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ALGEBRAICALLY INVARIANT EXTENSIONS OF σ-FINITE MEASURES ON EUCLIDEAN SPACE

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ABSTRACT. Let G be a group of algebraic transformations of $\mathbf{R}^n$, i.e., the group of functions generated by bijections of $\mathbf{R}^n$ of the form $(f_1, \ldots, f_n)$ where each f_i is a rational function with coefficients in $\mathbf{R}$ in n-variables. For a function $\gamma: G \to (0, \infty)$ we say that a measure μ on $\mathbf{R}^n$ is γ-invariant when $\mu(g[A]) = \gamma(g) \cdot \mu(A)$ for every $g \in G$ and every μ-measurable set A. We will examine the question: "Does there exist a proper γ-invariant extension of μ?" We prove that if μ is σ-finite then such an extension exists whenever G contains an uncountable subset of rational functions $H \subset (\mathbf{R}(X_1, \ldots, X_n))^n$ such that $\mu(\{x: h_1(x) = h_2(x)\}) = 0$ for all $h_1, h_2 \in H$, $h_1 \neq h_2$. In particular if G is any uncountable subgroup of affine transformations of $\mathbf{R}^n$, $\gamma(g)$ is the absolute value of the Jacobian of $g \in G$ and μ is a γ-invariant extension of the n-dimensional Lebesgue measure then μ has a proper γ-invariant extension. The conclusion remains true for any σ-finite measure if G is a transitive group of isometries of $\mathbf{R}^n$. An easy strengthening of this last corollary gives also an answer to a problem of Harazisvili.

0. INTRODUCTION: NOTATION AND HISTORY

Our terminology related to algebra, measure theory, set theory and model theory follows [La, Ru, Je and CK] respectively.

Throughout the paper a measure on a set X will stand for a nontrivial positive σ-additive measure, i.e., a function $\mu: \mathscr{M} \to [0, \infty]$ defined on a σ-algebra $\mathscr{M}$ of subsets of X containing all *singletons* such that

(i) $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ for all pairwise disjoint sets A_i from $\mathscr{M}$,
(ii) $\mu(\{x\}) = 0$ for all $x \in X$,
(iii) $0 < \mu(A) < \infty$ for some $A \in \mathscr{M}$.

If $\mu: \mathscr{M} \to [0, \infty]$ is a measure on X and $A \subset X$ then the inner measure of A is defined in the standard way: $\mu_*(A) = \sup\{\mu(B): B \subset A \& B \in \mathscr{M}\}$.

A measure on X is said to be σ-finite if X is a countable union of sets of finite measure. A measure μ is complete if all subsets of every set of μ measure zero are μ-measurable.

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If G is a group of bijections of a set X then a measure μ on X is said to be G-invariant provided μ is γ-invariant where $\gamma(g) = 1$ for all $g \in G$.

For example, if A_n is a group of affine transformations of $\mathbf{R}^n$ then every element of A_n is uniquely represented as a superposition $T \circ L$ where T is a translation and L is a linear transformation of $\mathbf{R}^n$. Let $\gamma: A_n \to (0, \infty)$, where $\gamma(T \circ L)$ is defined as the absolute value of the Jacobian of L. Then m, the n-dimensional Lebesgue measure, is γ-invariant. Moreover, if G_n is a group of isometries of $\mathbf{R}^n$ then $G_n \subset A_n$ and m is G_n-invariant.

We say that a measure $\nu: \mathcal{N} \to [0, \infty]$ on a set X is an extension of a measure $\mu: \mathcal{M} \to [0, \infty]$ defined on the same set X if $\mathcal{M} \subset \mathcal{N}$ and $\nu(A) = \mu(A)$ for every $A \in \mathcal{M}$. Moreover, an extension is proper if $\mathcal{M} \neq \mathcal{N}$.

For a group G of bijections of a set X we say that a set $N \subset X$ is G-absolutely negligible if for every G-invariant σ-finite measure μ on X and for every countable set $\{g_r: r = 0, 1, 2, \ldots\} \subset G$ we have $\mu_*(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ (or, equivalently, if for every G-invariant σ-finite measure μ on X there exists a G-invariant extension ν of μ such that $\nu(N) = 0$; compare Proposition 1.2(b)).

We say that a bijection g of $\mathbf{R}^n$ is an algebraic transformation of $\mathbf{R}^n$ if g is generated by bijections of $\mathbf{R}^n$ from the set $(\mathbf{R}(X_1, \ldots, X_n))^n$. For an algebraic transformation g of $\mathbf{R}^n$ we say that g is defined over the field $L \subset \mathbf{R}$ if g is generated by some bijections of $\mathbf{R}^n$ from $(L(X_1, \ldots, X_n))^n$. For example, the functions

$$f(x, y) = (x^3 + 1, (y + 7)^5), \quad g(x, y) = \left(x, y + \frac{1}{x^2 + 1}\right)$$

and

$$(f^{-1} \circ g)(x, y) = \left((x - 1)^{1/3}, \left(y + \frac{1}{x^2 + 1}\right)^{1/5} - 7\right)$$

are algebraic transformations of $\mathbf{R}^2$ defined over $\mathbf{Q}$. Notice also that isometries and, more generally, nonsingular affine transformations of $\mathbf{R}^n$ are algebraic transformations of $\mathbf{R}^n$ that belong to the set $(\mathbf{R}(X_1, \ldots, X_n))^n$.

