{1t})|Z_{1t})+\epsilon_t-\hat{E}(\epsilon_t|Z_{1t})+(X_t-\hat{E}(X_t|Z_{1t}))(\beta-\tilde{\beta})$$

$$\begin{aligned}\tilde{\epsilon}_t^2=\ &(m(Z_{1t})-\hat{E}(m(Z_{1t})|Z_{1t}))^2+(\epsilon_t-\hat{E}(\epsilon_t|Z_{1t}))^2+((X_t-\hat{E}(X_t|Z_{1t}))(\beta-\tilde{\beta}))^2\\&+2(m(Z_{1t})-\hat{E}(m(Z_{1t})|Z_{1t}))(\epsilon_t-\hat{E}(\epsilon_t|Z_{1t}))\\&+2(m(Z_{1t})-\hat{E}(m(Z_{1t})|Z_{1t}))(X_t-\hat{E}(X_t|Z_{1t}))(\beta-\tilde{\beta})\\&+2(\epsilon_t-\hat{E}(\epsilon_t|Z_{1t}))(X_t-\hat{E}(X_t|Z_{1t}))(\beta-\tilde{\beta})\\&=I_1+\cdots+I_6.\end{aligned}$$

So $\hat{\sigma}^2(\mathbf{Z_t})=\hat{E}(\tilde{\epsilon}_t^2|\mathbf{Z_t})=\hat{E}(I_1|\mathbf{Z_t})+\cdots+\hat{E}(I_6|\mathbf{Z_t})$.

We use Lemma 2(2) to have

$$\begin{aligned}&\hat{E}(I_1|\mathbf{Z_t})\\=\ &[\tfrac{f_z(\mathbf{Z_t})-\hat{f}(\mathbf{Z_t})}{f_z(\mathbf{Z_t})\hat{f}(\mathbf{Z_t})}+\tfrac{1}{f_z(\mathbf{Z_t})}]\tfrac{1}{nh_1^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_2},\tfrac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})(m(Z_{1i})-\hat{E}(m(Z_{1i})|Z_{1i}))^2\\\le\ &\sup_{Z_{1i}\in G_1}|m(Z_{1i})-\hat{E}(m(Z_{1i})|Z_{1i})|^2[o_p(1)+\tfrac{1}{f_z(\mathbf{Z_t})}]\underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})}_{I_{11}}.\end{aligned}$$

We notice that $|K_2(\cdot)|$ satisfies the Lipschitz condition given assumption A3(2). We apply Lemma 1 to obtain $\sup_{Z_{1t}\in G_1}|I_{11}-EI_{11}|=O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$.
$EI_{11}\to f_z(\mathbf{Z_t})\int|K_2(\psi_{1i},\psi_{2i})|d\psi_{1i}d\psi_{2i}<\infty$ uniformly in $\mathbf{Z_t}\in G$.
So $I_{11}=O_p(1)$ uniformly. With result in Lemma 2(5), we conclude
$\sup_{Z_{1t}\in G_1}|\hat{E}(I_1|\mathbf{Z_t})|=(O_p(h_1(\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})+O(h_1^{s+1}))^2=o_p(n^{-\frac{1}{2}})$ with assumption A5.

$$\begin{aligned}&\hat{E}(I_3|\mathbf{Z_t})\\=\ &[o_p(1)+\tfrac{1}{f_z(\mathbf{Z_t})}]\tfrac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_2},\tfrac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\sum_{k=1}^{K}\sum_{k'=1}^{K}(X_{i,k}-\hat{g}_{1,k}(Z_{1i}))\\&\times(X_{i,k'}-\hat{g}_{1,k'}(Z_{1i}))(\beta_k-\tilde{\beta}_k)(\beta_{k'}-\tilde{\beta}_{k'})\\=\ &O_p(n^{-1})[o_p(1)+\tfrac{1}{f_z(\mathbf{Z_t})}]\sum_{k=1}^{K}\sum_{k'=1}^{K}\tfrac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_2},\tfrac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\\&\times(e_{1,ki}+g_{1,k}(Z_{1i})-\hat{g}_{1,k}(Z_{1i}))(e_{1,k'i}+g_{1,k'}(Z_{1i})-\hat{g}_{1,k'}(Z_{1i}))\\=\ &O_p(n^{-1})[o_p(1)+\tfrac{1}{f_z(\mathbf{Z_t})}]\sum_{k=1}^{K}\sum_{k'=1}^{K}I_{31}\end{aligned}$$

$$\begin{aligned}I_{31}=\ &\tfrac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_2},\tfrac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})[e_{1,ki}e_{1,k'i}+(g_{1,k}(Z_{1i})-\hat{g}_{1,k}(Z_{1i}))e_{1,k'i}\\&+(g_{1,k'}(Z_{1i})-\hat{g}_{1,k'}(Z_{1i}))e_{1,ki}+(g_{1,k}(Z_{1i})-\hat{g}_{1,k}(Z_{1i}))(g_{1,k'}(Z_{1i})-\hat{g}_{1,k'}(Z_{1i}))]\\=\ &I_{311}+\cdots+I_{314}\end{aligned}$$

$I_{311}=\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})e_{1,ki}e_{1,k'i}$. Given assumption A6(3), we have $E(|e_{1,ki}e_{1,k'i}|^{2+\delta_1}|Z_i)\le[E(|e_{1,ki}|^{4+\delta}|Z_i)E(|e_{1,k'i}|^{4+\delta}|Z_i)]^{\frac{1}{2}}<\infty$, so we have $\sup_{\mathbf{Z_t}\in G}|I_{311}-EI_{311}|=O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$. Furthermore,
$|EI_{311}|\le\frac{1}{h_2^{l_{1c}+l_{2c}}}E|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|X_{i,k}-g_{1,k}(Z_{1i})||X_{i,k'}-g_{1,k'}(Z_{1i})|=O_p(1)$ given result on I_{11} above and $E(|X_{i,k}X_{i,k'}||Z_i)<\infty$ with assumption A4(1). So $\sup_{\mathbf{Z_t}\in G}|I_{311}|=O_p(1)$.

$$I_{312} \leq \sup_{Z_{1i}\in G_1} |(g_{1,k}(Z_{1i} - \hat{g}_{1,k}(Z_{1i})| \underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} |K_2(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2})| I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) |e_{1,k'i}|}_{I_{3121}}.$$

With assumption A2(3) and A6(3), we apply Lemma 1 to have $\sup_{\mathbf{Z_t}\in G} |I_{3121} - EI_{3121}| = O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$. Furthermore, $EI_{3121} \leq \frac{1}{h_2^{l_{1c}+l_{2c}}} E|K_2(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2})| I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) [E(|X_{i,k'}||Z_i) + |g_{1,k'}(Z_{1i})] = O(1)$ with result I_{11} above and assumption A4(1). So $I_{312} = o_p(1)$ uniformly in $\mathbf{Z_t} \in G$. Similarly, $I_{313} = o_p(1)$ uniformly in $\mathbf{Z_t} \in G$.
$I_{314} = o_p(1)$ uniformly in $Z_{1t} \in G_1$ with result on I_{11} above and Lemma 2(3).
So in all, we have $I_{31} = O_p(1)$ uniformly in $\mathbf{Z_t} \in G$ and $\hat{E}(I_3|\mathbf{Z_t}) = O_p(n^{-1})$ uniformly.

$$\begin{aligned}
\hat{E}(I_4|\mathbf{Z_t}) = \; & [o_p(1) + \tfrac{1}{f_z(\mathbf{Z_t})}] \tfrac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} K_2(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_2}, \tfrac{Z_{2i}^c - Z_{2t}^c}{h_2}) I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) \\
& \times 2(m(Z_{1i}) - \hat{E}(m(Z_{1i})|Z_{1i}))(\epsilon_i - \hat{E}(\epsilon_i|Z_{1i})). \\
\leq \; & [O_p(h_1(\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + O_p(h_1^{s+1})] \\
& \times [\underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} |K_2(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2})| I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) |\epsilon_i|}_{I_{41}} + o_p(1)]
\end{aligned}$$

uniformly in $\mathbf{Z_t} \in G$ with Lemma 2(5) and (8). With assumptions A4(2) and A4(4), we apply Lemma 1 to obtain $\sup_{Z_{1t}\in G_1} |I_{41} - EI_{41}| = O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$. Furthermore,

$$\begin{aligned}
EI_{41} = \; & \int |K_2(\psi_{1i}, \psi_{2i})| E(|\epsilon_i||Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d) \\
& \times f_z(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d) d\psi_{1i} d\psi_{2i} < \infty
\end{aligned}.$$

So we have $\sup_{\mathbf{Z_t}\in G} |I_{41}| = O_p(1)$ and $\sup_{\mathbf{Z_t}\in G} |\hat{E}(I_4|\mathbf{Z_t})| = O_p(h_1(\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + O_p(h_1^{s+1})$.

$$\begin{aligned}
\hat{E}(I_5|\mathbf{Z_t}) = \; & [o_p(1) + \tfrac{1}{f_z(\mathbf{Z_t})}] \tfrac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} K_2(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_2}, \tfrac{Z_{2i}^c - Z_{2t}^c}{h_2}) I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) 2(m(Z_{1i}) - \hat{E}(m(Z_{1i})|Z_{1i})) \\
& \times \sum_{k=1}^{K} (X_{i,k} - \hat{g}_{1,k}(Z_{1i}))(\beta_k - \tilde{\beta}_k) \\
\leq \; & [O_p(h_1(\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + O_p(h_1^{s+1})] O_p(n^{-\frac{1}{2}}) \\
& \times \sum_{k=1}^{K} \{ \underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} |K_2(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2})| I(Z_i^d = \mathbf{Z_t^d}) |X_{i,k} - g_{1,k}(Z_{1i})|}_{I_{51}} \\
& + \underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}} \sum_{i=1}^{n} |K_2(\frac{Z_{1i}^c - Z_{1t}^c}{h_2}, \frac{Z_{2i}^c - Z_{2t}^c}{h_2})| I(\mathbf{Z_i^d} = \mathbf{Z_t^d}) |g_{1,k}(Z_{1i}) - \hat{g}_{1,k}(Z_{1i})| \}}_{I_{52}}
\end{aligned}$$

We obtain easily that $\sup_{\mathbf{Z_t}\in G} |I_{52}| = o_p(1)$ from result on I_{11} and Lemma 2(3). Furthermore, $\sup_{\mathbf{Z_t}\in G} |I_{51}| = O_p(1)$ as argued for term I_{3121}. So we conclude $\sup_{\mathbf{Z_t}\in G} |\hat{E}(I_5|Z_{1t})| = o_p(n^{-\frac{1}{2}})$.

$$\begin{aligned}
\hat{E}(I_6|\mathbf{Z_t}) = & [o_p(1)+\frac{1}{f_z(\mathbf{Z_t})}]\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\sum_{k=1}^{K}2(\epsilon_i-\hat{E}(\epsilon_i|Z_{1i}))\\
& \times(X_{i,k}-\hat{g}_{1,k}(Z_{1i}))(\beta_k-\tilde{\beta}_k)\\
\leq & O_p(n^{-\frac{1}{2}})\sum_{k=1}^{K}[\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|\epsilon_i||X_{i,k}-g_{1,k}(Z_{1i})|\\
& +\sup_{Z_{1i}\in G_1}|g_{1,k}(Z_{1i})-\hat{g}_{1,k}(Z_{1i})|\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|\epsilon_i|\\
& +\sup_{Z_{1i}\in G_1}|\hat{E}(\epsilon_i|Z_{1i})|\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|X_{i,k}-g_{1,k}(Z_{1i})|\\
& +\sup_{Z_{1i}\in G_1}|\hat{E}(\epsilon_i|Z_{1i})|\sup_{Z_{1i}\in G_1}|g_{1,k}(Z_{1i})-\hat{g}_{1,k}(Z_{1i})|\\
& \times\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|]\\
= & O_p(n^{-\frac{1}{2}})[I_{61}+\cdots+I_{64}]
\end{aligned}$$

It is easy to see that $\sup_{\mathbf{Z_t}\in G}|I_{64}|=o_p(1)$ with result on term I_{11}, Lemma 2(3) and (8). Similarly, with Lemma 2, we have $\sup_{Z_{1t}\in G_1}|I_{62}|=o_p(1)$ and $\sup_{Z_{1t}\in G_1}|I_{63}|=o_p(1)$ with results on I_{41} and I_{3121}.

$$\begin{aligned}
I_{61}\leq & \frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|\epsilon_i||X_{i,k}|\\
& +\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|\epsilon_i||g_{1,k}(Z_{1i})|\\
= & I_{611}+I_{612}
\end{aligned}$$

With assumption A6(3), $E(|\epsilon_i|^{2+\delta_1}|X_{i,k}|^{2+\delta_1}|Z_i)\leq[E(|\epsilon_i|^{4+2\delta}|Z_i)E(|X_{i,k}|^{4+2\delta}|Z_i)]^{\frac{1}{2}}<\infty$, so we apply Lemma 1 to obtain $\sup_{\mathbf{Z_t}\in G}|I_{611}-EI_{611}|=O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$.

$EI_{611}=\frac{1}{h_2^{l_{1c}+l_{2c}}}E[|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})E(|\epsilon_i||X_{i,k}||Z_i)]<\infty$ with assumption A6(3), so we have

$\sup_{Z_{1t}\in G_1}|I_{611}|=O_p(1)$.

$I_{612}\leq c\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}|K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})|I(\mathbf{Z_i^d}=\mathbf{Z_t^d})|\epsilon_i|=cI_{41}$, so we conclude $\sup_{\mathbf{Z_t}\in G}|I_{612}|=O_p(1)$. So $\sup_{\mathbf{Z_t}\in G}|I_{61}|=O_p(1)$ and in all $\sup_{\mathbf{Z_t}\in G}|\hat{E}(I_6|Z_{1t})|=O_p(n^{-\frac{1}{2}})$.

