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SUM AND DIFFERENCE FREE PARTITIONS OF VECTOR SPACES

BY

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1. Preliminaries. In this paper V will stand for a vector space over the rationals $\mathbb{Q}$ and S for a subset of V. All the partitions $\mathcal{P}$ of a set $S \subseteq V$ will be countable, i.e., $|\mathcal{P}| \le \omega$. They will often be identified with colorings $f : S \to \omega$ of S, via $\mathcal{P} = \{f^{-1}(n) : n < \omega\}$. For a cardinal $\kappa > 0$ and a partition $\mathcal{P}$ of $S \subseteq V$ we say that $\mathcal{P}$ is κ *sum free* if for every $a \in V$ the equation $x + y = a$ has less than κ solutions with x and y from the same element of the partition $\mathcal{P}$, i.e., such that $f(x) = f(y)$. We consider the solutions $\langle x, y\rangle$ and $\langle y, x\rangle$ identical and ignore the solution $\langle x, y\rangle = \langle a/2, a/2\rangle$. We say that a set S is κ sum free if $\mathcal{P} = \{S\}$ is κ sum free. In particular, if a partition $\mathcal{P}$ of $S \subseteq V$ is κ sum free then $\mathcal{P}$ *partitions* S *into* κ *sum free sets*. Thus, a partition $\mathcal{P} = \{P_n \subseteq S : n < \omega\}$ of a set $S \subseteq V$ is κ sum free if

$$(\forall a \in V)\ \Big|\bigcup_{n<\omega}\{\{x, y\} \in [P_n]^2 : x + y = a\}\Big| < \kappa,$$

while $\mathcal{P}$ partitions S into κ free sets if

$$(\forall a \in V)(\forall n < \omega)\ |\{\{x, y\} \in [P_n]^2 : x + y = a\}| < \kappa.$$

Similarly, we say that a partition $\mathcal{P}$ of S is κ *difference free* if for every $a \in V$, $a \ne 0$, the equation $x - y = a$ has less than κ solutions $\langle x, y\rangle$, with x and y from the same element of $\mathcal{P}$. A set S is κ difference free if $\mathcal{P} = \{S\}$ is κ difference free.

These notions lead in a natural way to the following cardinal invariants, in which $\mathcal{P}$ is a countable partition:

$\widehat{\sigma}(\kappa) = \min\{|V| : \text{there is no } \kappa \text{ sum free partition } \mathcal{P} \text{ of } V\}$,

$\sigma(\kappa) = \min\{|V| : \text{there is no partition } \mathcal{P} \text{ of } V \text{ into } \kappa \text{ sum free sets}\}$,

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$$\widehat{\delta}(\kappa) = \min\{|V| : \text{there is no } \kappa \text{ difference free partition } \mathcal{P} \text{ of } V\},$$
$$\delta(\kappa) = \min\{|V| : \text{there is no partition } \mathcal{P} \text{ of } V \text{ into } \kappa \text{ difference free sets}\}.$$

Note that the above minima are taken over non-empty sets by (1) and the facts that

(a) if $\lambda \to (\kappa+1)^2_\omega$ then $\sigma(\kappa) \le \lambda$;
(b) if $\binom{\lambda}{\lambda} \to \binom{\kappa}{2}^{1,1}_\omega$ then $\delta(\kappa) \le \lambda$.

(See [4] for the definitions and basic facts concerning the above partition relations.) The proofs of these implications are similar to the proofs of Theorems 2.2 and 2.4, respectively.

Clearly, for any cardinal numbers $\lambda < \kappa$,

$$\widehat{\sigma}(\kappa) \le \sigma(\kappa) \ \&\ \widehat{\delta}(\kappa) \le \delta(\kappa) \tag{1}$$

and

$$\widehat{\sigma}(\lambda) \le \hat{\sigma}(\kappa) \ \&\ \sigma(\lambda) \le \sigma(\kappa) \ \&\ \widehat{\delta}(\lambda) \le \hat{\delta}(\kappa) \ \&\ \delta(\lambda) \le \delta(\kappa). \tag{2}$$

Note also that if $\kappa \ge \mathrm{cf}(\kappa) > \omega$ then $\widehat{\sigma}(\kappa) = \sigma(\kappa)$ and $\widehat{\delta}(\kappa) = \delta(\kappa)$.