Now let G be the group of all isometries of $\mathbf{R}^n$ and let μ be a G-invariant σ-finite measure on $\mathbf{R}^n$. Can we find a proper G-invariant extension of μ?

This question has been discussed several times in the literature. In 1935 Szpilrajn proved that Lebesgue measure on $\mathbf{R}^n$ has a proper isometrically invariant extension (see [Sz]). In the same paper, he stated Sierpiński's question: "Does there exist a maximal isometrically invariant extension of Lebesgue measure on $\mathbf{R}^n$?" A negative answer to this question, i.e., the theorem "every isometrically invariant measure that extends Lebesgue measure on $\mathbf{R}^n$ has a proper isometrically invariant extension," was proved by several mathematicians. The first result of that kind was obtained independently by Pkhakadze (in 1958, see [Pk]) and Hulanicki (in 1962, see [Hu]) under the additional set-theoretical assumption that there does not exist a real measurable cardinal less

than or equal to continuum 2^{ω}, i.e., that there is no measure on $\mathbf{R}$ defined on all subsets of $\mathbf{R}$. In 1977, Harazisvili got the full result stated above without any set-theoretical assumptions for the one dimensional case, i.e., for $n = 1$ (see [Ha1]). Finally in 1983, Ciesielski and Pelc generalized Harazisvili's result to all n-dimensional Euclidean spaces $\mathbf{R}^n$ (see [CP]; for more historical details of this issue see also [Ci]). In the same paper Ciesielski and Pelc stated the problem of characterizing those groups G of isometries of $\mathbf{R}^n$ for which every σ-finite G-invariant measure has a proper G-invariant extension (see [CP, p. 6]). A more technical version of the same problem, i.e., the problem of characterizing those groups G of isometries of $\mathbf{R}^n$ for which $\mathbf{R}^n$ is a union of countable many G-absolutely negligible sets, was also stated by Harazisvili in [Ha2].

In the present paper we will consider a generalization of this problem to the case of γ-invariant measure where $\gamma: G \to (0, \infty)$ and G is a group of algebraic transformations of $\mathbf{R}^n$. In particular our main theorem (see Abstract, or Theorem 3.1) implies that

"if G is a transitive group of isometries of $\mathbf{R}^n$ then $\mathbf{R}^n$ is a countable union of G-absolutely negligible sets."

The above fact has been proved earlier by Harazisvili under the assumption of the continuum hypothesis (see [Ha2]). He also asked whether it is possible to remove this assumption from his theorem. Our results give an affirmative answer to this question.

The proof of our main theorem 3.1 uses a generalization of the technique of Ciesielski and Pelc [CP, Theorem 2.1, pp. 4–6]. The author wishes to thank Jan Mycielski for numerous important remarks about former versions of this paper. In particular it was Mycielski's suggestion to replace in the proof of [CP, Theorem 2.1] the linear basis of $\mathbf{R}$ over $\mathbf{Q}$ by a transcendence basis of $\mathbf{R}$ over $\mathbf{Q}$ and to study in this way algebraic transformations of $\mathbf{R}^n$. Compare also the paper of Weglorz [We, Theorem 2.4] which was influenced by Mycielski in a similar way.

The author wishes also to thank Piotr Zakrzewski for calling his attention to the paper of Harazisvili [Ha2] and for other helpful remarks.

1. Measure theoretic preliminaries

In what follows we will need the following proposition essentially due to Szpilrajn (see [Sz, §2]).

Proposition 1.1. *Let $\gamma: G \to (0, \infty)$ where G is a group of bijections of a set X and let $\mu: \mathscr{M} \to [0, \infty]$ be a γ-invariant measure on X. If a family $\mathscr{A}$ of subsets of X is such that*

(i) *$\mathscr{A}$ is closed under countable union,*
(ii) *if $A \in \mathscr{A}$ and $g \in G$ then $g[A] \in \mathscr{A}$,*
(iii) *every $A \in \mathscr{A}$ has μ inner measure zero,*

then μ *has a* γ*-invariant extension* $\nu:\mathscr{N} \to [0,\infty]$ *such that* $\mathscr{A} \subset \mathscr{N}$ *and* $\nu(A) = 0$ *for every* $A \in \mathscr{A}$.

The construction of such an extension is very simple. If $\mathscr{I}$ is an ideal of subsets of X generated by the family $\mathscr{A}$, and $\mathscr{N}$ stands for a σ-algebra generated by $\mathscr{M} \cup \mathscr{I}$ then all elements of $\mathscr{N}$ are of the form $(M \cup I_1)\backslash I_2$ where $M \in \mathscr{M}$ and $I_1, I_2 \in \mathscr{I}$. It is easy to see that $\nu:\mathscr{N} \to [0,\infty]$ such that $\nu((M \cup I_1)\backslash I_2) = \mu(M)$ is a well-defined γ-invariant measure on X extending μ.