With Lemma 2(7), we obtain uniformly for $\mathbf{Z_t}\in G$,

$$\begin{aligned}
\hat{E}(I_2|\mathbf{Z_t}) = & [o_p(1)+\frac{1}{f_z(\mathbf{Z_t})}]\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\\
& \times\{\epsilon_i^2-2\epsilon_i\hat{E}(\epsilon_i|Z_{1i})+(\hat{E}(\epsilon_i|Z_{1i}))^2\}\\
= & \frac{1}{f_z(\mathbf{Z_t})}\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\epsilon_i^2+O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}).
\end{aligned}$$

(b) So we have from above

$\hat{\sigma}^2(\mathbf{Z_t})=\hat{E}(\tilde{\epsilon}_t^2|\mathbf{Z_t})$

$$=\frac{1}{f_z(\mathbf{Z_t})}\underbrace{\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2})I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\epsilon_i^2}_{I}+O_p(n^{-\frac{1}{2}})+O_p(L_{1n}).$$

With A6(3) and A4(4), we apply Lemma 1 to obtain $\sup_{\mathbf{Z_t}\in G}|I-EI|=O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{ln(n)})^{-\frac{1}{2}})$. With a change of variable and using A6(1)(ii) and A2(1),

$$
\begin{aligned}
&EI = \int K_2(\Psi_{1i},\Psi_{2i})\sigma^2(Z^c_{1t}+h_2\Psi_{1i},Z^d_{1t},Z^c_{2t}+h_2\Psi_{2i},Z^d_{2t})\\
&\times f_z(Z^c_{1t}+h_2\Psi,Z^d_{1t},Z^c_{2t}+h_2\Psi_{2i},Z^d_{2t})d\Psi_{1i}d\Psi_{2i}\\
= \; & \int K_2(\Psi_{1i},\Psi_{2i})[\sigma^2(\mathbf{Z_t})+\textstyle\sum_{|j|=1}^{s_1}\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t})\frac{h_2^{|j|}(\Psi_{1i},\Psi_{2i})^j}{j!}\\
&+\textstyle\sum_{|j|=s_1}(\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t}))\frac{h_2^{|j|}(\Psi_{1i},\Psi_{2i})^j}{j!}]\\
&\times[f_z(\mathbf{Z_t})+\textstyle\sum_{|l|=1}^{s}\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t})\frac{h_2^{|l|}(\Psi_{1i},\Psi_{2i})^l}{l!}\\
&+\textstyle\sum_{|l|=s_1}(\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t}))\frac{h_2^{|l|}(\Psi_{1i},\Psi_{2i})^l}{l!}]d\Psi_{1i}d\Psi_{2i}\\
= \; & \sigma^2(\mathbf{Z_t})f_z(\mathbf{Z_t})+\sigma^2(\mathbf{Z_t})\textstyle\sum_{|l|=1}^{s_1}\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t})\frac{h_2^{|l|}}{l!}\int K_2(\Psi_{1i},\Psi_{2i})(\Psi_{1i},\Psi_{2i})^l d\Psi_{1i}d\Psi_{2i}\\
&+\sigma^2(\mathbf{Z_t})\textstyle\sum_{|l|=s_1}\int K_2(\Psi_{1i},\Psi_{2i})(\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t}))\frac{h_2^{|l|}(\Psi_{1i},\Psi_{2i})^l}{l!}d\Psi_{1i}d\Psi_{2i}\\
&+\textstyle\sum_{|j|=1}^{s_1}\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t})\frac{h_2^{|j|}}{j!}f_z(\mathbf{Z_t})\int K_2(\Psi_{1i},\Psi_{2i})(\Psi_{1i},\Psi_{2i})^j d\Psi_{1i}d\Psi_{2i}\\
&+\textstyle\sum_{|j|=1}^{s_1}\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t})\frac{h_2^{|j|}}{j!}\sum_{|l|=1}^{s_1}\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t})\frac{h_2^{|l|}}{l!}\int K_2(\Psi_{1i},\Psi_{2i})(\Psi_{1i},\Psi_{2i})^{j+l}d\Psi_{1i}d\Psi_{2i}\\
&+\textstyle\sum_{|j|=1}^{s_1}\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t})\frac{h_2^{|j|}}{j!}(\sum_{|l|=s_1}\frac{h_2^{|l|}}{l!}\int K_2(\Psi_{1i},\Psi_{2i})\\
&\times(\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t}))(\Psi_{1i},\Psi_{2i})^{j+l}d\Psi_{1i}d\Psi_{2i})\\
&+f_z(\mathbf{Z_t})\textstyle\sum_{|j|=s_1}\frac{h_2^{|j|}}{j!}\int K_2(\Psi_{1i},\Psi_{2i})(\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(Z_{2t}))(\Psi_{1i},\Psi_{2i})^j d\Psi_{1i}d\Psi_{2i}\\
&+\textstyle\sum_{|j|=s_1}\sum_{|l|=1}^{s_1}\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t})\frac{h_2^{|l|}}{l!}\frac{h_2^{|j|}}{j!}\int K_2(\Psi_{1i},\Psi_{2i})\\
&\times(\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t}))(\Psi_{1i},\Psi_{2i})^{j+l}d\Psi_{1i}d\Psi_{2i}\\
&+\textstyle\sum_{|j|=s_1}\sum_{|l|=s_1}\frac{h_2^{|l|}}{l!}\frac{h_2^{|j|}}{j!}\int K_2(\Psi_{1i},\Psi_{2i})(\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^l}{\partial(\mathbf{Z^c_t})^l}f_z(\mathbf{Z_t}))\\
&\times(\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(Z^c_{1t}*,Z^d_{1t},Z^c_{2t}*,Z^d_{2t})-\frac{\partial^j}{\partial(\mathbf{Z^c_t})^j}\sigma^2(\mathbf{Z_t}))(\Psi_{1i},\Psi_{2i})^{j+l}d\Psi_{1i}d\Psi_{2i}\\
= \; & \sigma^2(\mathbf{Z_t})f_z(\mathbf{Z_t})+O(h_2^{s+1}),\ \text{with the additional assumption A6(1)(ii).}
\end{aligned}
$$

The claim in (2) above follows from (a) and (b).

Theorem 1: *Proof.*

We denote $g_{2,k}(Z_{1t}) \equiv (E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}E(\frac{X_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$, $\hat{g}^I_{2,k}(Z_{1t}) \equiv (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{X_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$. Since

$$
\begin{aligned}
&\hat{E}(Y_t|\mathbf{Z_t})-\hat{E}^{*I}(Y_t|Z_{1t})\\
= \; & \textstyle\sum_{k=1}^{K}(\hat{E}(X_{t,k}|\mathbf{Z_t})-(\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{X_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))\beta_k\\
&+\hat{E}(m(Z_{1t})|\mathbf{Z_t})-(\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\\
&+\hat{E}(\epsilon_t|\mathbf{Z_t})-(\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\tilde{\beta}^I-\beta\\
= \; & [(\tfrac{1}{n}\tilde{W}^{I'}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1}-(E(\tilde{W}'_t\tfrac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t))^{-1}+(E(\tilde{W}'_t\tfrac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t))^{-1}]\\
&\times\underbrace{\frac{1}{n}\tilde{W}^{I'}\Omega^{-1}(\vec{Z})[\hat{E}(m(Z_1)|\vec{Z})-\hat{E}^{*I}(m(Z_1)|\vec{Z}_1)+\hat{E}(\epsilon|\vec{Z})-\hat{E}^{*I}(\epsilon|\vec{Z}_1)]}_{C},
\end{aligned}
$$

where the $\hat{E}(m(Z_1)|\vec{Z})$ is a $n\times 1$ vector with the $t-th$ element $\hat{E}(m(Z_{1t})|\mathbf{Z_t})$, $\hat{E}^{*I}(m(Z_1)|\vec{Z}_1)$ is $n\times 1$ with the $t-th$ element $\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) \equiv (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$, $\hat{E}(\epsilon|\vec{Z})$ is $n\times 1$ with the $t-th$ element $\hat{E}(\epsilon_t|\mathbf{Z_t})$, and $\hat{E}^{*I}(\epsilon|\vec{Z}_1)$ is $n\times 1$ with the $t-th$ element $\hat{E}^{*I}(\epsilon_t|Z_{1t}) \equiv (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$.

(1) For $k,\tau=(1,2,\cdots,K)$, the $(k,\tau)th$ element in $\frac{1}{n}\tilde{W}^{I'}\Omega^{-1}(\vec{Z})\tilde{W}^I$ is

$$
\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\tilde{W}_{t,k}^{I}\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_{t,\tau}^{I}\\
=\;&\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t})-g_k(\mathbf{Z_t})+g_{2,k}(Z_{1t})-\hat{g}_{2,k}^{I}(Z_{1t})+g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})]\\
&\qquad\qquad\times[\hat{g}_\tau(\mathbf{Z_t})-g_\tau(\mathbf{Z_t})+g_{2,\tau}(Z_{1t})-\hat{g}_{2,\tau}^{I}(Z_{1t})+g_\tau(\mathbf{Z_t})-g_{2,\tau}(Z_{1t})]\\
=\;&\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t})-g_k(\mathbf{Z_t})][\hat{g}_\tau(\mathbf{Z_t})-g_\tau(\mathbf{Z_t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t})-g_k(\mathbf{Z_t})][g_{2,\tau}(Z_{1t})-\hat{g}_{2,\tau}^{I}(Z_{1t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t})-g_k(\mathbf{Z_t})][g_\tau(\mathbf{Z_t})-g_{2,\tau}(Z_{1t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t})-\hat{g}_{2,k}^{I}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t})-g_\tau(\mathbf{Z_t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t})-\hat{g}_{2,k}^{I}(Z_{1t})][g_{2,\tau}(Z_{1t})-\hat{g}_{2,\tau}^{I}(Z_{1t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t})-\hat{g}_{2,k}^{I}(Z_{1t})][g_\tau(\mathbf{Z_t})-g_{2,\tau}(Z_{1t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t})-g_\tau(\mathbf{Z_t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})][g_{2,\tau}(Z_{1t})-\hat{g}_{2,\tau}^{I}(Z_{1t})]\\
&+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})][g_\tau(\mathbf{Z_t})-g_{2,\tau}(Z_{1t})]\\
=\;&A_1+A_2+\cdots+A_9.
\end{aligned}
$$

(a) We show $\sup_{z_{1t}\in G_1}|\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})-E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})|=O_p(L_{1n})$.

$$
\begin{aligned}
&\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})-E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\\
=\;&[\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\tfrac{1}{\sigma^2(\mathbf{Z_i})}-f_1(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})+f_1(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\\
&-\hat{f}_1(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})][\tfrac{f_1(Z_{1t})-\hat{f}_1(Z_{1t})}{\hat{f}_1(Z_{1t})f_1(Z_{1t})}+\tfrac{1}{f_1(Z_{1t})}]\\
=\;&[\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\tfrac{1}{\sigma^2(Z_i)}-f_1(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})+O_p(L_{1n})][O_p(L_{1n})+\tfrac{1}{f_1(Z_{1t})}]
\end{aligned}
$$

uniformly for all $Z_{1t}\in G_1$, where Lemma 2(1) and assumption A6(1) are used to obtain the last equality. The claim in (a) will be true if we show

$$
\sup_{Z_{1t}\in G_1}|\underbrace{\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}}_{I_a}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})|=O_p(L_{1n}).
$$

By Assumption A6(1), $\frac{1}{\sigma^2(\mathbf{Z_t})}<\infty$. We apply Lemma 1 to obtain $\sup_{z_{1t}\in G_1}|I_a-EI_a|=O_p((\frac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}})$. By A6(1) again, $\sigma^2(\mathbf{Z_t})\in C_{1,2}^{s_1}$, so easily we have $\frac{1}{\sigma^2(\mathbf{Z_t})}\in C_1^s$. So

$$
\begin{aligned}
&E(I_a|Z_{1t})=E\tfrac{1}{h_1^{l_{1c}}}K_1\left(\tfrac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\tfrac{1}{\sigma^2(\mathbf{Z_i})}\\
=\;&\sum_{Z_{2i}^d}\int K_1(\psi_i)\{\tfrac{1}{\sigma^2(Z_{1t},Z_{2i})}+\sum_{|j|=1}^{s}\tfrac{\partial^j}{\partial(Z_{1t}^c)^j}\tfrac{1}{\sigma^2(Z_{1t},Z_{2i})}\tfrac{(h_1\psi_i)^j}{j!}\\
&+\sum_{|j|=s}[\tfrac{\partial^j}{\partial(Z_{1t}^c)^j}\tfrac{1}{\sigma^2(Z_{1t*}^c,Z_{1t}^d,Z_{2i})}-\tfrac{\partial^j}{\partial(Z_{1t}^c)^j}\tfrac{1}{\sigma^2(Z_{1t},Z_{2i})}]\tfrac{(h_1\psi_i)^j}{j!}\}\{f_z(Z_{1t},Z_{2i})\\
&+\sum_{|l|=1}^{s}\tfrac{\partial^l}{\partial(Z_{1t}^c)^l}f_z(Z_{1t},Z_{2i})\tfrac{(h_1\psi_i)^l}{l!}+\sum_{|l|=s}[\tfrac{\partial^l}{\partial(Z_{1t}^c)^l}f_z(Z_{1t*}^c,Z_{1t}^d,Z_{2i})-\tfrac{\partial^l}{\partial(Z_{1t}^c)^l}f_z(Z_{1t},Z_{2i})]\tfrac{(h_1\psi_i)^l}{l!}\}d\psi_i dZ_{2i}^c,
\end{aligned}
$$

where Z_{1t*}^c is between Z_{1t}^c and $Z_{1t}^c+h_1\psi_i$. With assumption A3 and A2(4), we obtain

$\sup_{z_{1t}\in G_1} |E(I_a|Z_{1t}) - f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O(h_1^{s+1})$. So in all, $\sup_{z_{1t}\in G_1} |I_a - f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$ as claimed.

