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Category theorems concerning $\mathcal{I}$-density continuous functions

by

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Abstract. The $\mathcal{I}$-density topology $\mathcal{T}_\mathcal{I}$ on $\mathbb{R}$ is a refinement of the natural topology. It is a category analogue of the density topology [9, 10]. This paper is concerned with $\mathcal{I}$-density continuous functions, i.e., the real functions that are continuous when the $\mathcal{I}$-density topology is used on the domain and the range. It is shown that the family $\mathcal{C}_\mathcal{I}$ of ordinary continuous functions $f : [0,1] \to \mathbb{R}$ which have at least one point of $\mathcal{I}$-density continuity is a first category subset of $\mathcal{C}([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ equipped with the uniform norm. It is also proved that the class $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ of $\mathcal{I}$-density continuous functions, equipped with the topology of uniform convergence, is of first category in itself. These results remain true when the $\mathcal{I}$-density topology is replaced by the deep $\mathcal{I}$-density topology.

1. Introduction. The ordinary density topology on $\mathbb{R}$ is defined to be the collection of all subsets of $\mathbb{R}$ which have full Lebesgue density at every point [1]. The collection of all sets open in the density topology is denoted by $\mathcal{T}_\mathcal{N}$. The open sets in the ordinary topology are denoted by $\mathcal{T}_\mathcal{O}$. A function $f : \mathbb{R} \to \mathbb{R}$ is *approximately continuous* at a point x if it is continuous at x with the ordinary topology on the range and the density topology on the domain, and it is *density continuous* at x if it is continuous at x when $\mathcal{T}_\mathcal{N}$ is used on both the domain and the range. The spaces of everywhere ordinary continuous, approximately continuous and density continuous functions $f : \mathbb{R} \to \mathbb{R}$ are denoted by $\mathcal{C}_{\mathcal{O}\mathcal{O}}$, $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ and $\mathcal{C}_{\mathcal{N}\mathcal{N}}$, respectively.

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The structures of $\mathcal{C}_{\mathcal{O}\mathcal{O}}$ and $\mathcal{C}_{\mathcal{N}\mathcal{O}}$ are quite well understood, but $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is more difficult to study, mainly because it is closed neither under addition nor uniform convergence [4]. In particular, the relationship between density continuity and ordinary continuity is quite complicated. The definitions yield at once that $\mathcal{C}_{\mathcal{O}\mathcal{O}} \subset \mathcal{C}_{\mathcal{N}\mathcal{O}} \supset \mathcal{C}_{\mathcal{N}\mathcal{N}}$, but it is not hard to construct examples showing that

$$\mathcal{C}_{\mathcal{O}\mathcal{O}} \not\subset \mathcal{C}_{\mathcal{N}\mathcal{N}} \not\subset \mathcal{C}_{\mathcal{O}\mathcal{O}} \tag{1}$$

[4, 5]. The following theorem is known [6].

Theorem 1. *Let $\mathcal{C}_{\mathcal{O}\mathcal{O}}$ be given the topology of uniform convergence. If $\mathcal{C}$ is the subset of $\mathcal{C}_{\mathcal{O}\mathcal{O}}$ consisting of functions with at least one point of density continuity*, *then $\mathcal{C}$ is a first category subset of $\mathcal{C}_{\mathcal{O}\mathcal{O}}$.*

A combination of this theorem with the fact that every density continuous function is continuous on a dense open set can be used to show the following corollary [6].

Corollary 2. *If $\mathcal{C}_{\mathcal{N}\mathcal{N}}$ is given the topology of uniform convergence*, *then it is a first category subset of itself.*

Let $\mathcal{I}$ be the collection of all first category subsets of $\mathbb{R}$ and $E \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is an *$\mathcal{I}$-dispersion point* of E if for every increasing sequence of natural numbers $\{t_n\}$ there is a subsequence $\{t_{n_m}\}$ such that

$$\limsup_{m\in\mathbb{N}} t_{n_m}(E-x)\cap(-1,1) \in \mathcal{I}\,.$$

The point x is an *$\mathcal{I}$-density point* of E if it is an $\mathcal{I}$-dispersion point of E^c. Using this category density instead of Lebesgue density, the *$\mathcal{I}$-density topology*, $\mathcal{T}_{\mathcal{I}}$, is defined to consist of all Baire sets $E \subset \mathbb{R}$ such that every point of E is an $\mathcal{I}$-density point of E [9, 10].