In the proof of the next proposition, we use a method which goes back to Harazisvili's paper [Ha1] (see also [CP, Proposition 1.9, p. 4]).

Proposition 1.2. *Let* G *be a group of bijections of* X, $\gamma: G \to (0,\infty)$ *and let* μ *be a* γ*-invariant* σ*-finite measure on* X.

(a) *If* $N \subset X$ *is such that there is an uncountable set* $H \subset G$ *such that* $\mu_*(h_1[N] \cap h_2[N]) = 0$, *for distinct* $h_1, h_2 \in H$, *then* $\mu_*(N) = 0$.

(b) *If* $N \subset X$ *is such that for every countable set* $\{g_r: r = 0,1,2,\dots\} \subset G$ *we have* $\mu_*(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ *then there exists a* γ*-invariant extension* ν *of* μ *such that* $\nu(N) = 0$.

(c) *Moreover if* $X = \bigcup_{k=0}^{\infty} N_k$ *where each* N_k *satisfies the assumption of* (b) *then* μ *has a proper* γ*-invariant extension.*

Proof. (a) If $M \in \mathscr{M}$ is a subset of N then $\mu(h_1[M] \cap h_2[M]) = 0$ for every distinct h_1, h_2 from H. But $\mu(h[M]) = \gamma(h) \cdot \mu(M)$ and $\gamma(h) \neq 0$ for every h from H. Hence, σ-finiteness of μ implies that $\mu(M) = 0$ and so $\mu_*(N) = 0$.

(b) By Proposition 1.1 it is enough to notice that every element of the family $\mathscr{A} = \{\bigcup_{r=0}^{\infty} g_r[N]: g_r \in G \text{ for } r = 0,1,2,\dots\}$ has μ inner measure 0.

(c) By part (b), for each $k = 0,1,2,\dots$ there is a γ-invariant extension ν_k of μ such that $\nu_k(N_k) = 0$. But all N_k's cannot have μ measure zero. So some ν_k must be a proper extension of μ.

In what follows, we will also use the following well-known fact. For the complex case the proof (using the Jensen's Inequality) can be found in [GR, p. 9]. The direct proof follows also from Fubini's theorem.

Proposition 1.3. *If* $f: \mathbf{R}^n \to \mathbf{R}$ *is a nonzero real analytic function then the set* $Z = \{a \in \mathbf{R}^n: f(a) = 0\}$ *has Lebesgue measure zero. In particular, if* $h, g \in (\mathbf{R}(X_1, \dots, X_n))^n$ *are different algebraic transformations of* $\mathbf{R}^n$ *then the set* $\{a \in \mathbf{R}^n: h(a) = g(a)\}$ *has Lebesgue measure zero.*

2. Algebraic preliminaries

A field $L \subset \mathbf{R}$ is said to be algebraically closed in $\mathbf{R}$ if $L = M \cap \mathbf{R}$ where $M \subset \mathbf{C}$ is an algebraic closure of L. Notice, that an algebraically closed field in $\mathbf{R}$ is real closed (i.e. satisfies the theory of real closed fields) in the sense defined in [CK or Ro]. The smallest field algebraically closed in $\mathbf{R}$ containing $L \subset \mathbf{R}$ is called a real closure of L and it will be denoted by $\mathrm{cl}_{\mathbf{R}}(L)$. The algebraic closure of a field K will be denoted by $\mathrm{cl}(K)$.

The next proposition will be used only in the case of algebraic transformation g such that $g^{-1} \in (\mathbf{R}(X_1, \ldots, X_n))^n$. In this case this is a well-known fact and can be proved using standard algebraic technic. However we like to prove it in more general form (that possibly can be used to answer Problem 3 stated in the end of the paper). For this we will need the following model-theoretic definition (compare e.g. [CK]).

A model $\mathscr{L}$ is said to be an elementary submodel of a model $\mathscr{R}$ if $\mathscr{L} \subset \mathscr{R}$ and for every first order formula $\varphi(x_1, \ldots, x_m)$ and any parameters $a_1, \ldots, a_m$ from $\mathscr{L}$ the model $\mathscr{L}$ satisfies $\varphi(a_1, \ldots, a_m)$ if and only if $\mathscr{R}$ satisfies $\varphi(a_1, \ldots, a_m)$.

A theory T is said to be model complete if and only if for all models $\mathscr{L}$ and $\mathscr{R}$ of T, if $\mathscr{L} \subset \mathscr{R}$ then $\mathscr{L}$ is an elementary submodel of $\mathscr{R}$.

We need the following important theorem of A. Robinson (see [CK, p. 110] or [Ro, §3.3]).