In our studies we will concentrate on these cardinals in the cases when $2 \le \kappa \le \omega$.

2. $\widehat{\sigma}(\omega) = \sigma(\omega) = (2^\omega)^+$ and $\widehat{\delta}(\omega) = \delta(\omega) = \omega_2$. The equality $\sigma(\omega) = (2^\omega)^+$ has been proved by Komjáth in [5, Thm. 2] and the inequality $\widehat{\sigma}(\omega) \ge (2^\omega)^+$ by Ciesielski and Larson in [1, Thm. 1.1].

More precisely, the equality $\widehat{\sigma}(\omega) = \sigma(\omega) = (2^\omega)^+$ follows from the inequalities $(2^\omega)^+ \le \widehat{\sigma}(\omega) \le \sigma(\omega) \le (2^\omega)^+$. The inequalities $\widehat{\sigma}(\omega) \le \sigma(\omega)$, $(2^\omega)^+ \le \widehat{\sigma}(\omega)$ and $\sigma(\omega) \le (2^\omega)^+$ follow from (1) and the next two theorems, respectively.

THEOREM 2.1 (Ciesielski and Larson [1, Thm. 1.1]). *If $|V| \le 2^\omega$ then there is a countable ω sum free partition of V.* ∎

THEOREM 2.2 (Komjáth [5, Thm. 2(b)]). *If $|V| > 2^\omega$ then V is not a countable union of ω_1 sum free sets.* ∎

Next we turn our attention to the equality $\widehat{\delta}(\omega) = \delta(\omega) = \omega_2$. Once again, by (1), it is enough to prove only two inequalities: $\widehat{\delta}(\omega) \ge \omega_2$ and $\delta(\omega) \le \omega_2$. They are proved in the next two theorems. Notice that the equality $\delta(\omega) = \omega_2$ has been proved by Komjáth in [5, Thm. 1].

THEOREM 2.3. *If $|V| \le \omega_1$ then there is a countable ω difference free partition of V.*

P r o o f. Represent V as the union of a continuous increasing sequence $\{V_\alpha : \alpha < \omega_1\}$ of countable subspaces. In particular, $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for every limit ordinal $\lambda < \omega_1$. For convenience we also assume that $V_0 = \emptyset$. Thus,

$\{V_{\alpha+1} \setminus V_\alpha : \alpha < \omega_1\}$ is a partition of V into countable sets. For $\alpha < \omega_1$ let $\{p_n^\alpha : n < \omega\}$ be an enumeration of $V_{\alpha+1} \setminus V_\alpha$.

A coloring function $f : V \to \omega$ generating the desired partition is constructed by defining, by induction on $\alpha < \omega_1$, its one-to-one restrictions $f|(V_{\alpha+1} \setminus V_\alpha) : V_{\alpha+1} \setminus V_\alpha \to \omega$ such that the following inductive condition holds:

$$f(p_n^\alpha) \in \omega \setminus \{f(p) : p \in V_\alpha \ \&\ p = p_n^\alpha \pm p_j^\alpha \text{ for some } j \le n\}.$$

To show that the partition generated by f is ω difference free choose an arbitrary $a = p_n^\alpha \neq 0$ and consider the pairs $\langle x, y\rangle$ satisfying $x - y = a$ and such that $f(x) = f(y)$. It is enough to show that $\{x, y\} \cap \{p_j^\alpha : j \le n\} \neq \emptyset$.

Let $x = p_m^\beta$ and $y = p_k^\gamma$. Then $p_m^\beta - p_k^\gamma = p_n^\alpha$. Notice that $\delta = \max\{\alpha, \beta, \gamma\}$ must be equal to at least two of α, β and γ since otherwise the number p associated with the index δ would belong to V_δ. Moreover, $\beta \neq \gamma$, since otherwise $f(p_m^\beta) = f(x) = f(y) = f(p_k^\beta)$, contradicting the fact that f is one-to-one on $V_{\alpha+1} \setminus V_\alpha$.

We are left with two cases.

If $\alpha = \beta > \gamma$ then $p_m^\alpha - p_n^\alpha = x - a = y = p_k^\gamma \in V_\alpha$. So, $f(p_m^\alpha) = f(x) = f(y) = f(p_m^\alpha - p_n^\alpha)$ implies that $m \le n$ and $y = p_m^\alpha \in \{p_j^\alpha : j \le n\}$.