$\mathcal{T}_{\mathcal{I}}$ has many properties in common with $\mathcal{T}_{\mathcal{N}}$, but $\mathcal{T}_{\mathcal{N}}$ is completely regular while $\mathcal{T}_{\mathcal{I}}$ is not. To remedy this, a topology coarser than $\mathcal{T}_{\mathcal{I}}$, called the *deep $\mathcal{I}$-density topology*, is introduced in the following way. A point x is a *deep $\mathcal{I}$-density point* of the set $E \subset \mathbb{R}$ if there is an ordinary closed set $F \subset E \cup \{x\}$ such that x is an $\mathcal{I}$-density point of F. Using the idea of deep $\mathcal{I}$-density, the *deep $\mathcal{I}$-density topology*, $\mathcal{T}_{\mathcal{D}}$, is defined in the by now familiar way [7]. $\mathcal{T}_{\mathcal{D}}$ is completely regular [7].

Given these two topologies based on $\mathcal{I}$-density, the $\mathcal{I}$-density continuous functions, $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, and deep $\mathcal{I}$-density continuous functions, $\mathcal{C}_{\mathcal{D}\mathcal{D}}$, are defined in the natural way.

It is reasonable to ask if the known properties of the density continuous functions can be proved in the case of the $\mathcal{I}$-density and deep $\mathcal{I}$-density continuous functions. The purpose of this paper is to establish Theorem 1 and Corollary 2 using these topologies in place of the density topology.

2. Comparison with $\mathcal{C}_{\mathcal{O}\mathcal{O}}$. The purpose of this section is to prove that the $\mathcal{I}$-density continuous and deep $\mathcal{I}$-density continuous functions have the same relationship to the ordinary continuous functions as do the density continuous functions. First, in analogy to (1), it is known [5] that

$$\mathcal{C}_{\mathcal{I}\mathcal{I}} \subset \mathcal{C}_{\mathcal{D}\mathcal{D}} \not\supset \mathcal{C}_{\mathcal{O}\mathcal{O}} \quad \text{and} \quad \mathcal{C}_{\mathcal{I}\mathcal{I}} \not\subset \mathcal{C}_{\mathcal{O}\mathcal{O}}\,. \tag{2}$$

Moreover, the containment in (2) is proper [5]. To give some idea of just how delicate the situation is, note the following lemma [2], [5, Example 5.7].

LEMMA 3. *There exists a convex C^∞ function that is not deep $\mathcal{I}$-density continuous.*

To proceed further toward the proof of a theorem similar to Theorem 1, some more definitions must be introduced.

If A is a measurable subset of $\mathbb{R}$, then its measure is denoted by $\mathrm{m}(A)$. A set of the form $\bigcup_{n\in\mathbb{N}}[a_n,b_n]$ or $\bigcup_{n\in\mathbb{N}}(a_n,b_n)$ is a *right interval set* if $b_n > a_n > b_{n+1} > 0$ for all $n \in \mathbb{N}$ and $a_n \to 0$. The definition of a *left interval set* is obvious. Any set which is the union of a right and left interval set is just called an *interval set*. The following lemmas give useful techniques for constructing $\mathcal{I}$-density open sets [3], [10, Theorem 2].

LEMMA 4. *If $B = \bigcup_{n\in\mathbb{N}}(a_n,b_n)$ is a right interval set and there exists a positive number c such that*

$$(b_n - a_n)/b_n > c$$

for every $n \in \mathbb{N}$, then 0 is not an $\mathcal{I}$-dispersion point of B.