Theorem 2.1. *The theory T of real closed fields is model complete. In particular if $L \subset \mathbf{R}$ is a real closed field then L is an elementary submodel of $\mathbf{R}$.*

As a corollary of this fact we easily obtain

Proposition 2.1. *If g is an algebraic transformation of $\mathbf{R}^n$ defined over a real closed field $L \subset \mathbf{R}$ then*

$$g[L^n] = L^n. \tag{2.1}$$

Proof. A first order formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n)$ defined by $g(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ has as its parameters only elements from L. If $a = (a_1, \ldots, a_n) \in L^n$ then $\mathbf{R}$ satisfies $\exists y_1 \cdots \exists y_n \varphi(a_1, \ldots, a_n, y_1, \ldots, y_n)$ and so does L (by Theorem 2.1), i.e. $g(a_1, \ldots, a_n) \in L^n$. This proves $g[L^n] \subset L^n$. To show the converse inclusion it is enough to consider the formula $\exists x_1 \cdots \exists x_n \varphi(x_1, \ldots, x_n, a_1, \ldots, a_n)$.

3. The main theorem

From now on let $\mathscr{B}$ denote a transcendence base of $\mathbf{R}$ over $\mathbf{Q}$.

Now we are ready to prove our main lemma.

Lemma 3.1. *Let $H \subset (\mathbf{R}(X_1, \ldots, X_n))^n$ be an uncountable set of algebraic transformations of $\mathbf{R}^n$. Then there exists an uncountable set $H' \subset H$, a finite set $A \subset \mathscr{B}$ and, for every $h \in H'$, a finite set $A_h \subset \mathscr{B} \setminus A$ with the following properties:*

(1) *each $h \in H'$ (and so h^{-1}) is defined over the field $\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(A \cup A_h))$;*

(2) *$A_{h_1} \cap A_{h_2} = \emptyset$ for distinct $h_1, h_2 \in H'$;*

(3) *for every $h_1, h_2 \in H'$ if $L = \mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B} \setminus (A_{h_1} \cup A_{h_2})))$ then $a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n]$ implies $h_1(a) = h_2(a)$, i.e., $h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\}$.*

Proof. In the definition of each $h \in H$ we use only finitely many parameters (i.e. coefficients) so for every $h \in H$ there exists a finite set $B_h \subset \mathscr{B}$ such that

$$h = (h_1, \dots, h_n) \in [\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(B_h))(X_1, \dots, X_n)]^n.$$

Using for the family $\{B_h : h \in H\}$ the Δ-system argument (see e.g. [Je, Lemma 22.6, p. 226]) we can find an uncountable set $H_0 \subset H$, a finite set $A \subset \mathscr{B}$, a natural number m and, for every $h \in H_0$, a set A_h such that

(i) $B_h = A \cup A_h$, and $A \cap A_h = \emptyset$,
(ii) $A_{h_1} \cap A_{h_2} = \emptyset$ for distinct $h_1, h_2 \in H_0$,
(iii) A_h has exactly m elements.

Thus for the family H_0, the sets A, A_h $(h \in H_0)$ already satisfy (1) and (2). Therefore it is enough to find an uncountable $H' \subset H_0$ which satisfies (3). We will do this in such a way that all elements of H' will have the same definitions with parameters from $\mathscr{B}$.

Let $Z = \{Z_1, \dots, Z_m\}$ be a set of variables and, for $h \in H_0$, let $\sigma'_h : A_h \to Z$ be a bijection. Then we can extend σ'_h to a field isomorphism σ''_h from $\mathrm{cl}(\mathbf{Q}(\mathscr{B})) = \mathbf{C}$ to $\mathrm{cl}(\mathbf{Q}(\mathscr{B}\setminus A_h)(Z))$ in such a way that $\sigma''_h(a) = a$ for every $a \in \mathrm{cl}(\mathbf{Q}(\mathscr{B}\setminus A_h))$. Let us extend σ''_h to $\sigma_h : [\mathrm{cl}(\mathbf{Q}(\mathscr{B}))(X_1, \dots, X_n)]^n \to [\mathrm{cl}(\mathbf{Q}(\mathscr{B}\setminus A_h)(Z))(X_1, \dots, X_n)]^n$. But $\sigma_h(h) \in [\mathrm{cl}(\mathbf{Q}(A \cup Z))(X_1, \dots, X_n)]^n$ and the field $\mathrm{cl}(\mathbf{Q}(A \cup Z))$ is countable.

Define $H' \subset H_0$ as an uncountable set with the property

$$(*) \qquad \sigma_{h_1}(h_1) = \sigma_{h_2}(h_2) \quad \text{for every } h_1, h_2 \in H'.$$

We prove that H' satisfies (3).