If $\alpha = \gamma > \beta$ then $p_k^\alpha + p_n^\alpha = y + a = x = p_m^\beta \in V_\alpha$. So, $f(p_k^\alpha) = f(y) = f(x) = f(p_k^\alpha + p_n^\alpha)$ implies that $k \le n$ and $x = p_k^\alpha \in \{p_j^\alpha : j \le n\}$. ■

THEOREM 2.4. *If $|V| = \kappa \ge \omega_2$ and $\mathrm{cf}(\kappa) > \omega_1$ then V is not a countable union of κ difference free sets.*

Proof. For $\kappa = \omega_2$ this has been proved by Komjáth in [5, Thm. 1(b)]. The proof of the general case is essentially identical and follows from the following partition theorem $\binom{\kappa}{\omega_1} \to \binom{\kappa}{2}^{1,1}_{\omega}$ of Erdős and Hajnal [2, p. 129]: if $f : \kappa \times \omega_1 \to \omega$ then there are $\eta < \xi < \omega_1$ and $K \in [\kappa]^\kappa$ such that $f(\alpha, \eta) = f(\beta, \xi)$ for every $\alpha, \beta \in K$. ■

It is interesting how important is the assumption $\mathrm{cf}(\kappa) > \omega_1$ in the last theorem. It certainly cannot be completely removed, since for $|V| = \kappa$ with $\mathrm{cf}(\kappa) = \omega$ the space V is a countable union of κ difference free sets as V can be partitioned into countably many sets of size $< \kappa$ each.

PROBLEM 2.1. *Let $|V| = \kappa > \mathrm{cf}(\kappa) = \omega_1$. Does this imply that V is not a countable union of κ difference free sets, or at least that there is no κ difference free partition of V?*

PROBLEM 2.2. *Let $|V| = \kappa > \mathrm{cf}(\kappa) = \omega$. Does this imply that there is no κ difference free partition of V?*

3. $\delta(n) = \omega_2$ and $\widehat{\delta}(n) = \omega_1$ for $2 \le n < \omega$. To see that $\delta(n) = \omega_2$ notice that the inequality $\delta(n) \le \delta(\omega) \le \omega_2$ follows from condition (2) and Theorem 2.4. The inequality $\delta(n) \geq \omega_2$ follows immediately from the following theorem of Erdős and Kakutani [2]. (See also [5, Thm. 1(a)].)

THEOREM 3.1. *If* $|V| \leq \omega_1$ *then* V *is a union of countably many bases.* ∎

To see that $\widehat{\delta}(n) = \omega_1$ notice first that the inequality $\widehat{\delta}(n) \geq \omega_1$ is obvious, since any countable V can be partitioned into singletons, and such a partition is clearly 1 difference free. The inequality $\widehat{\delta}(n) \leq \omega_1$ follows from the next theorem.

THEOREM 3.2. *If* $|V| \geq \omega_1$ *then there is no* k *difference free partition of* V *for any* $k < \omega$.

Proof. Let $f : V \to \omega$ be a coloring generating a partition $\mathcal{P}$. We will show that $\mathcal{P}$ is not k difference free for any $k < \omega$.

Take a linear base H of V over $\mathbb{Q}$ and choose disjoint sets $A_0 \in [H]^{\omega_1}$ and $B \in [H]^{\omega}$. Let $B = \{b_1, b_2, \ldots\}$. Define a sequence $\langle A_n \in [A_0]^{\omega_1} : 0 < n < \omega\rangle$ such that for all $n > 0$,

$$f(a + b_n) = f(a' + b_n) \quad \text{for all } a, a' \in A_n,$$

and that $A_{n+1} \subseteq A_n$ for all $n < \omega$. The set A_{n+1} can be chosen to be one of the sets $\{a \in A_n : f(a + b_n) = i\}$ for $i < \omega$.