LEMMA 5. *If $\bigcup_{n\in\mathbb{N}}[a_n,b_n]$ is a right interval set with*

$$\lim_{n\to\infty}(b_n - a_n)/b_n = 0\,,$$

then there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers such that 0 is an $\mathcal{I}$-dispersion point of $\bigcup_{m\in\mathbb{N}}[a_{n_m},b_{n_m}]$.

THEOREM 6. *Let $\mathcal{C}_{\mathcal{I}}$ denote the class of all continuous functions $f : [0,1] \to \mathbb{R}$ which have at least one point of $\mathcal{I}$-density continuity. Then $\mathcal{C}_{\mathcal{I}}$ is a first category subset of $\mathcal{C}([0,1])$.*

Proof. We will show that there exists a dense $\mathbf{G}_\delta$ subset E of $\mathcal{C} = \mathcal{C}([0,1])$ such that every $f \in E$ is nowhere $\mathcal{I}$-density continuous.

For every $n \in \mathbb{N}$ denote by D_n the set of all $f \in \mathcal{C}$ such that for every $i = 1,2,\ldots,2^n$, f is linear and nonconstant on every interval $[(i-1)2^{-n}, i2^{-n}]$. Note that $D_{n+1} \supset D_n$ for every $n \in \mathbb{N}$ and $D = \bigcup_{n\in\mathbb{N}} D_n$ is a dense subset of $\mathcal{C}$.

For $f \in \mathcal{C}$ define

$$\|f\|_n = \max_{i=1,2,\ldots,2^n} |f(i2^{-n}) - f((i-1)2^{-n})|\,.$$

We claim that for each open set U in $\mathcal{C}$, there exists an $n \in \mathbb{N}$ and a function $f \in D_n$ such that the ball in $\mathcal{C}$ centered at f of radius $\|f\|_n$ is entirely contained in U. To see this, first find an $m \in \mathbb{N}$ and an $f \in D_m$ such that $f \in U$. Since U is open, there is a $\delta > 0$ such that the open ball of radius δ centered at f is contained in U. Using the uniform continuity of f, we can find an $n > m$ such that if $|x - y| < 2^{-n}$, then $|f(x) - f(y)| < \delta$. From this it is clear that $f \in D_n$ and $\|f\|_n < \delta$. The claim becomes evident.

We now start the construction of the $\mathbf{G}_\delta$ set E as the intersection of dense open sets W_k.

Let $k \geq 1$ be an integer and let U be a nonempty open subset of $\mathcal{C}$. Choose f and $n \geq k$ as above. For $j = 0, 1, 2, \ldots, 2^{n+1}$, define

$$g(j/2^{n+1}) = f(j/2^{n+1}) .$$

If $i2^{-n} \leq j2^{-n-1} < (j+1)2^{-n-1} \leq (i+1)2^{-n}$, where $i \in \{0, 1, 2, \ldots, 2^n - 1\}$, put

$$L_i = (i2^{-n}, (i+1)2^{-n}) , \qquad M_j = (j2^{-n-1}, (j+1)2^{-n-1})$$

and let $K_j = [a_j, b_j]$ be an interval concentric with M_j such that

$$\mathrm{m}(K_j)/\mathrm{m}(M_j) = 1 - 1/2^n = 2\mathrm{m}(K_j)/\mathrm{m}(L_i) .$$

Choose $I_j^0 = [c_j, d_j]$ concentric with the interval $f(M_j)$ and such that

$$\mathrm{m}(I_j^0)/\mathrm{m}(f(M_j)) = 1/2^n .$$

Define the function g to be linear on each of the intervals $[j2^{-n-1}, a_j]$, $[a_j, b_j]$, and $[b_j, (j+1)2^{-n-1}]$ in such a way that $g([a_j, b_j]) = [c_j, d_j] = I_j^0$. (See Fig. 1.) Thus, if

$$J_j = f(M_j) = g(M_j) ,$$

then

$$\mathrm{m}(g(K_j))/\mathrm{m}(g(M_j)) = \mathrm{m}(I_j^0)/\mathrm{m}(J_j) = 1/2^n ,$$
$$\mathrm{m}(g^{-1}(I_j^0))/\mathrm{m}(g^{-1}(J_j)) = \mathrm{m}(K_j)/\mathrm{m}(M_j) = 1 - 1/2^n .$$

Note that g is contained in the open ball centered at f of radius $\|f\|_n$. Thus, $g \in U$.