From above, we also obtain

$$\sup_{Z_{1t}\in G_1} |\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \tfrac{1}{f_1(Z_{1t})}I_a + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n}^2).$$

With assumption A6(1), we have $E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) > C > 0$, so

$\inf_{Z_{1t}\in G_1} \hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \geq \inf_{Z_{1t}\in G_1} [\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] + \inf_{Z_{1t}\in G_1} E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) > 0$, since $\inf_{Z_{1t}\in G_1} [\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \leq \sup_{Z_{1t}\in G_1} |\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = o_p(1)$. Thus,

$$\begin{aligned}
&\sup_{Z_{1t}\in G_1} \left|(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\right| \\
= &\sup_{Z_{1t}\in G_1} \left|\frac{E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})}{\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})}\right| = O_p(1)\sup_{Z_{1t}\in G_1}\left|E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right| = O_p(L_{1n}).
\end{aligned}$$

(b) We show $\sup_{Z_{1t}\in G_1} |\hat{E}(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$.

$$\begin{aligned}
&\hat{E}(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
= &[\underbrace{\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^n K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{X_{i,k}}{\sigma^2(\mathbf{Z_i})}}_{I_b} - f_1(Z_{1t})E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + f_1(Z_{1t})E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
&-\hat{f}_1(Z_{1t})E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})][\tfrac{f_1(Z_{1t})-\hat{f}_1(Z_{1t})}{\hat{f}_1(Z_{1t})f_1(Z_{1t})} + \tfrac{1}{f_1(Z_{1t})}] \\
= &[I_b - f_1(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + O_p(L_{1n})][O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}]
\end{aligned}$$

uniformly for all $Z_{1t} \in G_1$, where Lemma 2(1) and assumption A6(1) are used to obtain the last equality. The claim in (b) is proved if we show $\sup_{Z_{1t}\in G_1} |I_b - f_1(Z_{1t})E(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$.

$$\begin{aligned}
I_b &= \tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^n K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\tfrac{g_k(\mathbf{Z_i})}{\sigma^2(\mathbf{Z_i})} + \tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^n K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\tfrac{e_{ki}}{\sigma^2(\mathbf{Z_i})} \\
&= I_{b1} + I_{b2}.
\end{aligned}$$

Since $\left|\frac{g_k(\mathbf{Z_i})}{\sigma^2(\mathbf{Z_i})}\right| < \infty$ by A4(1) and A6(1), we apply Lemma 1 to obtain

$$\sup_{Z_{1t}\in G_1} |I_{b1} - E(I_{b1}|Z_{1t})| = O_p((\tfrac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}}).$$

$$\begin{aligned}
E(I_{b1}|Z_{1t}) &= \sum_{Z_{2i}^d}\int K_1(\psi_i)\tfrac{g_k(Z_{1t}^c + h_1\psi_i, Z_{1t}^d, Z_{2i})}{\sigma^2(Z_{1t}^c + h_1\psi_i, Z_{1t}^d, Z_{2i})} f_z(Z_{1t}^c + h_1\psi_i, Z_{1t}^d, Z_{2i})d\psi_i dZ_{2i}^c \\
&= f_1(Z_{1t})E(\tfrac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + O(h_1^{s+1}) \text{ uniformly in } Z_{1t} \in G_1 \text{ with A2(4),(6), A3 and A6(1).}
\end{aligned}$$

So in all, $\sup_{Z_{1t}\in G_1} |I_{b1} - f_1(Z_{1t})E(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$, since $E(\frac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) = E(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$.

For I_{b2}: since $E(e_{ki}|\mathbf{Z_i}) = 0$, $|\frac{1}{\sigma^2(\mathbf{Z_i})}| < \infty$, with A2(6) and A4(1) we apply lemma 1 to obtain $\sup_{Z_{1t}\in G_1} |I_{b2}| = O_p((\frac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}})$. So $\sup_{Z_{1t}\in G_1} |I_b - f_1(Z_{1t})E(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$.

From above, we also obtain

$$\sup_{Z_{1t}\in G_1} |\hat{E}(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \tfrac{1}{f_1(Z_{1t})}I_b + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n}^2).$$

$$\begin{aligned}
(c)\quad &\hat{g}_{2,k}^I(Z_{1t}) - g_{2,k}(Z_{1t}) \\
= \quad &[(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}][\hat{E}(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \\
&+[(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}]E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
= \quad &[o_p(1) + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}]O_p(L_{1n}) + E(\tfrac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})O_p(L_{1n}) \\
= \quad &O_p(L_{1n}) \text{ uniformly for all } Z_{1t} \in G_1.
\end{aligned}$$

(d) Note Lemma 2(4) gives $\sup_{\mathbf{Z_t}\in G} |\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})| = o_p(1)$. $\frac{1}{n}\sum_{t=1}^n \frac{1}{\sigma^2(\mathbf{Z_t})} \xrightarrow{p} E\frac{1}{\sigma^2(\mathbf{Z_t})} < \infty$, $\frac{1}{n}\sum_{t=1}^n \frac{1}{\sigma^2(\mathbf{Z_t})}|g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})| \xrightarrow{p} E\frac{1}{\sigma^2(\mathbf{Z_t})}|g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})| < \infty$ since

$E\frac{1}{\sigma^2(\mathbf{Z_t})}|g_k(\mathbf{Z_t})| < CE|E(X_{t,k}|\mathbf{Z_t})| < CE(E(X_{t,k}^2|\mathbf{Z_t}))^{\frac{1}{2}} < \infty$ by A4(1) and A6(1), and $E\frac{1}{\sigma^2(\mathbf{Z_t})}|g_{2,k}(Z_{1t})| < CE(E(X_{t,k}^2|\mathbf{Z_t}))^{\frac{1}{2}} < \infty$ similarly. Results in (a)-(c) and above observations give $A_i = o_p(1)$ for $i = 1, \cdots, 8$.

$$\begin{aligned} A_9 \xrightarrow{p} & E\{\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][g_\tau(\mathbf{Z_t}) - g_{2,\tau}(Z_{1t})]\} \\ \leq & E\tfrac{1}{\sigma^2(\mathbf{Z_t})}|g_k(\mathbf{Z_t})g_\tau(\mathbf{Z_t})| + E\tfrac{1}{\sigma^2(\mathbf{Z_t})}|g_k(\mathbf{Z_t})g_{2,\tau}(Z_{1t})| \\ & + E\tfrac{1}{\sigma^2(\mathbf{Z_t})}|g_{2,k}(Z_{1t})g_\tau(\mathbf{Z_t})| + E\tfrac{1}{\sigma^2(\mathbf{Z_t})}|g_{2,k}(\mathbf{Z_t})g_{2,\tau}(Z_{1t})| < \infty, \end{aligned}$$

where the last inequality is obtained with A4(1), A6(1) and

$$\begin{aligned} & E\tfrac{1}{\sigma^2(\mathbf{Z_t})}|g_{2,k}(\mathbf{Z_t})g_{2,\tau}(Z_{1t})| \\ \leq & CE|g_{2,k}(\mathbf{Z_t})g_{2,\tau}(Z_{1t})| = E|(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-2}E(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})E(\tfrac{X_{t,\tau}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| \\ \leq & CE[E(|X_{t,k}||Z_{1t})E(|X_{t,\tau}||Z_{1t})] \leq C(E(E[X_{t,k}^2|Z_{1t}]))^{\frac{1}{2}}(E(E[X_{t,\tau}^2|Z_{1t}]))^{\frac{1}{2}} < \infty. \end{aligned}$$

So $\frac{1}{n}\sum_{t=1}^{n}\tilde{W}_{t,k}^I\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_{t,\tau}^I \xrightarrow{p} E(\tilde{W}_{t,k}\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_{t,\tau})$, thus $\frac{1}{n}\,\tilde{W}^{I\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I \xrightarrow{p} E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t)$. By A6(2), we obtain

$$[\frac{1}{n}\,\tilde{W}^{I\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I]^{-1} \xrightarrow{p} [E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t)]^{-1}.$$

(2) The $k-th$ element in C is

$$\begin{aligned} C_k = & \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_{t,k}^I[\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - \hat{E}^{*I}(m(Z_{1t})|Z_{1t}) + \hat{E}(\epsilon_t|\mathbf{Z_t}) - \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\ = & \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - \hat{E}^{*I}(m(Z_{1t})|Z_{1t})] \\ & + \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t}) - \hat{g}_{2,k}^I(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - \hat{E}^{*I}(m(Z_{1t})|Z_{1t})] \\ & + \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - \hat{E}^{*I}(m(Z_{1t})|Z_{1t})] \\ & + \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{E}(\epsilon_t|\mathbf{Z_t}) - \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\ & + \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t}) - \hat{g}_{2,k}^I(Z_{1t})][\hat{E}(\epsilon_t|\mathbf{Z_t}) - \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\ & + \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}(\epsilon_t|\mathbf{Z_t}) - \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\ = & C_{1k} + \cdots + C_{6k} \end{aligned}$$

(a) Let $I_1 = \frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\frac{m(Z_{1i})}{\sigma^2(\mathbf{Z_i})}$.

We show $\sup_{Z_{1t}\in G_1}|I_1 - f_1(Z_{1t})m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})$.

By A2(7) and A6(1), we have $|\frac{m(Z_{1i})}{\sigma^2(\mathbf{Z_i})}| < \infty$, thus we apply Lemma 1 to obtain

$$\sup_{Z_{1t}\in G_1}|I_1 - E(I_1|Z_{1t})| = O_p((\tfrac{nh_1^{l_{1c}}}{ln(n)})^{-\frac{1}{2}}).$$

$$\begin{aligned} E(I_1|Z_{1t}) & = \sum_{Z_{2i}^d}\int K_1(\psi_i)\tfrac{m(Z_{1t}^c + h_1\psi_i, Z_{1t}^d)}{\sigma^2(Z_{1t}^c + h_1\psi_i, Z_{1t}^d, Z_{2i})}f_z(Z_{1t}^c + h_1\psi_i, Z_{1t}^d, Z_{2i})d\psi_i dZ_{2i}^c \\ & = f_1(Z_{1t})m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + O(h_1^{s+1}) \text{ uniformly in } Z_{1t}\in G_1, \end{aligned}$$

with A2(4),(7), A3 and A6(1). Combining above two results, we obtain the claim.

$$\begin{aligned} (b) \quad & \hat{E}(\tfrac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\ = \; & [I_1 - f_1(Z_{1t})m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + (f_1(Z_{1t}) - \hat{f}_1(Z_{1t}))m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \\ & \times[\tfrac{f_1(Z_{1t}) - \hat{f}_1(Z_{1t})}{\hat{f}_1(Z_{1t})f_1(Z_{1t})} + \tfrac{1}{f_1(Z_{1t})}] \\ = \; & [I_1 - f_1(Z_{1t})m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + O_p(L_{1n})][O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}] \\ = \; & O_p(L_{1n}) \end{aligned}$$

uniformly for all $Z_{1t}\in G_1$, where Lemma 2(1) is used to obtain the second to last equality.
From above, we also obtain

$$\sup_{Z_{1t}\in G_1}|\hat{E}(\frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \frac{1}{f_1(Z_{1t})}I_1 + \frac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n}^2).$$

(c) Claim: $\sup_{Z_{1t}\in G_1}|\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t})| = O_p(L_{1n})$.

$$\begin{aligned}
& \hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) \\
= \; & [(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}] \\
& \times[\hat{E}(\tfrac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \\
& +[(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}]m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
= \; & O_p(L_{1n}) \text{ uniformly for all } Z_{1t}\in G_1, \text{ with results (a) above, and (1)(a).}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) \\
= \; & [(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}] \\
& \times[\hat{E}(\tfrac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \tfrac{1}{f_1(Z_{1t})}I_1 + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
& +\tfrac{1}{f_1(Z_{1t})}I_1 - \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \\
& +[(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}](E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
& \times[E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + \tfrac{1}{f_1(Z_{1t})}I_a - \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
& -\tfrac{1}{f_1(Z_{1t})}I_a + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]]m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}),
\end{aligned}$$

Thus, from (1)(a) and (2)(b) we obtain

$$\begin{aligned}
& \sup_{Z_{1t}\in G_1}|\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
& \times\{-\tfrac{1}{f_1(Z_{1t})}I_1 + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + m(Z_{1t})[\tfrac{1}{f_1(Z_{1t})}I_a - \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]\}| \\
= \; & O_p(L_{1n}^2).
\end{aligned}$$

(d) We claim: $\sup_{Z_{1t}\in G_1}|\hat{E}^{*I}(\epsilon_t|Z_{1t})| = O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})$.

$$\hat{E}^{*I}(\epsilon_t|Z_{1t}) = [(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}]\hat{E}(\tfrac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$$

with result (1)(a) and Lemma 2(1), we obtain uniformly for all $Z_{1t}\in G_1$

$$\begin{aligned}
\hat{E}^{*I}(\epsilon_t|Z_{1t}) = \; & [O_p(L_{1n}) + (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}] \\
& \times\tfrac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{\epsilon_i}{\sigma^2(\mathbf{Z_i})}[O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}].
\end{aligned}$$