Now, for $k < \omega$ pick different $a, a' \in A_k$. Then $f(a+b_n) = f(a'+b_n)$ and $(a + b_n) - (a' + b_n) = a - a'$ for every $n \leq k$. So, the equation $x - y = a - a'$ has at least k different solutions with x and y from the same element of the partition. ∎

4. $\omega_2 \leq \widehat{\sigma}(m) \leq \sigma(n) = \min\{\omega_{k+2}, (2^{\omega})^{+}\}$ **for** $2^k < n \leq 2^{k+1} \leq m < 2^{k+2}$ **and** $k < \omega$**.** The equality $\sigma(n) = \min\{\omega_{k+2}, (2^{\omega})^{+}\}$ will be proved by showing that for every $k < \omega$,

$$\min\{\omega_{k+2}, (2^{\omega})^{+}\} \leq \sigma(2^k + 1) \leq \sigma(2^{k+1}) \leq \min\{\omega_{k+2}, (2^{\omega})^{+}\}.$$

The inequalities

$$\sigma(2^k + 1) \leq \sigma(2^{k+1}) \leq \sigma(\omega) \leq (2^{\omega})^{+}$$

are consequences of (2) and Theorem 2.2. The remaining two inequalities: $\sigma(2^{k+1}) \leq \omega_{k+2}$ and $\sigma(2^k + 1) \geq \min\{\omega_{k+2}, (2^{\omega})^{+}\}$ follow respectively from the following two theorems.

THEOREM 4.1. *If* $|V| \geq \omega_{k+2}$ *for* $k < \omega$ *then there is no partition* $\mathcal{P}$ *of* V *into* 2^{k+1} *sum free sets.*

Proof. The argument is included implicitly in the proof of [6, Thm. 5]. ∎

THEOREM 4.2. *If* $|V| \leq \min\{\omega_{k+1}, 2^{\omega}\}$ *for* $k < \omega$ *then there is a countable partition* $\mathcal{P}$ *of* V *into* $2^k + 1$ *sum free sets.*

P r o o f. For $k = 0$ this follows immediately from Theorem 3.1. The proof of the general case, presented below, is essentially due to Baumgartner (private communication).

Let $|V| = \min\{\omega_{k+1}, 2^\omega\}$. Using induction on $k < \omega$, if necessary, we can assume that $|V| = \omega_{k+1} \le 2^\omega$. We will construct a coloring $f : V \to R$ into a countable set R which will generate the right partition.

So, let H be a linear base of V over $\mathbb{Q}$. Thus, for every $v \in V$ there exists a unique mapping $h \mapsto q_h^v$ from a finite set $H_v \subset H$ into $\mathbb{Q} \setminus \{0\}$ such that $v = \sum_{h\in H_v} q_h^v h$.

Now, let $e : H \to \{0,1\}^\omega$ be a one-to-one mapping and for every $S \subset H$ choose a bijection $w_S : S \to |S|$. Thus, w_S establishes a well ordering $\le_S$ of S in order type $|S|$ by $h_1 \le_S h_2$ if and only if $w_S(h_1) \le w_S(h_2)$. Moreover, choose $l_v < \omega$ such that $e(h_1)|_{l_v} \ne e(h_2)|_{l_v}$ for all different $h_1, h_2 \in H_v$.

We define $f(v)$ as a sequence $\langle s_i^v : i < |H_v|\rangle$ by induction on $i < |H_v|$ as follows. Let h_0^v be the $\le_H$-maximal element of H_v and put $s_0^v = \langle e(h_0^v)|_{l_v}, q^v_{h_0^v}, 0\rangle$. Moreover, let $S_0^v = \{h \in H : h <_H h_0^v\}$. Thus, $|S_0^v| \le \omega_k$. Now, if $i+1 < |H_v|$ is such that s_i^v and S_i^v are already defined, we define s_{i+1}^v and S_{i+1}^v as follows. Let h_{i+1}^v be the $\le_{S_i^v}$-maximal element of $H_v \setminus \{h_j^v : j \le i\}$. If $|S_i^v| \le \omega$ we put $s_{i+1}^v = \langle e(h_{i+1}^v)|_{l_v}, q^v_{h_{i+1}}, w_{S_i^v}(h_{i+1}^v)\rangle$ and $S_{i+1}^v = S_i^v$. If $|S_i^v| > \omega$ we put $s_{i+1}^v = \langle e(h_{i+1}^v)|_{l_v}, q^v_{h_{i+1}^v}, 0\rangle$ and $S_{i+1}^v = \{h \in S_i^v : h <_{S_i^v} h_{i+1}^v\}$.

Thus, the range R of f is a subset of a countable set $\{0,1\}^{<\omega} \times \mathbb{Q} \times \omega$. It is enough to show that f has the desired properties.