Let W_U^k be the open ball centered at g of radius

$$\varepsilon_k = 2^{-n-1} \min_{i=1,2,\ldots,2^n} |f(i/2^n) - f((i-1)/2^n)| > 0 . \tag{3}$$

Obviously $W_k = \bigcup\{W_U^k : U \text{ is open and nonempty in } \mathcal{C}\}$ is open and dense in $\mathcal{C}$, so that $E = \bigcap_{k \in \mathbb{N}} W_k$ is a residual set in $\mathcal{C}$. We will show that if $h \in E$ then h is nowhere $\mathcal{I}$-density continuous.

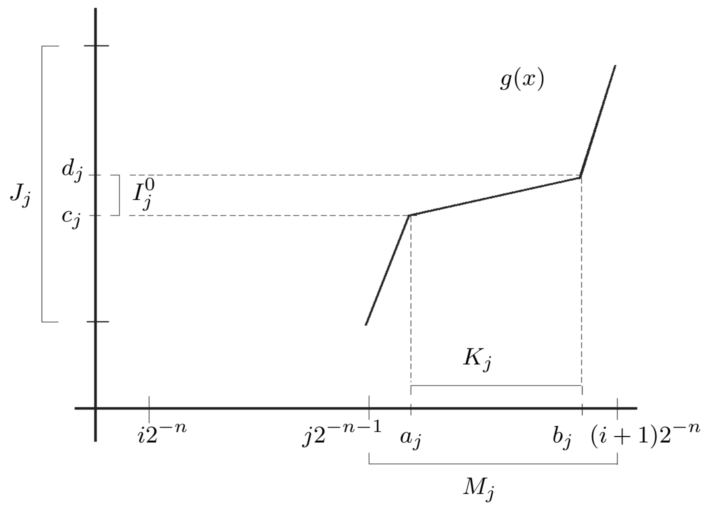

Fig. 1. The function $g(x)$

Let $x \in [0,1]$ be arbitrary. We will choose intervals I_m, $m \in \mathbb{N}$, such that $h(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{m\in\mathbb{N}} I_m$, but x is not an $\mathcal{I}$-dispersion point of $h^{-1}(\bigcup_{m\in\mathbb{N}} I_m)$. This will prove that h is not $\mathcal{I}$-density continuous at x.

Let $m \in \mathbb{N}$. We have $h \in W_m$, so there exists a set U, open in $\mathcal{C}$, such that $h \in W_U^m$. Let g be the center of W_U^m. Let $n \geq m$ be the number given in the construction of W_U^m. Let $i \in \{0,1,2,\ldots,2^n-1\}$ be such that $x \in [i2^{-n},(i+1)2^{-n}]$. Put

$$L_m = [i2^{-n},(i+1)2^{-n}].$$

Let

$$M^1 = ((2i)2^{-n-1},(2i+1)2^{-n-1})\,, \quad M^2 = ((2i+1)2^{-n-1},2(i+1)2^{-n-1})\,,$$

and let $M_m \in \{M^1,M^2\}$ be such that $h(x) \notin g(M_m)$. Put $J_m = g(M_m)$ and let $I_m^0 = [c_j,d_j]$, $K_m = [a_j,b_j]$ be as in the construction of g.