Let $a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n]$, where $L = \mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}\setminus(A_{h_1} \cup A_{h_2})))$ and $h_1, h_2 \in H'$. Notice that $a \in L^n$ as, by Proposition 2.1, (1) and (2),

$$\begin{aligned} a \in h_1^{-1}[L^n] \cap h_2^{-1}[L^n] &\subset h_1^{-1}[(\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}\setminus A_{h_2})))^n] \cap h_2^{-1}[(\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}\setminus A_{h_1})))^n] \\ &= (\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}\setminus A_{h_2})))^n \cap (\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}\setminus A_{h_1})))^n = L^n. \end{aligned}$$

Put $h_1(a) = b_1$ and $h_2(a) = b_2$. Thus $b_1, b_2 \in L^n$. We have to prove that $b_1 = b_2$. But, by $(*)$ and the fact that $\sigma_{h_1}(c) = c = \sigma_{h_2}(c)$ for every $c \in L^n$,

$$\begin{aligned} b_1 &= \sigma_{h_1}(b_1) = \sigma_{h_1}(h_1(a)) = \sigma_{h_1}(h_1)(\sigma_{h_1}(a)) = \sigma_{h_1}(h_1)(a) \\ &= \sigma_{h_2}(h_2)(a) = \sigma_{h_2}(h_2)(\sigma_{h_2}(a)) = \sigma_{h_2}(h_2(a)) = \sigma_{h_2}(b_2) = b_2. \end{aligned}$$

This finishes the proof of Lemma 3.1.

As a next step we will prove an essential part of the assumptions of Proposition 1.2.

Lemma 3.2. *If G is a group of algebraic transformations of $\mathbf{R}^n$ and $H \subset (\mathbf{R}(X_1, \dots, X_n))^n$ is an uncountable subset of G then there exists a countable*

family of sets $\{N_k: k = 0, 1, 2, \dots\}$ *such that* $\mathbf{R}^n = \bigcup_{k=0}^{\infty} N_k$ *and that each* N_k *satisfies the condition*:

for every countable set $\{g_r: r = 0, 1, 2, \dots\} \subset G$ *there is an uncountable set* $H_0 \subset H$ *such that for every distinct* $h_1, h_2 \in H_0$

$$h_1^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \cap h_2^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \subset \{a \in \mathbf{R}^n: h_1(a) = h_2(a)\}. \tag{3.1}$$

Proof. Let $\mathscr{B}$ be a transcendence base of $\mathbf{R}$ over Q and let $H' \subset H$, A and A_h be as in Lemma 3.1. We choose an increasing sequence $\mathscr{B}_0 \subset \mathscr{B}_1 \subset \mathscr{B}_2 \subset \cdots$ of subsets of $\mathscr{B}$ in such a way that $\mathscr{B} = \bigcup_{k=1}^{\infty} \mathscr{B}_k$ and for every k the set

$$H^k = \{h \in H': A_h \subset \mathscr{B}_{k+1} \setminus \mathscr{B}_k\} \tag{$*$}$$

is uncountable.

Define $N_k = [\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B}_k))]^n$. Then $\bigcup_{k=0}^{\infty} N_k = \mathbf{R}^n$.

Let us fix $\{g_r: r = 0, 1, 2, \dots\} \subset G$ and a natural number k. Choose also a countable set $\mathscr{A} \subset \mathscr{B}$ such that $A \subset \mathscr{A}$ and every g_r is defined over $\mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{A}))$. Let $H_0 = \{h \in H^{k+1}: A_h \cap \mathscr{A} = \varnothing\}$.

By $(*)$ the set H_0 is uncountable.

Let us fix arbitrary distinct $h_1, h_2 \in H_0$ and let $L = \mathrm{cl}_{\mathbf{R}}(\mathbf{Q}(\mathscr{B} \setminus (A_{h_1} \cup A_{h_2})))$. Then, by $(*)$ and definitions of H_0 and N_k, we can conclude that $N_k \subset L^n$ and the g_r's are defined over L. Hence, by Proposition 2.1,

$$h_1^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \cap h_2^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \subset h_1^{-1}\left[\bigcup_{r=0}^{\infty} g_r[L^n]\right] \cap h_2^{-1}\left[\bigcup_{r=0}^{\infty} g_r[L^n]\right] = h_1^{-1}[L^n] \cap h_2^{-1}[L^n]$$

and, by (3) of Lemma 3.1, $h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\}$.

Therefore

$$h_1^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \cap h_2^{-1}\left[\bigcup_{r=0}^{\infty} g_r[N_k]\right] \subset h_1^{-1}[L^n] \cap h_2^{-1}[L^n] \subset \{a: h_1(a) = h_2(a)\}.$$

This finishes the proof of Lemma 3.2.

Theorem 3.1. *Let* G *be a group of algebraic transformations of* $\mathbf{R}^n$, $\gamma: G \to (0, \infty)$ *and let* μ *be a* γ*-invariant* σ*-finite measure on* $\mathbf{R}^n$. *If* G *has an uncountable subset* $H \subset (\mathbf{R}(X_1, \dots, X_n))^n$ *with the property*

$$\mu_*(\{a: h_1(a) = h_2(a)\}) = 0 \quad \text{for every } h_1, h_2 \in H, h_1 \neq h_2 \tag{3.2}$$

then μ *has a proper* γ*-invariant extension.*

Proof. By (3.2) and Lemma 3.2 we have $\mathbf{R}^n = \bigcup_{k=0}^{\infty} N_k$ where, by Proposition 1.2(a), $\mu_*(\bigcup_{r=0}^{\infty} g_r[N_k]) = 0$ for every countable set $\{g_r: r = 0, 1, 2, \dots\} \subset G$

and every $k = 0,1,2,\dots$. Hence, by Proposition 1.2(c), μ has a proper γ-invariant extension.