So, fix $a \in V$ and let $x, y \in V$ be different such that $x + y = a$ and $f(x) = f(y)$, i.e., $\langle s_i^x : i < |H_x|\rangle = \langle s_i^y : i < |H_y|\rangle = \langle s_i : i < n\rangle$ for some $\sigma = \langle s_i : i < n\rangle \in R$. First notice that

$$H_x \cup H_y = H_a.$$

The inclusion $H_a \subset H_x \cup H_y$ is obvious, since $\sum_{h\in H_a} q_h^a h = \sum_{h\in H_x} q_h^x h + \sum_{h\in H_y} q_h^y h$. To see the other inclusion let $h \in H_x \cup H_y$. If $h \notin H_x \cap H_y$, then clearly $h \in H_a$. So, assume that $h \in H_x \cap H_y$. Then $h = h_i^x = h_j^y$ for some $i, j < n$ and $e(h_i^x)|_{l_x} = e(h_j^y)|_{l_y} = e(h_j^x)|_{l_x}$. Hence, $j = i$ by the choice of l_x. But then $q^x_{h_i^x} h_i^x + q^y_{h_j^y} h_j^y = 2q^x_{h_i^x} h$, i.e., $h \in H_a$, as $2q^x_{h_i^x} \ne 0$.

Next notice that

$$\text{if } h_i^x = h_i^y \text{ for } i \le k \text{ then } x = y. \tag{3}$$

To see this, it is enough to show that if $h_i^x = h_i^y$ for $i \le k$ then $h_i^x = h_i^y$ for all $i < n$. If $n \le k+1$ then there is nothing to prove. So, assume that $n > k+1$. It is easy to see by our construction that $|S_i^x| > |S_{i+1}^x|$ provided $|S_i^x| > \omega$. This easily implies that $|S_k^x| \le \omega$ and $S_i^x = S_i^y$ for every $i < n$. So,

$$w_{S_k^x}(h_{k+1}^x) = w_{S_k^y}(h_{k+1}^y) = w_{S_k^x}(h_{k+1}^y)$$

and, since $w_{S^x_k}$ is one-to-one, $h^x_{k+1} = h^y_{k+1}$. Continuing this by induction, we obtain $h^x_i = h^y_i$ for every $i < n$. This finishes the proof of (3).

Now, for fixed $\sigma = \langle s_i : i < n\rangle = \langle\langle e_i, q_i, m_i\rangle \in \{0,1\}^l \times \mathbb{Q} \times \omega : i < n\rangle$ and for every $i < n$ there are at most two $h \in H_a$ such that $e(h)|_l = e_i$, one from each H_x and H_y. Thus, for $x+y = a$ and $f(x) = f(y) = \sigma$ each h^x_i can be chosen in at most two ways. So, we have at most 2^{k+1} possible sequences $\langle h^x_i : i \le k\rangle$. Since, by (3), each such sequence determines x, we have at most 2^{k+1} numbers x satisfying the equation. So, the number of different pairs cannot exceed 2^k. This finishes the proof. ∎

To argue for the inequalities $\omega_2 \le \widehat{\sigma}(m) \le \min\{\omega_{k+2}, (2^\omega)^+\}$ for $2^{k+1} \le m < 2^{k+2}$ and $k < \omega$ notice first that $\widehat{\sigma}(m) \le \widehat{\sigma}(\omega) \le (2^\omega)^+$ follows from (1) and Theorem 2.2. The inequalities $\widehat{\sigma}(m) \ge \omega_2$ and $\widehat{\sigma}(m) \le \omega_{k+2}$ follow from the next two theorems, respectively.

Theorem 4.3 (Komjáth and Shelah [6, Thm. 5]). *If $|V| \ge \omega_{k+2}$ for $k < \omega$ then there is no $2^{k+2} - 1$ sum free partition $\mathcal{P}$ of V.* ∎

Theorem 4.4 (Komjáth and Shelah [6, Thm. 3]). *If $|V| \le \omega_1$ then there is a 2 sum free partition $\mathcal{P}$ of V.* ∎

The proof of Theorem 4.4 can also be easily obtained by a slight modification of the proof of Theorem 4.2.

The above inequalities give us, in particular, the following equalities.

Corollary 4.5. $\widehat{\sigma}(2) = \widehat{\sigma}(3) = \omega_2$. ∎

Problem 4.1. *Find the exact values of $\widehat{\sigma}(n)$ for $3 < n < \omega$.*

Note that it is consistent with ZFC that $\widehat{\sigma}(2^{k+1}) = \omega_{k+2} = (2^\omega)^+$ for every $k < \omega$. This can be deduced from the next theorem in the same way as [6, Thm. 5] was deduced from [6, Thm. 4].