Thus we have

$$\frac{\mathrm{m}(I_m^0)}{\mathrm{m}(J_m)} = \frac{1}{2^n} \leq \frac{1}{2^m} \quad \text{and} \quad \frac{\mathrm{m}(K_m)}{\mathrm{m}(M_m)} = 1 - \frac{1}{2^n} \geq 1 - \frac{1}{2^m}\,.$$

Define $I_m = [c_j-\varepsilon_m, d_j+\varepsilon_m]$. As $h(x) \notin J_m$, we can choose a subsequence $\{I_{m_i}\}_{i\in\mathbb{N}}$ of $\{I_m\}_{m\in\mathbb{N}}$ such that the union of all intervals in the sequence $\{I_{m_i}\}_{i\in\mathbb{N}}$ is a left or right interval set at $h(x)$. Without loss of generality we may assume that it is a right interval set at $h(x)$. As, for each $i \in \mathbb{N}$, I_{m_i} and J_{m_i} have a common center and

$$\lim_{i\to\infty} \mathrm{m}(I_{m_i})/\mathrm{m}(J_{m_i}) = 0\,,$$

Lemma 5 says that we can choose a subsequence $\{I_{m_{i_j}}\}_{j\in\mathbb{N}}$ of $\{I_{m_i}\}_{i\in\mathbb{N}}$ such that $h(x)$ is an $\mathcal{I}$-dispersion point of $\bigcup_{j\in\mathbb{N}} I_{m_{i_j}}$.

On the other hand, by the way ε_n was chosen in (3), $K_n \subset h^{-1}(I_n)$. Thus, using Lemma 4, the fact that $x \in L_m$ for every $m \in \mathbb{N}$ and

$$\lim_{j\to\infty} \mathrm{m}(K_{n_{i_j}})/\mathrm{m}(L_{n_{i_j}}) = \lim_{j\to\infty} \mathrm{m}(K_{n_{i_j}})/(2\mathrm{m}(M_{n_{i_j}})) = 1/2 > 0$$

we conclude that x is not an $\mathcal{I}$-dispersion point of $\bigcup_{j\in\mathbb{N}} K_{n_{i_j}}$. Thus x is not an $\mathcal{I}$-dispersion point of $h^{-1}(\bigcup_{j\in\mathbb{N}} I_{n_{i_j}})$. This finishes the proof of Theorem 6.

3. Comparison of $\mathcal{C}_{\mathcal{I}\mathcal{I}}$ and $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ to themselves. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is in the class Baire*1 if for each nonempty perfect set P there exists an open interval I such that $I \cap P \neq \emptyset$ and the restricted function $f|_{I\cap P}$ is continuous [8]. It is clear from the definition that any $f \in$ Baire*1 must be continuous at each point of a dense open set. This useful property is true of the functions in $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ [3], [5, Theorem 4.1(iv)].

THEOREM 7. $\mathcal{C}_{\mathcal{D}\mathcal{D}} \subset$ Baire*1.

THEOREM 8. *The spaces $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ and $\mathcal{C}_{\mathcal{I}\mathcal{I}}$, equipped with the topology of uniform convergence, are of the first category in themselves.*

Proof. We only prove this for the class $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ as the other case is essentially the same.

Let $\{I_n\}_{n\in\mathbb{N}}$ be the sequence of all open intervals with rational endpoints and let C_n be the family of all deep $\mathcal{I}$-density continuous functions that are continuous on I_n in the ordinary sense. By Theorem 7, $\mathcal{C}_{\mathcal{D}\mathcal{D}} = \bigcup_{n\in\mathbb{N}} C_n$. Also, it is evident that the sets C_n are closed in $\mathcal{C}_{\mathcal{D}\mathcal{D}}$ equipped with the topology of uniform convergence. Finally, for any function $f \in C_n$ and any of its neighborhoods $U \subset \mathcal{C}_{\mathcal{D}\mathcal{D}}$, it is easy to slightly modify a function g such as in Lemma 3 in such a way that $g \in U \setminus C_n$. Thus, the sets C_n are nowhere dense.

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