Note $E(\epsilon_i|\mathbf{Z_i}) = 0$ and $\frac{1}{\sigma^2(\mathbf{Z_i})} < \infty$. With assumptions A1(2), A4(2), (4), A3 and A6(1), we apply Lemma 1 to obtain

$$\sup_{Z_{1t}\in G_1}\left|\tfrac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{\epsilon_i}{\sigma^2(\mathbf{Z_i})}\right| = O_p((\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}). \text{ Thus,}$$

$\sup_{Z_{1t}\in G_1}|\hat{E}^{*I}(\epsilon_t|Z_{1t})| = O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})$ as claimed. Furthermore, we have

$$\sup_{Z_{1t}\in G_1}|\hat{E}^{*I}(\epsilon_t|Z_{1t}) - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\tfrac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n})O_p((\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}).$$

(e) With Lemma 2 (4) and (6) and result (c) above, we obtain

$$\begin{aligned}
C_{1k} = \; & \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) - (\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}))] \\
= \; & O_p(L_{2n}^2) + O_p(L_{2n}L_{1n}) = o_p(n^{-\frac{1}{2}}) \text{ with assumption A5.}
\end{aligned}$$

With result (1)(c), we obtain

$$\begin{aligned}
C_{2k} = \; & \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t}) - \hat{g}_{2,k}^I(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) - (\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}))] \\
= \; & O_p(L_{1n}L_{2n}) + O_p(L_{1n}^2) = o_p(n^{-\frac{1}{2}}).
\end{aligned}$$

With Lemma 2(6) and result (2)(c) above, let's define

$$I_c = \tfrac{1}{f_z(\mathbf{Z_t})nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_2}, \tfrac{Z_{2i}^c - Z_{2t}^c}{h_2}\right)I(\mathbf{Z_i^d} = \mathbf{Z_t^d})[m(Z_{1i}) - m(Z_{1t})], \text{ and}$$

$$\begin{aligned}
I_d = \; & (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\{-\tfrac{1}{f_1(Z_{1t})}I_1 + \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
& +m(Z_{1t})[\tfrac{1}{f_1(Z_{1t})}I_a - \tfrac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]\}.
\end{aligned}$$

$$
\begin{aligned}
C_{3k} = & \ \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t})-\hat{E}^{*I}(m(Z_{1t})|Z_{1t})]\\
= & \ \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})]\{\hat{E}(m(Z_{1t})|\mathbf{Z_t})-m(Z_{1t})\\
& -I_c+I_c-[\hat{E}^{*I}(m(Z_{1t})|Z_{1t})-m(Z_{1t})+I_d-I_d]\\
= & \ O_p(L_{1n}^2)+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})]\{I_c+I_d\}\\
= & \ O_p(L_{1n}^2)+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})]\\
& \times\frac{1}{f_z(\mathbf{Z_t})nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2}\right)I(\mathbf{Z_i^d}=\mathbf{Z_t^d})[m(Z_{1i})-m(Z_{1t})]\\
& +\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})](E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\{-\frac{1}{f_1(Z_{1t})}I_1+\frac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\\
& +\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})](E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}m(Z_{1t})[\frac{1}{f_1(Z_{1t})}I_a-\frac{\hat{f}_1(Z_{1t})}{f_1(Z_{1t})}E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]\\
= & \ O_p(L_{1n}^2)+C_{31k}+C_{32k}+C_{33k}.\\
C_{32k} = & \ -\frac{1}{n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\frac{1}{h_1^{l_{1c}}f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})](E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\\
& \times K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)[\frac{m(Z_{1i})}{\sigma^2(\mathbf{Z_i})}-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))m(Z_{1t})]\\
= & \ -\frac{1}{n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\psi_n(\mathbf{Z_i},\mathbf{Z_t})=-\frac{1}{n^2}\sum_{t=1}^{n}\psi_n(\mathbf{Z_t},\mathbf{Z_t})-\frac{1}{n^2}\sum_{\substack{t=1\\t\neq i}}^{n}\sum_{i=1}^{n}\psi_n(\mathbf{Z_t},\mathbf{Z_i})\\
= & \ -\frac{1}{n^2}\sum_{t=1}^{n}\psi_n(\mathbf{Z_t},\mathbf{Z_t})-\frac{1}{2n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\underbrace{(\psi_n(\mathbf{Z_i},\mathbf{Z_t})+\psi_n(\mathbf{Z_t},\mathbf{Z_i}))}_{\phi_n(\mathbf{Z_t},\mathbf{Z_i})}\\
= & \ C_{321k}+C_{322k}\\
C_{321k} = & \ -\frac{1}{n^2}\sum_{t=1}^{n}\frac{1}{h_1^{l_{1c}}f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})](E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\\
& K_1(0)[\frac{1}{\sigma^2(\mathbf{Z_t})}-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))]m(Z_{1t})]\\
= & \ O_p((nh_1^{l_{1c}})^{-1})\text{ given A2(2), (7), A4(1) and A6(1).}
\end{aligned}
$$

By construction, $\phi_n(\mathbf{Z_i},\mathbf{Z_t})$ is symmetric, so C_{322k} is a two dimensional U-statistic. By Lemma 1 in Yao and Ullah (2011), we define

$$
\hat{U}_n=\frac{2}{n}\sum_{t=1}^{n}E(\phi_n(\mathbf{Z_t},\mathbf{Z_i})|\mathbf{Z_t})-E(\phi_n(\mathbf{Z_t},\mathbf{Z_i})),
$$

then $C_{322k}=-\frac{1}{2}\hat{U}_n+O_p(n^{-1}(E(\phi_n^2(\mathbf{Z_t},\mathbf{Z_i}))^{\frac{1}{2}})$.
We show $C_{322k}=O_p(h_1^{s+1})+O_p((n^2h_1^{l_{1c}})^{-\frac{1}{2}})$.

(i) $E(\phi_n^2(\mathbf{Z_t},\mathbf{Z_i}))=E(\psi_n(\mathbf{Z_i},\mathbf{Z_t})+\psi_n(\mathbf{Z_i},\mathbf{Z_t}))^2\leq 4E\psi_n^2(\mathbf{Z_i},\mathbf{Z_t})=O(h_1^{-l_{1c}})$, since

$$
\begin{aligned}
h_1^{l_{1c}}E\psi_n^2(\mathbf{Z_i},\mathbf{Z_t}) = & \sum_{\mathbf{Z_t^d}}\sum_{Z_{2i}^d}\int K_1^2(\psi_i)[\frac{m(Z_{1t}^c+h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})}-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))m(Z_{1t})]^2\frac{(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))^2}{(\sigma^2(\mathbf{Z_t}))^2}\\
& (E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-2}f_z(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})\frac{f_z(\mathbf{Z_t})}{f_1^2(Z_{1t})}d\psi_i dZ_{2i}^c dZ_{1t}^c dZ_{2t}^c\\
= & \ O(1)\text{ given assumptions A2(4), (7), A3, A4(1) and A6(1).}
\end{aligned}
$$

$$
\begin{aligned}
(ii)\quad & E(\psi_n(\mathbf{Z_i},\mathbf{Z_t})|\mathbf{Z_t})\\
= & \sum_{Z_{2i}^d}\int K_1(\psi_i)[\frac{m(Z_{1t}^c+h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})}-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))m(Z_{1t})]f_z(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})d\psi_i dZ_{2i}^c\\
& \times\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})f_1(Z_{1t})}(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\\
= & \ I_{3221k}-I_{3222k}.
\end{aligned}
$$

$$
\begin{aligned}
& I_{3221k} \\
= \;& \{\sum_{Z_{2i}^d}\int K_1(\psi_i)\frac{m(Z_{1t}^c+h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})}f_z(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})d\psi_i dZ_{2i}^c\} \\
& \times\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})f_1(Z_{1t})}(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
= \;& \{\sum_{Z_{2i}^d}\int K_1(\psi_i)[\frac{m(Z_{1t})}{\sigma^2(Z_{1t},Z_{2i})}+\sum_{|j|=1}^{s}\frac{\partial^j}{\partial(Z_{1t}^c)^j}\frac{m(Z_{1t})}{\sigma^2(Z_{1t},Z_{2i})}\frac{(h_1\psi_i)^j}{j!} \\
& +\sum_{|j|=s}[\frac{\partial^j}{\partial(Z_{1t*}^c)^j}\frac{m(Z_{1t})}{\sigma^2(Z_{1t*}^c,Z_{1t}^d,Z_{2i})}-\frac{\partial^j}{\partial(Z_{1t}^c)^j}\frac{m(Z_{1t})}{\sigma^2(Z_{1t},Z_{2i})}]\frac{(h_1\psi_i)^j}{j!}] \\
& \times[f_z(Z_{1t},Z_{2i})+\sum_{|l|=1}^{s}\frac{\partial^l}{\partial(Z_{1t}^c)^l}f_z(Z_{1t},Z_{2i})\frac{(h_1\psi_i)^l}{l!} \\
& +\sum_{|j|=s}[\frac{\partial^l}{\partial(Z_{1t*}^c)^l}f_z(Z_{1t},Z_{2i})-\frac{\partial^l}{\partial(Z_{1t}^c)^l}f_z(Z_{1t},Z_{2i})]\frac{(h_1\psi_i)^l}{l!}]d\psi_i dZ_{2i}^c\} \\
& \times\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})f_1(Z_{1t})}(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
= \;& m(Z_{1t})\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})}+O(h_1^{s+1}) \text{ uniformly for all } \mathbf{Z_t}\in G, \text{ with A2(2), (4), (7), A3, A4(1)}
\end{aligned}
$$

and A6(1), where $Z_{1t*}^c=\lambda Z_{1t}^c+(1-\lambda)(Z_{1t}^c+h_1\psi_i)$ for some $\lambda\in(0,1)$.

$$
\begin{aligned}
& I_{3222k} \\
= \;& \{\sum_{Z_{2i}^d}\int K_1(\psi_i)(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))m(Z_{1t})f_z(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})d\psi_i dZ_{2i}^c\} \\
& \times\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})f_1(Z_{1t})}(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
= \;& m(Z_{1t})\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})}+O(h_1^{s+1}) \text{ uniformly for all } \mathbf{Z_t}\in G \text{ with similar argument.}
\end{aligned}
$$

So we conclude $E(\psi_n(\mathbf{Z_i},\mathbf{Z_t})|\mathbf{Z_t})=O(h_1^{s+1})$ uniformly for all $\mathbf{Z_t}\in G$.

$$
\begin{aligned}
& E(\psi_n(\mathbf{Z_t},\mathbf{Z_i})|\mathbf{Z_t}) \\
= \;& \sum_{Z_{2i}^d}\frac{1}{h_1^{l_{1c}}}\int K_1\left(\frac{Z_{1t}^c-Z_{1i}^c}{h_1}\right)I(Z_{1t}^d=Z_{1i}^d)[\frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}-(E(\frac{1}{\sigma^2(\mathbf{Z_i})}|Z_{1i}))m(Z_{1i})] \\
& \times\frac{g_k(\mathbf{Z_i})-g_{2,k}(Z_{1i})}{\sigma^2(\mathbf{Z_i})f_1(Z_{1i})}(E(\frac{1}{\sigma^2(\mathbf{Z_i})}|Z_{1i}))^{-1}f_z(Z_i)dZ_{1i}^c dZ_{2i}^c \\
= \;& I_{3223k}-I_{3224k}. \\
& I_{3223k} \\
= \;& \frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}\sum_{Z_{2i}^d}\int K_1(\psi_i)\frac{g_k(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})-g_{2,k}(Z_{1t}^c-h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})} \\
& \times(E(\frac{1}{\sigma^2(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})}|Z_{1t}^c-h_1\psi_i,Z_{1t}^d))^{-1}\frac{f_z(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})}{f_1(Z_{1t}^c-h_1\psi_i,Z_{1t}^d)}d\psi_i dZ_{2i}^c \\
= \;& \frac{m(Z_{1t})}{\sigma^2(\mathbf{Z_t})}[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))]^{-1}E[\frac{g_k(Z_{1t},Z_{2i})-g_{2,k}(Z_{1t})}{\sigma^2(Z_{1t},Z_{2i})}|Z_{1t}]+O(h_1^{s+1}) \\
= \;& O(h_1^{s+1}) \text{ uniformly in } \mathbf{Z_t}\in G, \text{ by definition of } g_{2,k}(Z_{1t}).
\end{aligned}
$$

Note this illustrates the importance of constructing the estimator as in Equation (7), which implies the bias of the estimation disappears asymptotically. Similar argument shows

$$
\begin{aligned}
& I_{3224k} \\
= \;& \sum_{Z_{2i}^d}\int K_1(\psi_i)\frac{g_k(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})-g_{2,k}(Z_{1t}^c-h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})} \\
& \times m(Z_{1t}^c-h_1\psi_i,Z_{1t}^d)\frac{f_z(Z_{1t}^c-h_1\psi_i,Z_{1t}^d,Z_{2i})}{f_1(Z_{1t}^c-h_1\psi_i,Z_{1t}^d)}d\psi_i dZ_{2i}^c \\
= \;& O(h_1^{s+1}) \text{ uniformly in } Z_{1t}\in G_1.
\end{aligned}
$$

$$
\begin{aligned}
& (iii) \quad E\phi_n(\mathbf{Z_t},\mathbf{Z_i})=2E\psi_n(\mathbf{Z_i},\mathbf{Z_t}) \\
= \;& \sum_{\mathbf{Z_t^d}}\sum_{Z_{2i}^d}\int K_1(\psi_i)[\frac{m(Z_{1t}^c+h_1\psi_i,Z_{1t}^d)}{\sigma^2(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})}-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))m(Z_{1t})]\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})} \\
& (E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}f_z(Z_{1t}^c+h_1\psi_i,Z_{1t}^d,Z_{2i})\frac{f_z(\mathbf{Z_t})}{f_1(Z_{1t})}d\psi_i dZ_{2i}^c dZ_{1t}^c dZ_{2t}^c \\
= \;& O(h_1^{s+1}) \text{ with similar arguments in (ii).}
\end{aligned}
$$

So $C_{322k}=O_p(h_1^{s+1})+O_p((n^2h_1^{l_{1c}})^{-\frac{1}{2}})$. Thus, $C_{32k}=O_p(h_1^{s+1})+O_p((nh_1^{l_{1c}})^{-1})$.