Corollary 3.1. *Let G be a group of algebraic transformations of $\mathbf{R}^n$, $\gamma: G \to (0,\infty)$ and let μ be a γ-invariant σ-finite measure on $\mathbf{R}^n$. If at least one of the following conditions holds*

(C1) *G contains uncountably many translations;*

(C2) *μ extends the n-dimensional Lebesgue measure and the set $G \cap (\mathbf{R}(X_1,\dots,X_n))^n$ is uncountable;*

then μ has a proper γ-invariant extension.

Proof. It is enough to show that both (C1) and (C2) imply (3.2).

If (C1) holds and H is an uncountable set of translations then for every $h_1, h_2 \in H$, $h_1 \neq h_2$ the set $\{a: h_1(a) = h_2(a)\}$ is empty, so (3.2) is satisfied.

If (C2) holds then (3.2) is implied by Proposition 1.3.

To solve Harazisvili's problem we will need the following lemma due to Harazisvili (see [Ha2, Remark 2, p. 507]).

Lemma 3.3. *Let G be a transitive group of isometries of $\mathbf{R}^n$, i.e., such that for every $a, b \in \mathbf{R}^n$ there exists $g \in G$ with the property $g(a) = b$. If $A \subset \mathbf{R}^n$ is a countable union of proper affine hyperplanes of $\mathbf{R}^n$ than A is G-absolutely negligible.*

Proof. For $k \leq n$ let $\mathscr{A}_k$ denote the family of countable unions of affine hyperplanes of $\mathbf{R}^n$ of dimension less than k. We prove by induction on $k \leq n$ that elements of $\mathscr{A}_k$ are G-absolutely negligible.

So let $k < n$ be such that the elements of $\mathscr{A}_k$ are G-absolutely negligible.

Let us fix an arbitrary $A \in \mathscr{A}_{k+1}$, a G-invariant σ-finite measure μ on $\mathbf{R}^n$ and a countable set $\{g_r: r = 0,1,2,\dots\} \subset G$. By Proposition 1.2(a) it is enough to find a sequence $\{h_\zeta: \zeta < \omega_1\} \subset G$ such that for every $\zeta < \eta < \omega_1$

$$\text{(a)} \qquad \mu_*\left(h_\zeta\left[\bigcup_{r=0}^{\infty} g_r[A]\right] \cap h_\eta\left[\bigcup_{r=0}^{\infty} g_r[A]\right]\right) = 0.$$

We will construct it by transfinite induction.

So let us assume that for some $\xi < \omega_1$ we have already constructed $\{h_\zeta: \zeta < \xi\} \subset G$ such that the condition (a) is satisfied for every $\zeta < \eta < \xi$. Let A_i and H_j $(i, j = 0,1,2,\dots)$ be affine hyperplanes of $\mathbf{R}^n$ of dimensions less than or equal to k and such that

$$\bigcup_{r=0}^{\infty} g_r[A] = \bigcup_{i=0}^{\infty} A_i \quad \text{and} \quad \bigcup_{\zeta<\xi} h_\zeta\left[\bigcup_{r=0}^{\infty} g_r[A]\right] = \bigcup_{j=0}^{\infty} H_j.$$

We have to find h_ξ such that

$$\mu_*\left(h_\xi\left[\bigcup_{i=0}^{\infty} A_i\right] \cap \bigcup_{j=0}^{\infty} H_j\right) = 0.$$

But if $h_\xi[A_i] \neq H_j$ then $h_\xi[A_i] \cap H_j \in \mathscr{A}_k$, i.e., by inductive hypothesis, it is enough to construct $h_\xi \in G$ such that

(b) $$h_\xi[A_i] \neq H_j \quad \text{for every } i, j = 0, 1, 2, \ldots .$$

Let $w \in \mathbf{R}^n$ represents a vector in $\mathbf{R}^n$ such that w is not parallel to any H_j $(j = 0, 1, 2, \ldots)$. Then for different reals a, b the distances

$$\operatorname{dist}(0, a \cdot w + H_j) \neq \operatorname{dist}(0, b \cdot w + H_j) \quad \text{for every } j = 0, 1, 2, \ldots .$$

So we can choose $b \in \mathbf{R}$ such that

(c) $$\operatorname{dist}(0, -b \cdot w + H_j) \neq \operatorname{dist}(0, A_i) \quad \text{for every } i, j = 0, 1, 2, \ldots .$$

Now let $h_\xi \in G$ be such that $h_\xi(0) = b \cdot w$. We prove that such h_ξ satisfies (b).

By way of contradiction let us assume that for some i and j

(d) $$h_\xi[A_i] = H_j.$$

But $h_\xi = T \circ L$, where L is an isometry of $\mathbf{R}^n$ preserving origin and T is a translation such that $T(x) = x + b \cdot w$ for every $x \in \mathbf{R}^n$. Hence, by (d), $L[A_i] = T^{-1}[H_j] = -b \cdot w + H_j$ and so

$$\operatorname{dist}(0, -b \cdot w + H_j) = \operatorname{dist}(0, L[A_i]) = \operatorname{dist}(0, A_i)$$

contradicting (c).