We obtain with similar arguments that $C_{33k}=O_p(h_1^{s+1})+O_p((nh_1^{l_{1c}})^{-1})$.

$$\begin{aligned} C_{31k} &= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n} \frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{h_2^{l_{1c}+l_{2c}} f_z(\mathbf{Z_t})\sigma^2(\mathbf{Z_t})} K_2\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_2}, \frac{Z_{2i}^c-Z_{2t}^c}{h_2}\right) I(\mathbf{Z_i^d}=\mathbf{Z_t^d})(m(Z_{1i})-m(Z_{1t})) \\ &= \frac{1}{n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n} \psi_n(\mathbf{Z_i},\mathbf{Z_t}) = \frac{1}{2n^2}\sum_{t=1}^{n}\sum_{\substack{i=1\\ t\neq i}}^{n} \underbrace{(\psi_n(\mathbf{Z_i},\mathbf{Z_t})+\psi_n(\mathbf{Z_t},\mathbf{Z_i}))}_{\phi_n(\mathbf{Z_t},\mathbf{Z_i})} \end{aligned}$$

Since $\phi_n(\mathbf{Z_t},\mathbf{Z_i})$ is symmetric, C_{31k} is a two dimensional U-statistic. Following the proof in C_{32k} above, we show

(i) $E\phi_n^2(\mathbf{Z_t},\mathbf{Z_i}) \leq 2[E\psi_n^2(\mathbf{Z_i},\mathbf{Z_t})+E\psi_n^2(\mathbf{Z_t},\mathbf{Z_i})] = O(h_2^{-(l_{1c}+l_{2c})})$, since

$$\begin{aligned} h_2^{l_{1c}+l_{2c}} E\psi_n^2(\mathbf{Z_i},\mathbf{Z_t}) &= \sum_{\mathbf{Z_t^d}} \int K_2^2(\psi_{1i},\psi_{2i})[m(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d)-m(Z_{1t})]^2 \frac{(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))^2}{f_z^2(\mathbf{Z_t})(\sigma^2(\mathbf{Z_t}))^2} \\ &\quad \times f_z(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d,Z_{2t}^c+h_2\psi_{2i},Z_{2t}^d) f_z(\mathbf{Z_t}) d\psi_{1i}d\psi_{2i}dZ_{1t}^c dZ_{2t}^c \\ &= O(1). \end{aligned}$$

$$\begin{aligned} &(ii) \quad E(\psi_n(\mathbf{Z_i},\mathbf{Z_t})|\mathbf{Z_t}) \\ &= \int K_2(\psi_{1i},\psi_{2i})[[m(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d)-m(Z_{1t})] f_z(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d,Z_{2t}^c+h_2\psi_{2i},Z_{2t}^d) \\ &\quad d\psi_{1i}d\psi_{2i}\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_z(\mathbf{Z_t})\sigma^2(\mathbf{Z_t})} \\ &= O(h_2^{s_1+1}) \text{ uniformly at } \mathbf{Z_t}\in G, \text{ with A2(4), (5), (7), A3, A4(1) and A6(1).} \\ &\quad E(\psi_n(\mathbf{Z_t},\mathbf{Z_i})|\mathbf{Z_t}) \\ &= \int K_2(\psi_{1i},\psi_{2i})[[m(Z_{1t})-m(Z_{1t}^c-h_2\psi_{1i},Z_{1t}^d)] \\ &\quad \times \frac{g_k(Z_{1t}^c-h_2\psi_{1i},Z_{1t}^d,Z_{2t}^c-h_2\psi_{2i},Z_{2t}^d)-g_{2,k}(Z_{1t}^c-h_2\psi_{1i},Z_{1t}^d)}{\sigma^2(Z_{1t}^c-h_2\psi_{1i},Z_{1t}^d,Z_{2t}^c-h_2\psi_{2i},Z_{2t}^d)} d\psi_{1i}d\psi_{2i} \\ &= O(h_2^{s_1+1}) \text{ uniformly at } \mathbf{Z_t}\in G, \text{ with similar argument.} \\ &(iii) \quad E\phi_n(\mathbf{Z_t},\mathbf{Z_i}) = 2E\psi_n(\mathbf{Z_i},\mathbf{Z_t}) \\ &= \sum_{\mathbf{Z_t^d}} \int K_2(\psi_{1i},\psi_{2i})[m(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d)-m(Z_{1t})]\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})} \\ &\quad f_z(Z_{1t}^c+h_2\psi_{1i},Z_{1t}^d,Z_{2t}^c+h_2\psi_{2i},Z_{2t}^d)d\psi_{1i}d\psi_{2i}dZ_{1t}^c dZ_{2t}^c \\ &= O(h_1^{s+1}) \text{ with similar arguments in (ii).} \end{aligned}$$

So we conclude $C_{31k} = O(h_2^{s_1+1}) + O_p((n^2h_2^{l_{1c}+l_{2c}})^{-\frac{1}{2}})$. Thus, we obtain

$C_{3k} = O(h_2^{s_1+1}) + O_p(h_1^{s+1}) + o_p(n^{-\frac{1}{2}})$.

$$\begin{aligned} C_{4k} &= \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t})-g_k(\mathbf{Z_t})][\hat{E}(\epsilon_t|\mathbf{Z_t})-\hat{E}^{*I}(\epsilon_t|Z_{1t})] \\ &= O_p(L_{1n})[O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})+O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})] \end{aligned}$$

with lemma 2(4), (7) and result (2)(d) above.

$C_{5k} = O_p(L_{1n})[O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})+O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})]$ could be shown in a similar fashion given result (1)(c).

With Lemma 2(4), (7) and result (2)(d) above, we have

$$\begin{aligned} C_{6k} &= \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})][\hat{E}(\epsilon_t|\mathbf{Z_t})-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\ &\quad -(\hat{E}^{*I}(\epsilon_t|Z_{1t})-(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})] \\ &= O_p(L_{1n})O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})] \\ &\quad \times\{\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2}\right)I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\epsilon_i[O_p(L_{2n})+\frac{1}{f_z(\mathbf{Z_t})}] \\ &\quad -(E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{\epsilon_i}{\sigma^2(\mathbf{Z_i})} \\ &\quad \times[O_p(L_{1n})+\frac{1}{f_1(Z_{1t})}]\}. \end{aligned}$$

By Lemma 2(7), we have $\sup_{Z_{1t}\in G_1}|\frac{1}{nh_2^{l_{1c}+l_{2c}}}\sum_{i=1}^{n}K_2\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_2},\frac{Z_{2i}^c-Z_{2t}^c}{h_2}\right)I(\mathbf{Z_i^d}=\mathbf{Z_t^d})\epsilon_i| = O_p((\frac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}})$, and by result (2)(d) above, $\sup_{Z_{1t}\in G_1}|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{\epsilon_i}{\sigma^2(\mathbf{Z_i})}| = O_p((\frac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}})$, so we obtain

$$
\begin{aligned}
C_{6k} = \;& O_p(L_{1n})O_p((\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + O_p(L_{2n})O_p((\tfrac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}}) \\
& + \tfrac{1}{n^2}\sum_{t=1}^{n}\sum_{i=1}^{n} \tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})]\{\tfrac{1}{h_2^{l_{1c}+l_{2c}} f_z(\mathbf{Z_t})} K_2\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_2}, \tfrac{Z_{2i}^c - Z_{2t}^c}{h_2}\right) I(\mathbf{Z_i^d} = \mathbf{Z_t^d})\epsilon_i \\
& - \tfrac{1}{h_1^{l_{1c}} f_1(Z_{1t})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\tfrac{\epsilon_i}{\sigma^2(\mathbf{Z_i})}\} \\
= \;& O_p(L_{1n})O_p((\tfrac{nh_1^{l_{1c}}}{lnn})^{-\frac{1}{2}}) + O_p(L_{1n})O_p((\tfrac{nh_2^{l_{1c}+l_{2c}}}{lnn})^{-\frac{1}{2}}) + \underbrace{\frac{1}{n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\psi_n(S_i, S_t)}_{C_{61k}},
\end{aligned}
$$

where $S_i = (\mathbf{Z_i}, \epsilon_i)$. We note

$$
\begin{aligned}
C_{61k} = \;& \tfrac{1}{2n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}(\psi_n(S_i, S_t) + \psi_n(S_t, S_i)) \\
= \;& \tfrac{1}{n^2}\sum_{t=1}^{n}\psi_n(S_i, S_i) + \tfrac{1}{2n^2}\sum_{t=1}^{n}\sum_{i=1}^{n}\underbrace{(\psi_n(S_i, S_t) + \psi_n(S_t, S_i))}_{\phi_n(S_i,S_t)} \\
= \;& C_{611k} + C_{612k}.
\end{aligned}
$$

$$
\begin{aligned}
C_{611k} = \;& \tfrac{1}{2n^2}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})] \\
& \times\{\tfrac{1}{h_2^{l_{1c}+l_{2c}} f_z(\mathbf{Z_t})} K_2(0)\,\epsilon_t - \tfrac{1}{h_1^{l_{1c}} f_1(Z_{1t})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} K_1(0)\,\tfrac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}\}
\end{aligned}
.
$$

Since $E(\epsilon_t|\mathbf{Z_t}) = 0$, with assumption A2(2), A2(5), A4(1) and A6(1), we have

$C_{611k} = O_p((n^{\frac{3}{2}}h_2^{l_{1c}+l_{2c}})^{-1}) + O_p((n^{\frac{3}{2}}h_1^{l_{1c}})^{-1})$.

Since $\phi_n(S_i, S_t)$ is symmetric, C_{612k} is an U-statistic. By Lemma 1 in Yao et al. (2010), we have

$C_{612k} = \frac{1}{n}\sum_{t=1}^{n}E(\phi_n(S_i, S_t)|S_t) - \frac{1}{2}E\phi_n(S_i, S_t) + O_p(n^{-1}(E\phi_n^2(S_i, S_t))^{\frac{1}{2}})$.

Since $E(\epsilon_i|\mathbf{Z_i}) = 0$, we have $E\phi_n(S_i, S_t) = 0$.

$E\phi_n^2(S_i, S_t) \le 2[E\psi_n^2(S_i, S_t) + E\psi_n^2(S_t, S_i)] = 4E\psi_n^2(S_i, S_t)$.

$$
\begin{aligned}
E\psi_n^2(S_i, S_t) \le \;& 2\{E[\tfrac{(g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t}))^2}{(\sigma^2(\mathbf{Z_t}))^2 h_2^{2(l_{1c}+l_{2c})} f_z^2(\mathbf{Z_t})} K_2^2\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_2}, \tfrac{Z_{2i}^c - Z_{2t}^c}{h_2}\right) I(\mathbf{Z_i^d} = \mathbf{Z_t^d})\sigma^2(\mathbf{Z_i})] \\
& + E[\tfrac{(g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t}))^2}{(\sigma^2(\mathbf{Z_t}))^2 h_1^{2l_{1c}} f_1^2(Z_{1t})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-2} K_1^2\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\tfrac{\sigma^2(\mathbf{Z_i})}{(\sigma^2(\mathbf{Z_i}))^2}]\} \\
= \;& O(h_2^{-(l_{1c}+l_{2c})} + h_1^{-l_{1c}}) \text{ with assumptions A2(2), (4), A3, A4(1) and A6(1).}
\end{aligned}
$$

So $O_p(n^{-1}(E\phi_n^2(S_i, S_t))^{\frac{1}{2}}) = O(n^{-1}(h_2^{-\frac{l_{1c}+l_{2c}}{2}} + h_1^{-\frac{l_{1c}}{2}}))$.

So $C_{612k} = \frac{1}{n}\sum_{t=1}^{n}E(\phi_n(S_i, S_t)|S_t) + O(n^{-1}(h_2^{-\frac{l_{1c}+l_{2c}}{2}} + h_1^{-\frac{l_{1c}}{2}})) = C_{6121k} + o_p(n^{-\frac{1}{2}})$.

$$
\begin{aligned}
& \tfrac{1}{\sqrt{n}}E(\phi_n(S_i, S_t)|S_t) = \tfrac{1}{\sqrt{n}}E(\psi_n(S_t, S_i)|S_t) \\
= \;& \tfrac{1}{\sqrt{n}}\epsilon_t \int \tfrac{g_k(\mathbf{Z_i}) - g_{2,k}(Z_{1i})}{\sigma^2(\mathbf{Z_i}) h_2^{l_{1c}+l_{2c}} f_z(\mathbf{Z_i})} K_2\left(\tfrac{Z_{1t}^c - Z_{1i}^c}{h_2}, \tfrac{Z_{2t}^c - Z_{2i}^c}{h_2}\right) I(\mathbf{Z_t^d} = \mathbf{Z_i^d}) f_z(\mathbf{Z_i}) d\mathbf{Z_i^c} \\
& - \tfrac{1}{\sqrt{n}}\tfrac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}\sum_{Z_{2i}^d}\int \tfrac{1}{h_1^{l_{1c}} f_1(Z_{1i})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}))^{-1} K_1\left(\tfrac{Z_{1t}^c - Z_{1i}^c}{h_1}\right) I(Z_{1t}^d = Z_{1i}^d) \\
& \times \tfrac{g_k(\mathbf{Z_i^c}) - g_{2,k}(Z_{1i})}{\sigma^2(\mathbf{Z_i^c})} f_z(\mathbf{Z_i}) d\mathbf{Z_i^c} \\
= \;& \tfrac{1}{\sqrt{n}}\epsilon_t C_{61211k} - \tfrac{1}{\sqrt{n}}\epsilon_t C_{61212k} \\
= \;& S_{tn}.
\end{aligned}
$$

$\sqrt{n}C_{6121k} = \sum_{t=1}^{n}S_{tn}$. S_{tn} forms an independent triangular array and $ES_{tn} = 0$. Furthermore, $\sum_{t=1}^{n}ES_{tn}^2 = E[E(\psi_n(S_t, S_i)|S_t)]^2$.

$$
\begin{aligned}
C_{61211k} = \;& \int K_2(\psi_{1i}, \psi_{2i})\tfrac{g_k(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d) - g_{2,k}(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d)}{\sigma^2(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d)} d\psi_{1i}d\psi_{2i} \\
\to \;& \tfrac{g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})}{\sigma^2(\mathbf{Z_t})}
\end{aligned}
$$

uniformly over $\mathbf{Z_t} \in G$.

$$
\begin{aligned}
C_{61212k} = \;& \tfrac{1}{\sigma^2(\mathbf{Z_t})}\sum_{Z_{2i}^d}\int K_1(\psi_{1i})(E(\tfrac{1}{\sigma^2(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)}|Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d))^{-1}\tfrac{f_z(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d, Z_{2i})}{f_1(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)} \\
& \times \tfrac{g_k(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d, Z_{2i}) - g_{2,k}(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)}{\sigma^2(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d, Z_{2i})} d\psi_{1i}dZ_{2i}^c \\
\to \;& \tfrac{1}{\sigma^2(\mathbf{Z_t})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})})^{-1}E(\tfrac{g_k(Z_{1t}, Z_{2i}) - g_{2,k}(Z_{1t})}{\sigma^2(Z_{1t}, Z_{2i})}|Z_{1t}) = 0
\end{aligned}
$$

uniformly over $\mathbf{Z_t} \in G$ by definition of $g_{2,k}(Z_{1t})$. Thus,

$$
\begin{aligned}
E[E(\psi_n(S_t,S_i)|S_t)]^2 &\to E\sigma^2(\mathbf{Z_t})[\tfrac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})} - \tfrac{1}{\sigma^2(\mathbf{Z_t})}(E(\tfrac{1}{\sigma^2(\mathbf{Z_t})})^{-1}E(\tfrac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]^2 \\
&= E(\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - (E(\tfrac{1}{\sigma^2(\mathbf{Z_t})})^{-1}E(\tfrac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]^2).
\end{aligned}
$$

By Liapounov's Central Limit Theorem, provided $\lim_{n\to\infty}\sum_{t=1}^{n}E\left|\frac{S_{tn}}{(\sum_{t=1}^{n}ES_{tn}^2)^{\frac{1}{2}}}\right|^{2+\delta} = 0$ for some $\delta > 0$, we have

$$
\sum_{t=1}^{n}S_{tn} \xrightarrow{d} N(0, E(\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - (E(\frac{1}{\sigma^2(\mathbf{Z_t})})^{-1}E(\frac{g_k(\mathbf{Z_t})}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]^2)).
$$

So we only need to show $\sum_{t=1}^{n}E\left|\frac{S_{tn}}{(\sum_{t=1}^{n}ES_{tn}^2)^{\frac{1}{2}}}\right|^{2+\delta} = (\sum_{t=1}^{n}ES_{tn}^2)^{-1-\frac{\delta}{2}}n^{-\frac{\delta}{2}}E|E(\psi_n(S_t,S_i)|S_t)|^{2+\delta} \to 0$. From above, we know that $\sum_{t=1}^{n}ES_{tn}^2 = O(1)$.

Furthermore, since $\sigma^2(\mathbf{Z_i^c}) > C > 0$, we have

$$
\begin{aligned}
C_{61211k} \le\ & C\int K_2(\psi_{1i},\psi_{2i})|g_k(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d) \\
& -g_{2,k}(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)|d\psi_{1i}d\psi_{2i} \\
\le\ & \int_A |g_k(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d)|d\psi_{1i}d\psi_{2i} \\
& + \int_A |g_{2,k}(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)|d\psi_{1i}
\end{aligned}
$$

where A is the bounded support for $K_2(\cdot)$. So

$$
\begin{aligned}
E|C_{61211k}|^{2+\delta} \le\ & C\int_A E|g_k(Z_{1t}^c + h_2\psi_{1i}, Z_{1t}^d, Z_{2t}^c + h_2\psi_{2i}, Z_{2t}^d)|^{2+\delta}d\psi_{1i}d\psi_{2i} \\
& +C\int_A E|g_{2,k}(Z_{1t}^c + h_1\psi_{1i}, Z_{1t}^d)|^{2+\delta}d\psi_{1i}
\end{aligned}
$$

Since $E|g_k(\mathbf{Z_t})|^{2+\delta} < C$ by A4(1), and $E|g_{2,k}(Z_{1t})|^{2+\delta} < CE|E(X_t|Z_{1t})|^{2+\delta} < C$ by A6(1) and A4(1), we conclude $E|C_{61211k}|^{2+\delta} < \infty$.
In a similar fashion with additional assumption A2(2) and (5), we obtain $E|C_{61212k}|^{2+\delta} < \infty$.

$E|E(\psi_n(S_t,S_i)|S_t)|^{2+\delta} = E\{E(|\epsilon_t|^{2+\delta}|\mathbf{Z_t})|C_{61211k} - C_{61212k}|^{2+\delta}\} < \infty$, with the results above and the assumption A4(2) that $E(|\epsilon_t|^{2+\delta}|\mathbf{Z_t}) < \infty$. So in all we conclude

$$
\lim_{n\to\infty}\sum_{t=1}^{n}E\left|\frac{S_{tn}}{(\sum_{t=1}^{n}ES_{tn}^2)^{\frac{1}{2}}}\right|^{2+\delta} = 0.
$$

Finally, we conclude $\sqrt{n}C_{612k} \xrightarrow{d} N(0, E(\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})]^2))$. With assumption A5, we have $\sqrt{n}C_{6k} \xrightarrow{d} N(0, E(\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})]^2))$.

Combining results on $C_{ik}, i = 1,\cdots,6$, result in (1) and assumption A5(3), we apply the Cramer-Rao's device to obtain

$$
\sqrt{n}(\tilde{\beta}^I - \beta) \xrightarrow{d} N(0, (E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t))^{-1}).
$$

Theorem 2: *Proof.*

$$
\begin{aligned}
(1)\quad & \tilde{\beta}^I - \tilde{\beta}^E = \tilde{\beta}^I - \beta - (\tilde{\beta}^E - \beta) \\
= \quad & (\tilde{W}^{I\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1}\ \tilde{W}^{I\prime}\Omega^{-1}(\vec{Z})[\hat{E}(m(Z_1)|\vec{Z}) + \hat{E}(\epsilon|\vec{Z}) - \hat{E}^{*I}(m(Z_1)|\vec{Z}_1) - \hat{E}^{*I}(\epsilon|\vec{Z}_1)] \\
& -(\tilde{W}^{F\prime}\hat{\Omega}^{-1}(\vec{Z})\tilde{W}^F)^{-1}\ \tilde{W}^{F\prime}\hat{\Omega}^{-1}(\vec{Z})[\hat{E}(m(Z_1)|\vec{Z}) + \hat{E}(\epsilon|\vec{Z}) - \hat{E}^{*}(m(Z_1)|\vec{Z}_1) - \hat{E}^{*}(\epsilon|\vec{Z}_1)].
\end{aligned}
$$

where $\hat{E}^*(m(Z_1)|\vec{Z}_1)$ is $n\times 1$ with the $t-th$ element $(\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{m(Z_{1t})}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})$, and $\hat{E}^*(\epsilon|\vec{Z}_1)$ is $n\times 1$ with the $t-th$ element $(\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{\epsilon_t}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})$.

So we write

$$
\begin{aligned}
&\sqrt{n}(\tilde{\beta}^I - \tilde{\beta}^E) \\
= \ &\sqrt{n}\{[(\tfrac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1} - (\tfrac{1}{n}\ \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})\tilde{W}^F)^{-1}] \\
&\underbrace{\times \frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})[\hat{E}(m(Z_1)|\vec{Z}) + \hat{E}(\epsilon|\vec{Z}) - \hat{E}^{*I}(m(Z_1)|\vec{Z}_1) - \hat{E}^{*I}(\epsilon|\vec{Z}_1)]}_{C} \\
&+(\tfrac{1}{n}\ \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})\tilde{W}^F)^{-1} \\
&\times[\tfrac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})(\hat{E}(m(Z_1)|\vec{Z}) + \hat{E}(\epsilon|\vec{Z}) - \hat{E}^{*I}(m(Z_1)|\vec{Z}_1) - \hat{E}^{*I}(\epsilon|\vec{Z}_1)) \\
&-\tfrac{1}{n}\ \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})(\hat{E}(m(Z_1)|\vec{Z}) + \hat{E}(\epsilon|\vec{Z}) - \hat{E}^{*}(m(Z_1)|\vec{Z}_1) - \hat{E}^{*}(\epsilon|\vec{Z}_1))]\}.
\end{aligned}
$$

Given results in Theorem 1, i.e., $\sqrt{n}C_k = O_p(1)$ for C_k the $k-th$ element in C, and $(\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1} - [E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t)]^{-1} = o_p(1)$, we only need to show

(i) $(\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1} - (\frac{1}{n}\ \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})\tilde{W}^F)^{-1} = o_p(1)$.

$(ii)\sqrt{n}\frac{1}{n}[\tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z}) - \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})][\hat{E}(m(Z_1)|\vec{Z}) - m(\vec{Z}_1) + \hat{E}(\epsilon|\vec{Z})] = o_p(1)$ where $m(\vec{Z}_1) = (m(Z_{11}), m(Z_{12}), \cdots, m(Z_{1n}))'$.

$(iii)\sqrt{n}\frac{1}{n}\{\tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})[\hat{E}^{*I}(m(Z_1)|\vec{Z}_1) - m(\vec{Z}_1) + \hat{E}^{*I}(\epsilon|\vec{Z}_1)] - \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})[\hat{E}^{*}(m(Z_1)|\vec{Z}_1) - m(\vec{Z}_1) + \hat{E}^{*}(\epsilon|\vec{Z}_1)]\} = o_p(1)$.

(i) Since we have $\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I \xrightarrow{p} E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t)$ and $(\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1} \xrightarrow{p} (E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t))^{-1}$ in the proof of Theorem 1, result (1), if we have

$$
\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I - \frac{1}{n}\ \tilde{W}^{F\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^F = o_p(1),
$$

then $\frac{1}{n}\ \tilde{W}^{F\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^F \xrightarrow{p} E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t)$ and $(\frac{1}{n}\ \tilde{W}^{F\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^F)^{-1} \xrightarrow{p} (E(\tilde{W}_t'\frac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}_t))^{-1}$. Thus, we have $(\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I)^{-1} - (\frac{1}{n}\ \tilde{W}^{F\,\prime}\hat{\Omega}^{-1}(\vec{Z})\tilde{W}^F)^{-1} = o_p(1)$.

Let's denote the $(k,\tau)-th$ element in $\frac{1}{n}\ \tilde{W}^{I\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^I - \frac{1}{n}\ \tilde{W}^{F\,\prime}\Omega^{-1}(\vec{Z})\tilde{W}^F$ as

$$
\begin{aligned}
&\tfrac{1}{n}\sum_{t=1}^{n}\tilde{W}^I_{t,k}\tfrac{1}{\sigma^2(\mathbf{Z_t})}\tilde{W}^I_{t,\tau} - \tfrac{1}{n}\sum_{t=1}^{n}\tilde{W}^F_{t,k}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}\tilde{W}^F_{t,\tau} \\
= \ &\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}^I_{2,\tau}(Z_{1t})] \\
&-\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}_{2,\tau}(Z_{1t})] \\
= \ &I_1 - I_2.
\end{aligned}
$$

$$
\begin{aligned}
I_2 = \ &\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t}) + \hat{g}^I_{2,k}(Z_{1t}) - \hat{g}_{2,k}(Z_{1t})] \\
&\times[\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}^I_{2,\tau}(Z_{1t}) + \hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})] \\
= \ &\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}^I_{2,\tau}(Z_{1t})] \\
&+\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})] \\
&+\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}^I_{2,k}(Z_{1t}) - \hat{g}_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}^I_{2,\tau}(Z_{1t})] \\
&+\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}^I_{2,k}(Z_{1t}) - \hat{g}_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})] \\
= \ &I_{21} + \cdots + I_{24}.
\end{aligned}
$$

$$
I_1 - I_{21} = \tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{1}{\sigma^2(\mathbf{Z_t})} - \tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}][\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}_\tau(\mathbf{Z_t}) - \hat{g}^I_{2,\tau}(Z_{1t})].
$$

(a) We first note

$$
\begin{aligned}
&\sup_{\mathbf{Z_t}\in G}|\tfrac{1}{\sigma^2(\mathbf{Z_t})} - \tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}| \\
\leq \ &[\inf_{\mathbf{Z_t}\in G}\sigma^2(\mathbf{Z_t})\inf_{\mathbf{Z_t}\in G}\hat{\sigma}^2(\mathbf{Z_t})]^{-1}\sup_{\mathbf{Z_t}\in G}|\hat{\sigma}^2(\mathbf{Z_t}) - \sigma^2(\mathbf{Z_t})|.
\end{aligned}
$$

With Lemma 3 and assumption A6(1), for large n, $\inf_{\mathbf{Z_t}\in G}\hat{\sigma}^2(\mathbf{Z_t}) > 0$, so

$$
\sup_{\mathbf{Z_t}\in G}|\frac{1}{\sigma^2(\mathbf{Z_t})} - \frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}| = O_p(L_n) + O_p(n^{-\frac{1}{2}}).
$$

With Lemma 2(4), and result (1)(c) in Theorem 1, we conclude $I_1 - I_{21} = o_p(1)$.