Thus we proved that h_ξ satisfies (b). This finishes the proof of the lemma.

Theorem 3.2. *If G is a transitive group of isometries of $\mathbf{R}^n$ then $\mathbf{R}^n$ is a countable union of G-absolutely negligible sets. In particular every σ-finite G-invariant measure on $\mathbf{R}^n$ has a proper G-invariant extension.*

Proof. Let $\{N_k : k = 0, 1, 2, \ldots\}$ be the family given in Lemma 3.2 where $H = G$. Then by Lemma 3.3 and Proposition 1.2(a) we have $\mu_*(\bigcup_{r=0}^{\infty} g_r[N]) = 0$ for every countable set $\{g_r : r = 0, 1, 2, \ldots\} \subset G$ and every $k = 0, 1, 2, \ldots$. Hence each N_k is G-absolutely negligible.

Generalizations, examples and problems

1. Let us remark first that although we have stated Theorem 3.1 only for measures on $\mathbf{R}^n$ the theorem can be generalized for measures on K^n where K is either a real closed or algebraically closed field, since the theory of algebraic closed fields is also model complete (see [CK, p. 110]). Moreover, in the case of algebraically closed fields, the assumptions that $H \subset (K(X_1, \ldots, X_n))^n$ may be dropped.

2. If $X \subset K^n$ where K is as above and we define algebraic transformations on X in natural way, i.e., by functions generated by bijections of X from $(K(X_1, \ldots, X_n))^n$, then we can prove Theorem 3.1 for measures on X. In particular we can conclude that it does not exist a maximal isometrically invariant extension of Lebesgue measure on n-dimensional sphere S^n.

3. Theorem 3.1 and its generalizations as in 1 and 2 can be also proved for complex measures (see [Ru, Chapter 6]).

4. For the cardinal number κ we say that a measure μ on a set X is κ-finite if X is a union of κ many sets of finite measure. Theorem 3.1 can be also generalized in the following way:

"Let κ be a cardinal number, G be a group of algebraic transformations of $\mathbf{R}^n$, $\gamma: G \to (0,\infty)$ and let μ be a γ-invariant κ-finite measure on $\mathbf{R}^n$. If G has a subset $H \subset (\mathbf{R}(X_1, \ldots, X_n))^n$ of power greater than κ with the property

$$(*) \qquad \{a: h_1(a) = h_2(a)\} = \varnothing \quad \text{for every } h_1, h_2 \in H,\ h_1 \neq h_2,$$

then μ has a proper γ-invariant extension."

5. In 4 condition $(*)$ can be replaced by the original condition (3.2) if we assume in addition that the measure μ is κ^+-additive.

6. We can also generalize the results from 4 and 5 in the way described in 1 and 2.

7. By 4, if in particular κ is less than continuum 2^ω, G is a group of all isometries of $\mathbf{R}^n$ and μ is a κ-finite G-invariant measure then there exists a proper G-invariant extension of μ. However for κ equal to continuum 2^ω this cannot be proved as it was shown in [CP, Theorem 3.1].

8. An interesting example, suggested to the author by Jan Mycielski, can be obtained by considering a hyperbolic space H^n for $n \geq 2$. If we identify H^n with the model $\{(a_1, \ldots, a_{n+1}) \in \mathbf{R}^{n+1}: a_{n+1} > 0\}$ then the group G of all isometries of H^n is a group of algebraic transformations of $\mathbf{R}^n$ and contains uncountably many translations. Moreover G is not a subgroup of a group of affine transformations of $\mathbf{R}^n$ (see [MW or Be]). Let ν be the hyperbolic invariant measure on H^n induced by the Haar measure on G. So ν is a G-invariant σ-finite measure on H^n. Using the previous remarks and Corollary 3.1 we may conclude that the measure ν does not have a maximal G-invariant extension.

9. Now we discuss the assumptions of Theorem 3.1, in particular condition (3.2).

First we prove that uncountability of $H \subset G$ is essential (compare [Pe, Proposition 2.3, p. 14]).

Let G_0 be a group of all translations of $\mathbf{R}^1$ by rational numbers and let V be a Vitali set, i.e., $V \cap H$ is a one element set for each orbit H of G_0. If we assume that there is a real measurable cardinal less than or equal to continuum (see [Je]) then there is a measure $\nu_0: \mathscr{P}(V) \to [0,1]$, where $\mathscr{P}(V)$ is a family of all subsets of the set V. Define a measure $\mu: \mathscr{P}(\mathbf{R}^1) \to [0,\infty]$ by

$$(4.1) \qquad \mu(A) = \sum_{g \in G_0} \nu_0(g^{-1}[g[V] \cap A]).$$

It is easy to see that μ is G_0-invariant and σ-finite. But μ is defined on all subsets of $\mathbf{R}^1$ so it cannot have any proper extension.

10. It can be also proved that if there is a real measurable cardinal less than or equal to the continuum then for every countable group G of bijections of $\mathbf{R}^1$ there exists a G-invariant measure defined on $\mathscr{P}(\mathbf{R}^1)$, however this needs a little more careful definition.

11. The group G_0 defined in 9 is related to an interesting open problem of Andrzej Pelc (see [Pe, p. 27]).

Problem 1. Let μ be a G_0-invariant extension of Lebesgue measure on $\mathbf{R}^1$. Does there exist a proper G_0-invariant extension of μ?

12. The next example shows that we have to assume about G something more than only uncountability.

Example. Let G' be the group of all rotations of $\mathbf{R}^2$ about the origin and let $\nu:\mathscr{P}(\mathbf{R}^2) \to [0,\infty]$ be such that $\nu(A) = 1$ when $(0,0) \in A$ and $\nu(A) = 0$ otherwise. ν does not vanish at points, but still it is a G'-invariant measure. To correct this let μ and G_0 be as in Example 2 and let $\mu_1:\mathscr{P}(\mathbf{R}^3) \to [0,\infty]$ be a product measure of ν and μ, i.e., $\mu_1(A) = \mu(\{x:(0,0,x) \in A\})$. Then μ is σ-finite and G_1-invariant, where the group $G_1 = \{(g',g''): g' \in G' \text{ and } g'' \in G_1\}$ is uncountable. It is also obvious that μ_1 does not have any proper extension.

13. The reason that this example works is that μ_1 is concentrated on a set $S = \{0\}\times\{0\}\times\mathbf{R}$ while $g[S] = S$ for every $g \in G_1$ and the group $\{g|_S: g \in G_1\}$ is countable. This suggests the following

Definition. Let G be a group of bijections of a set X and μ be a G-invariant measure on X. We say that G is μ-essentially countable if there is a set $S \subset X$ such that $\mu(X\setminus S) = 0$, $g[S] = S$ for all $g \in G$ and the group $\{g|_S: g \in G\}$ is countable.

Problem 2. Let G be a group of algebraic transformations of $\mathbf{R}^n$ and μ be a G-invariant σ-finite measure of $\mathbf{R}^n$ such that G is not μ-essentially countable. Does μ have a proper G-invariant extension?

Recently the author has been informed that Piotr Zakrzewski proved the following result connected with the Problem 2: "If G is a group of isometries of $\mathbf{R}^n$ and $\mu:\mathscr{P}(\mathbf{R}^n) \to [0,\infty]$ is G-invariant then the group G is μ-essentially countable."

14. In the next example we will construct a γ-invariant measure μ on $\mathbf{R}^1$ where γ will not be given in a classical way by Jacobian.

Example. Let $G_0 = \{x^{3^n}: n \in \mathbf{Z}\}$ be a group of transformations of $\mathbf{R}^1$ and let $V \subset \mathbf{R}^1\setminus\{0\}$ be such that $(V \cup \{0\}) \cap H$ contains exactly one element for every orbit H of G. Let $\mu_0:\mathscr{P}(V) \to [0,1]$ be a measure. For $n \in \mathbf{Z}$ let $g_n(x) = x^{3^n}$ and let $\mu_n:\mathscr{P}(g_n[V]) \to [0,2^n]$ be defined by $\mu_n(g_n[A]) = 2^n \cdot \mu_0(A)$. Define

$\mu: \mathscr{P}(\mathbf{R}^1) \to [0, \infty]$ by

$$\mu(A) = \sum_{n \in \mathbf{Z}} \mu_n(g_n[A_n]) = \sum_{n \in \mathbf{Z}} 2^n \cdot \mu_0(A_n)$$

where $A_n \subset V$ are such that $A \setminus \{0\} = \bigcup_{n \in \mathbf{Z}} g_n[A_n]$.

It is easy to see that μ is a σ-finite measure. Moreover,

$$\mu(g_m[A]) = \mu\left(\bigcup_{n \in \mathbf{Z}} (g_m \circ g_n)[A_n]\right) = \sum_{n \in \mathbf{Z}} 2^{m+n} \cdot \mu_0(A_n) = 2^m \cdot \mu(A),$$

i.e., μ is γ_0-invariant where $\gamma_0: G_0 \to (0, \infty)$ is defined by $\gamma_0(g_n) = 2^n$. It is easy to see that γ_0 has little to do with a classical Jacobian.

Our group G_0 is countable. But if we consider a measure ν being a product measure of μ and a one-dimensional Lebesgue measure m then ν is a σ-finite γ-invariant where $\gamma: G \to (0, \infty)$, $G = \{(g_n, i): g_n \in G_0$ and i is an isometry of $\mathbf{R}^1\}$, and $\gamma(g_n, i) = 2^n$. It is also obvious that G is uncountable. Moreover about ν we can prove that if f is a homeomorphism of $\mathbf{R}^2$ and the system $\langle \mathbf{R}^2, \mu_f, G_f, \gamma_f \rangle$ is induced by f from the system $\langle \mathbf{R}^2, \mu, G, \gamma \rangle$ then G is not a subgroup of affine transformations of $\mathbf{R}^2$.

15. **Problem 3.** Is the assumption $H \subset (\mathbf{R}(X_1, \ldots, X_n))^n$ essential in Theorem 3.1?

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