(b) We claim: $\sup_{Z_{1t}\in G_1} \left|\hat{g}_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})\right| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$.

$$\begin{aligned}
&\hat{g}_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})\\
=\ & [(\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}](\hat{E}(\tfrac{X_{t,k}}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))\\
&+(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}[\hat{E}(\tfrac{X_{t,k}}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})) - \hat{E}(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t})).
\end{aligned}$$

Given results (1)(a) and (b) in Theorem 1, if we further have

(A) $\sup_{Z_{1t}\in G_1} \left|\hat{E}(\frac{X_{t,k}}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})) - \hat{E}(\frac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))\right| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$ and

(B) $\sup_{Z_{1t}\in G_1} \left|(\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\right| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$,

then we have the claim in (b).

$$\begin{aligned}
(A)\quad & \sup_{Z_{1t}\in G_1} \left|\hat{E}(\tfrac{X_{t,k}}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})) - \hat{E}(\tfrac{X_{t,k}}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))\right|\\
=\ & \sup_{Z_{1t}\in G_1} \left|[o_p(1) + \tfrac{1}{f_1(Z_{1t})}]\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\tfrac{Z^c_{1i}-Z^c_{1t}}{h_1}\right) I(Z^d_{1i} = Z^d_{1t})X_{i,k}[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z^c_i})} - \tfrac{1}{\sigma^2(\mathbf{Z^c_i})}]\right|\\
=\ & O_p(L_n) + O_p(n^{-\frac{1}{2}}),
\end{aligned}$$

with result on (a) above, Lemma 3 terms I_{11} and I_{3121}, and with assumption A2(3), and A4(1).

$$\begin{aligned}
(B)\quad & \sup_{Z_{1t}\in G_1} \left|\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right|\\
=\ & \sup_{Z_{1t}\in G_1} \left|[o_p(1) + \tfrac{1}{f_1(Z_{1t})}]\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\tfrac{Z^c_{1i}-Z^c_{1t}}{h_1}\right) I(Z^d_{1i} = Z^d_{1t})[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z^c_i})} - \tfrac{1}{\sigma^2(\mathbf{Z^c_i})}]\right|\\
=\ & O_p(L_n) + O_p(n^{-\frac{1}{2}}),
\end{aligned}$$

using result on (a) above, Lemma 3 term I_{11}.

$$(\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} = \frac{\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})}{\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})}.$$

We have in Theorem 1 (1)(a), $\inf_{Z_{1t}\in G_1} \hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) > 0$ with assumption A6(1). Given result above, we follow similar argument there to obtain $\inf_{Z_{1t}\in G_1} \hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) > 0$. Thus,

$$\begin{aligned}
& \sup_{Z_{1t}\in G_1} \left|(\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\right|\\
\le\ & [\inf_{Z_{1t}\in G_1} \hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \inf_{Z_{1t}\in G_1} \hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})]^{-1} \sup_{Z_{1t}\in G_1} \left|\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right|\\
=\ & O_p(L_n) + O_p(n^{-\frac{1}{2}}).
\end{aligned}$$

$$\begin{aligned}
& I_{22}\\
=\ & \tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \tfrac{1}{\sigma^2(\mathbf{Z_t})} + \tfrac{1}{\sigma^2(\mathbf{Z_t})}][\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t}) + g_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t}) + g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})]\\
& \times[\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
=\ & \tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \tfrac{1}{\sigma^2(\mathbf{Z_t})}][\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
& +\tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \tfrac{1}{\sigma^2(\mathbf{Z_t})}][g_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
& +\tfrac{1}{n}\sum_{t=1}^{n}[\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \tfrac{1}{\sigma^2(\mathbf{Z_t})}][g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
& +\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
& +\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
& +\tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{g}^I_{2,\tau}(Z_{1t}) - \hat{g}_{2,\tau}(Z_{1t})]\\
=\ & o_p(1) \text{ with results in (a) and (b) above, result (1)(c) in Theorem 1 and lemma 2(4).}
\end{aligned}$$

Similarly, we have $I_{23} = o_p(1)$ and $I_{24} = o_p(1)$. So in all we have $I_1 - I_2 = o_p(1)$. So we have the claim in (i).

(ii) Let's denote the $k-th$ element of $\frac{1}{n}[\tilde{W}^{I'}\Omega^{-1}(\vec{Z}) - \tilde{W}^{F'}\hat{\Omega}^{-1}(\vec{Z})][\hat{E}(m(Z_1)|\vec{Z}) - m(\vec{Z}_1) + \hat{E}(\epsilon|\vec{Z})]$ by A_k. It suffices to show $\sqrt{n}A_k = o_p(1)$.

$$\begin{aligned}
A_k = & \frac{1}{n}\sum_{t=1}^{n}[\frac{1}{\sigma^2(\mathbf{Z_t})}(\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})) - \frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}(\hat{g}_k(\mathbf{Z_t}) - \hat{g}_{2,k}(Z_{1t}))] \\
& \times[\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) + \hat{E}(\epsilon_t|\mathbf{Z_t})] \\
= & \frac{1}{n}\sum_{t=1}^{n}[\frac{1}{\sigma^2(\mathbf{Z_t})} - \frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}][\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t}) + g_k(\mathbf{Z_t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) + \hat{E}(\epsilon_t|\mathbf{Z_t})] \\
& + \frac{1}{n}\sum_{t=1}^{n}[\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \frac{1}{\sigma^2(\mathbf{Z_t})}][\hat{g}^I_{2,k}(Z_{1t}) - g_{2,k}(Z_{1t}) + g_{2,k}(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) + \hat{E}(\epsilon_t|\mathbf{Z_t})] \\
& + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{E}(m(Z_{1t})|\mathbf{Z_t}) - m(Z_{1t}) + \hat{E}(\epsilon_t|\mathbf{Z_t})]
\end{aligned}$$

Given result in (i)(b) above, and result (1)(c) in Theorem 1, we have $\sup_{Z_{1t}\in G_1}|\hat{g}_{2,k}(Z_{1t}) - g_{2,k}(Z_{1t})| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$. Thus, using result (i)(a) above, Lemma 2(4), (6), (7) and assumption A5, we obtain $A_k = o_p(n^{-\frac{1}{2}})$.

(iii) Let's use B_k to denote the $k-th$ element of $\frac{1}{n}\{\tilde{W}^{I'}\Omega^{-1}(\vec{Z})[\hat{E}^{*I}(m(Z_1)|\vec{Z}_1) - m(\vec{Z}_1) + \hat{E}^{*I}(\epsilon|\vec{Z}_1)] - \tilde{W}^{F'}\hat{\Omega}^{-1}(\vec{Z})[\hat{E}^{*}(m(Z_1)|\vec{Z}_1) - m(\vec{Z}_1) + \hat{E}^{*}(\epsilon|\vec{Z}_1)]\}$. We note

$$\begin{aligned}
B_k = & \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\
& - \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - \hat{g}_{2,k}(Z_{1t})][\hat{E}^{*}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*}(\epsilon_t|Z_{1t})] \\
= & B_{1k} - B_{2k},
\end{aligned}$$

where $\hat{E}^{*I}(A_t|Z_{1t}) \equiv (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{A_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})$ and $\hat{E}^{*}(A_t|Z_{1t}) \equiv (\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\frac{A_t}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})$. It suffices to show

(a) $B_{1k} = o_p(n^{-\frac{1}{2}})$ and (b) $B_{2k} = o_p(n^{-\frac{1}{2}})$.

$$\begin{aligned}
(a)\ B_{1k} = & \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t})][\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\
& + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t})][\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\
& + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\
= & O_p(L_{2n}) * O_p(L_{1n}) + O_p(L^2_{1n}) + B_{11k} \\
= & o_p(n^{-\frac{1}{2}}) + B_{11k},
\end{aligned}$$

with Lemma 2(4), results (1)(c), (2)(c) and (d) in Theorem 1.

$$\begin{aligned}
(b)\ B_{2k} = & \frac{1}{n}\sum_{t=1}^{n}[\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})} - \frac{1}{\sigma^2(\mathbf{Z_t})} + \frac{1}{\sigma^2(\mathbf{Z_t})}][\hat{g}_k(\mathbf{Z_t}) - g_k(\mathbf{Z_t}) + g_{2,k}(Z_{1t}) - \hat{g}^I_{2,k}(Z_{1t}) \\
& + g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}^{*}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*}(\epsilon_t|Z_{1t})]
\end{aligned}$$

If we have

(A) $\sup_{Z_{1t}\in G_1}|\hat{E}^{*}(\epsilon_t|Z_{1t})| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$ and

(B) $\sup_{Z_{1t}\in G_1}|\hat{E}^{*}(m(Z_{1t})|Z_{1t}) - m(Z_{1t})| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$, then with results in (1)(i)(a), (b) above, and Lemma 2(4), we have

$$\begin{aligned}
B_{2k} = & \frac{1}{n}\sum_{t=1}^{n}[O_p(L_n) + O_p(n^{-\frac{1}{2}}) + \frac{1}{\sigma^2(\mathbf{Z_t})}][O_p(L_n) + O_p(n^{-\frac{1}{2}}) \\
& + g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}^{*}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*}(\epsilon_t|Z_{1t})] \\
= & o_p(n^{-\frac{1}{2}}) + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}^{*}(m(Z_{1t})|Z_{1t}) - m(Z_{1t})] \\
& + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})]\hat{E}^{*}(\epsilon_t|Z_{1t}) \\
= & o_p(n^{-\frac{1}{2}}) + B_{21k} + B_{22k}
\end{aligned}$$

$$\begin{aligned}
(A)\ \hat{E}^{*}(\epsilon_t|Z_{1t}) = & (\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})^{-1}\hat{E}(\frac{\epsilon_t}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) \\
= & [\hat{E}(\frac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t})^{-1} - (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})^{-1} + (\hat{E}(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})^{-1}] \\
& \times[\hat{E}(\frac{\epsilon_t}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) + \hat{E}(\frac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t})
\end{aligned}$$

$$\begin{aligned}
&\hat{E}(\tfrac{\epsilon_t}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - \hat{E}(\tfrac{\epsilon_t}{\sigma^2(\mathbf{Z_t})}|Z_{1t}) \\
= \;& \tfrac{1}{\hat{f}_1(Z_{1t})}\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\left(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})} - \tfrac{1}{\sigma^2(\mathbf{Z_i^c})}\right)\epsilon_i \\
\leq \;& [O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}]\sup_{\mathbf{Z_i}\in G}\left|\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})} - \tfrac{1}{\sigma^2(\mathbf{Z_i^c})}\right| \tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}\left|K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)\right| I(Z_{1i}^d = Z_{1t}^d)|\epsilon_i| \\
= \;& O_p(L_n) + O_p(n^{-\frac{1}{2}}),
\end{aligned}$$

with result (1)(i)(a) above, and similar argument as in term I_{41} in Lemma 3.
With results (1)(a), (2)(d) in Theorem 1, result in (1)(i)(b)(B) above and $E\left(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}\right) > 0$ with assumption A6(1), we conclude $\sup_{Z_{1t}\in G_1}\left|\hat{E}^*(\epsilon_t|Z_{1t})\right| = O_p(L_n) + O_p(n^{-\frac{1}{2}})$.

$$\begin{aligned}
(B)\;& \left|\hat{E}^*(m(Z_{1t})|Z_{1t}) - m(Z_{1t})\right| = \left|(\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\hat{E}(\tfrac{m(Z_{1t})}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}) - m(Z_{1t})\right| \\
= \;& \left|(\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\left[\tfrac{1}{\hat{f}_1(Z_{1t})}\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})}(m(Z_{1i}) - m(Z_{1t}))\right]\right| \\
= \;& \left|(\hat{E}(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_t})}|Z_{1t}))^{-1} - (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} + (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\right|[O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}] \\
&\times\left|\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)(\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})} - \tfrac{1}{\sigma^2(\mathbf{Z_i^c})} + \tfrac{1}{\sigma^2(\mathbf{Z_i^c})})(m(Z_{1i}) - m(Z_{1t})))\right| \\
\leq \;& [O_p(L_n) + O_p(n^{-\frac{1}{2}})] + (\hat{E}(\tfrac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}][O_p(L_{1n}) + \tfrac{1}{f_1(Z_{1t})}] \\
&\times[\sup_{\mathbf{Z_i}\in G}\left|\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i})} - \tfrac{1}{\sigma^2(\mathbf{Z_i})}\right|\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}\left|K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)\right|I(Z_{1i}^d = Z_{1t}^d)|m(Z_{1i}) - m(Z_{1t})| \\
&+\left|\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}(m(Z_{1i}) - m(Z_{1t}))\right|
\end{aligned}$$

We first note

$$\begin{aligned}
&\sup_{Z_{1t}\in G_1}\left[\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}\left|K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)\right|I(Z_{1i}^d = Z_{1t}^d)|m(Z_{1i}) - m(Z_{1t})|\right] \\
\leq \;& \sup_{Z_{1t}\in G_1}[\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}\left|K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)\right|I(Z_{1i}^d = Z_{1t}^d)]2\sup_{Z_{1t}\in G_1}|m(Z_{1t})| = O_p(1),
\end{aligned}$$

where we use I_{11} in Lemma 3 and assumption A2(7). Second, in result (2)(a) in Theorem 1, we obtain

$$\sup_{Z_{1t}\in G_1}|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i^c})}m(Z_{1i}) - f_1(Z_{1t})m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n}).$$

Third, following result (1)(a) in Theorem 1, we have

$$\sup_{Z_{1t}\in G_1}|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i^c})}m(Z_{1t}) - f_1(Z_{1t})m(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})| = O_p(L_{1n}).$$

With the three observations above, we conclude

$$\sup_{Z_{1t}\in G_1}\left|\hat{E}^*(m(Z_{1t})|Z_{1t}) - m(Z_{1t})\right| = O_p(L_n) + O_p(n^{-\frac{1}{2}}).$$

(c) We consider now

$$\begin{aligned}
B_{21k} = \;& \tfrac{1}{n}\sum_{t=1}^{n}\tfrac{1}{\sigma^2(\mathbf{Z_t})}(g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t}))\left[\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})}\right]^{-1} \\
&\times\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})}(m(Z_{1i}) - m(Z_{1t}))
\end{aligned}$$

(A) We first note the result in (b)(B) above implies

$$\sup_{Z_{1t}\in G_1}|\tfrac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\tfrac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\tfrac{1}{\hat{\sigma}^2(\mathbf{Z_i^c})}(m(Z_{1i}) - m(Z_{1t}))| = O_p(L_n) + O_p(n^{-\frac{1}{2}}).$$

(B) With result in (1)(a) in Theorem 1, we know

$$\sup_{Z_{1t}\in G_1}\left|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right)I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i^c})} - f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right| = O_p(L_{1n}).$$

With result in (1)(b)(B) above, we know

$$\sup_{Z_{1t}\in G_1}\left|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)[\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}-\frac{1}{\sigma^2(\mathbf{Z_i^c})}]\right|=O_p(L_n)+O_p(n^{-\frac{1}{2}}).$$

Combining them, we obtain

$$\sup_{Z_{1t}\in G_1}\left|\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right|=O_p(L_n)+O_p(n^{-\frac{1}{2}}).$$

(C) Let $B_{21k1}=\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}$, then we have

$$\begin{aligned}&\sup_{Z_{1t}\in G_1}\left|\left(\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}\right)^{-1}-(f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\right|\\ =&\sup_{Z_{1t}\in G_1}\frac{|f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})-B_{21k1}|}{|f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})B_{21k1}|}=O_p(L_n)+O_p(n^{-\frac{1}{2}}).\end{aligned}$$

Given result in (B) above and assumption A2(2) and A6(1), we only need to show $\inf_{Z_{1t}\in G_1}B_{21k1}>0$.

$\inf_{Z_{1t}\in G_1}B_{21k1}\geq\inf_{Z_{1t}\in G_1}[B_{21k1}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]+\inf_{Z_{1t}\in G_1}f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})>0$, since

$$\begin{aligned}&\inf_{Z_{1t}\in G_1}[B_{21k1}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})]\\ \leq&\inf_{Z_{1t}\in G_1}|B_{21k1}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})|\leq\sup_{Z_{1t}\in G_1}|B_{21k1}-f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})|=o_p(1)\end{aligned}$$

given result in (B) above. Thus we have the claimed result in (C).

With results in (A)-(C) above, we conclude

$$\begin{aligned}B_{21k}=&\ o_p(n^{-\frac{1}{2}})+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))\left[f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}\\ &\times\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}(m(Z_{1i})-m(Z_{1t}))\\ =&\ o_p(n^{-\frac{1}{2}})+B_{211k}.\end{aligned}$$

$$\begin{aligned}(d)\,B_{211k}=&\ \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\hat\sigma^2(\mathbf{Z_i})}\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\\ &\times\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}(m(Z_{1i})-m(Z_{1t}))\end{aligned}$$

We let $B_{211k1}=\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}m(Z_{1t})$.

Since with assumption A2(2), (7), A4(1) and A6(1), $|\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}m(Z_{1t})|<C<\infty$,

we could apply Lemma 1 to have $\sup_{Z_{1i}\in G_1}|B_{211k1}-E(B_{211k1}|Z_{1i})|=O_p(\left(\frac{nh_1^{l_{1c}}}{lnn}\right)^{-\frac{1}{2}})$.

With assumption A2 and A6(1) and (4), we have

$$
\begin{aligned}
E(B_{211k1}|Z_{1i}) = & \sum_{Z_{2t}^d}\int K_1(\psi)\frac{g_k(Z_{1i}^c-h_1\psi,Z_{1i}^d,Z_{2t})-g_{2,k}(Z_{1i}^c-h_1\psi,Z_{1i}^d)}{\sigma^2(Z_{1i}^c-h_1\psi,Z_{1i}^d,Z_{2t})f_1(Z_{1i}^c-h_1\psi,Z_{1i}^d)}m(Z_{1i}^c-h_1\psi,Z_{1i}^d) \\
& \times(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}^c-h_1\psi,Z_{1i}^d))^{-1}f_z(Z_{1i}^c-h_1\psi,Z_{1i}^d,Z_{2t})d\psi dZ_{2t}^c \\
= & \sum_{Z_{2t}^d}\int K_1(\psi)[\frac{g_k(Z_{1i},Z_{2t})-g_{2,k}(Z_{1i})}{\sigma^2(Z_{1i},Z_{2t})}m(Z_{1i})(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}))^{-1} \\
& +\sum_{|j|=1}^{s}\frac{\partial^j}{\partial(Z_{1t}^c)^j}\left(\frac{g_k(Z_{1i},Z_{2t})-g_{2,k}(Z_{1i})}{\sigma^2(Z_{1i},Z_{2t})}m(Z_{1i})(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}))^{-1}\right)\frac{(-h_1)^{|j|}\psi^j}{j!} \\
& +\sum_{|j|=s}[\frac{\partial^j}{\partial(Z_{1t}^c)^j}\left(\frac{g_k(Z_{1i*}^c,Z_{1i}^d,Z_{2t})-g_{2,k}(Z_{1i*}^c,Z_{1i}^d)}{\sigma^2(Z_{1i*}^c,Z_{1i}^d,Z_{2t})}m(Z_{1i*}^c,Z_{1i}^d)(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i*}^c,Z_{1i}^d))^{-1}\right) \\
& -\frac{\partial^j}{\partial(Z_{1t}^c)^j}\left(\frac{g_k(Z_{1i},Z_{2t})-g_{2,k}(Z_{1i})}{\sigma^2(Z_{1i},Z_{2t})}m(Z_{1i})(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}))^{-1}\right)]\frac{(-h_1)^{|j|}\psi^j}{j!}] \\
& \times[\frac{f_z(Z_{1i},Z_{2t})}{f_1(Z_{1i})}+\sum_{|l|=1}^{s}\frac{\partial^l}{\partial(Z_{1t}^c)^l}\frac{f_z(Z_{1i},Z_{2t})}{f_1(Z_{1i})}\frac{(-h_1)^{|l|}\psi^l}{l!} \\
& +\sum_{|l|=s}[\frac{\partial^l}{\partial(Z_{1t}^c)^l}\frac{f_z(Z_{1i*}^c,Z_{1i}^d,Z_{2t})}{f_1(Z_{1i*}^c,Z_{1i}^d)}-\frac{\partial^l}{\partial(Z_{1t}^c)^l}\frac{f_z(Z_{1i},Z_{2t})}{f_1(Z_{1i})}]\frac{(-h_1)^{|l|}\psi^l}{l!}]d\psi dZ_{2t}^c \\
= & \int K_1(\psi)d\psi E[\frac{g_k(Z_{1i},Z_{2t})-g_{2,k}(Z_{1i})}{\sigma^2(Z_{1i},Z_{2t})}m(Z_{1i})(E(\tfrac{1}{\sigma^2(\mathbf{Z_i^c})}|Z_{1i}))^{-1}|Z_{1i})+O(h_1^{s+1}) \\
= & O(h_1^{s+1}) \text{ uniformly at } Z_{1i}\in G_1\text{, where we use the definition of } g_{2,k}(Z_{1i}),
\end{aligned}
$$

and $Z_{1i*}^c=\lambda Z_{1i}^c+(1-\lambda)(Z_{1i}^c-h_1\psi)$ for some $\lambda\in(0,1)$.

So we have $\sup_{Z_{1i}\in G_1}|B_{211k1}|=O_p(L_{1n})$.

Similarly, define $B_{211k2}=\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}m(Z_{1i})$. We have $\sup_{Z_{1i}\in G_1}|B_{211k2}|=O_p(L_{1n})$. Thus,

$$
\begin{aligned}
B_{211k} = & \frac{1}{n}\sum_{i=1}^{n}[\frac{1}{\hat\sigma^2(\mathbf{Z_i})}-\frac{1}{\sigma^2(\mathbf{Z_i})}][B_{211k2}-B_{211k1}]+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_i})}[B_{211k2}-B_{211k1}] \\
= & [O_p(L_n)+O_p(n^{-\frac{1}{2}})]*O_p(L_{1n})+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_i})}[B_{211k2}-B_{211k1}] \\
= & o_p(n^{-\frac{1}{2}})+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_i})}\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t})}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})} \\
& \times\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1}(m(Z_{1i})-m(Z_{1t})) \\
= & o_p(n^{-\frac{1}{2}})+B_{2111k}.
\end{aligned}
$$

So in all, we have $B_{21k}=o_p(n^{-\frac{1}{2}})+B_{2111k}$.

(e) Following arguments in (b) and (c) above, we have

$$
\begin{aligned}
B_{22k} = & \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))\left[\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}\right]^{-1} \\
& \times\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}\epsilon_i \\
= & o_p(n^{-\frac{1}{2}})+\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))\left[f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1} \\
& \times\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}\epsilon_i \\
= & o_p(n^{-\frac{1}{2}})+B_{221k}.
\end{aligned}
$$

With result in (d) above, we write

$$
\begin{aligned}
B_{221k} = & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}\epsilon_i\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1} \\
= & \frac{1}{n}\sum_{i=1}^{n}[\frac{1}{\hat\sigma^2(\mathbf{Z_i^c})}-\frac{1}{\sigma^2(\mathbf{Z_i^c})}]\epsilon_i\frac{B_{211k2}}{m(Z_{1i})}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_i^c})}\epsilon_i\frac{B_{211k2}}{m(Z_{1i})} \\
= & o_p(n^{-\frac{1}{2}})+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_i^c})}\epsilon_i\frac{1}{nh_1^{l_{1c}}}\sum_{t=1}^{n}K_1\left(\frac{Z_{1i}^c-Z_{1t}^c}{h_1}\right)I(Z_{1i}^d=Z_{1t}^d)\frac{(g_k(\mathbf{Z_t})-g_{2,k}(Z_{1t}))}{f_1(Z_{1t})\sigma^2(\mathbf{Z_t})}\left[E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t})\right]^{-1} \\
= & o_p(n^{-\frac{1}{2}})+B_{2211k},
\end{aligned}
$$

where the second to the last inequality we use the fact that $\frac{1}{n}\sum_{i=1}^{n}|\epsilon_i|=O_p(1)$, result in (d) above, and result (1)(i)(a).

(f) We revisit the term B_{11k} in (a). Using results (1)(a), (2)(a), (2)(d) in Theorem 1, we obtain

$$
\begin{aligned}
& \hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) \\
= \; & [\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}]^{-1} \\
& \times \frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}(m(Z_{1i}) - m(Z_{1t})) \\
= \; & [O_p(L_{1n}) + (f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}] \\
& \times \frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}(m(Z_{1i}) - m(Z_{1t})) \\
= \; & o_p(n^{-\frac{1}{2}}) + (f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}(m(Z_{1i}) - m(Z_{1t})) \\
& \hat{E}^{*I}(\epsilon_t|Z_{1t}) \\
= \; & [\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}]^{-1}\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}\epsilon_i \\
= \; & [O_p(L_{1n}) + (f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}]\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}\epsilon_i \\
= \; & o_p(n^{-\frac{1}{2}}) + (f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1}\frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}\epsilon_i
\end{aligned}
$$

$$
\begin{aligned}
B_{11k} = \; & \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})][\hat{E}^{*I}(m(Z_{1t})|Z_{1t}) - m(Z_{1t}) + \hat{E}^{*I}(\epsilon_t|Z_{1t})] \\
= \; & o_p(n^{-\frac{1}{2}}) + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})](f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
& \times \frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}(m(Z_{1i}) - m(Z_{1t}) \\
& + \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma^2(\mathbf{Z_t})}[g_k(\mathbf{Z_t}) - g_{2,k}(Z_{1t})](f_1(Z_{1t})E(\frac{1}{\sigma^2(\mathbf{Z_t})}|Z_{1t}))^{-1} \\
& \times \frac{1}{nh_1^{l_{1c}}}\sum_{i=1}^{n} K_1\left(\frac{Z_{1i}^c - Z_{1t}^c}{h_1}\right) I(Z_{1i}^d = Z_{1t}^d)\frac{1}{\sigma^2(\mathbf{Z_i})}\epsilon_i \\
= \; & o_p(n^{-\frac{1}{2}}) + B_{2111k} + B_{2211k}
\end{aligned}
$$

So in all, we have $B_k = B_{1k} - B_{2k} = o_p(n^{-\frac{1}{2}})$, which concludes the proof.

(2) The result follows from (1) and Theorem 1.

Theorem 3: *Proof.* It follows from proof of Theorem 1 (1) and Theorem 2(1)(i).