Theorem 4.6 (Komjáth and Shelah [7, Thm. 1]). *For $1 \le n < \omega$ it is consistent that $2^\omega = \omega_n$ and there is a function $F : [\omega_n]^{<\omega} \to \omega$ such that for every $A \in [\omega_n]^{<\omega}$ there are at most $2^n - 1$ solutions of $A = H_0 \cup H_1$ with $H_0 \ne H_1$ and $F(H_0) = F(H_1)$.* ∎

5. Sum and difference free partitions of subsets of V. This section is motivated by the following theorems that deal with partitioning subsets of V into sum or difference free sets. We will try to examine their analogs for sum or difference free partitions.

Theorem 5.1 (Erdős [3]). *If $|V| \ge \omega_2$ then there exists a 3 sum free set $S \in [V]^{\omega_2}$ which does not admit a countable partition into 2 sum free sets.* ∎

Theorem 5.2 (Komjáth [5, Thms. 3 and 4]). (1) *If $S \in [V]^{\le\omega_2}$ is ω_2 difference free then S can be partitioned into countably many 2 difference free sets.*

(2) *If* $|V| \geq (2^{\mathfrak{c}})^+$ *then there is a* 3 *difference free set* $S \subseteq V$ *which cannot be partitioned into countably many* 2 *difference free sets.* ■

THEOREM 5.3 (Komjáth [5, Thm. 6]). *If* $S \subset V$ *is* ω_2 *difference free then* S *can be partitioned into countably many* ω *difference free and* ω *sum free sets.* ■

Theorem 5.1 has the following corollary.

COROLLARY 5.4. *The following conditions are equivalent.*

(i) $|V| \geq \omega_2$.
(ii) *There exists a* 3 *sum free set* $S \subseteq V$ *which does not admit a countable partition into* 2 *sum free sets.*
(iii) *There exists a* 3 *sum free set* $S \subseteq V$ *which does not admit a countable* 2 *sum free partition.*
(iv) V *does not admit a countable* 2 *sum free partition.*

Proof. (i)⇒(ii) follows from Theorem 5.1. (ii)⇒(iii) and (iii)⇒(iv) are obvious. (iv)⇒(i) follows from Theorem 4.4. ■

Since $\sigma(2) = \widehat{\sigma}(2) = \omega_2$ Corollary 5.4 suggests the following conjecture.

CONJECTURE 5.1. *Let* $1 < \kappa < \lambda$ *be cardinal numbers.*

(a) $|V| \geq \sigma(\kappa)$ *if and only if there exists a* λ *sum free set* $S \subseteq V$ *which does not admit a countable partition into* κ *sum free sets.*
(b) $|V| \geq \widehat{\sigma}(\kappa)$ *if and only if there exists a* λ *sum free set* $S \subseteq V$ *which does not admit a countable* κ *sum free partition.*

Notice that the implications from right to left are obvious. The following part of the conjecture follows from the results of Section 4.

THEOREM 5.5. *For* $k < \omega$ *such that* $2^\omega \geq \omega_{k+1}$ *the following conditions are equivalent*:

(i) $|V| \geq \sigma(2^{k+1}) = \omega_{k+2}$.
(ii) *There exists* $2^{k+1} + 1$ *sum free set* $S \subseteq V$ *which does not admit a countable* 2^{k+1} *sum free partition.*

Proof. (ii)⇒(i) follows by contraposition from Theorem 4.2.
To see (i)⇒(ii) take a subspace V_0 of V of cardinality ω_{k+2}. Then, by Theorem4.2, there exists a countable partition of V_0 into $2^{k+1} + 1$ sum free sets. At least one of these sets must satisfy (ii) by Theorem 4.1. ■

The difference free analog of Corollary 5.4 related to Theorem 5.2 reads as follows.

THEOREM 5.6. *The following conditions are equivalent*:

(i) $|V| \geq \omega_1$.

(ii) *There exists a* 3 *difference free set* $S \subseteq V$ *which does not admit a countable* 2 *difference free partition.*

Proof. (ii)$\Rightarrow$(i) is obvious, since every countable set can be partitioned into singletons.

To see the other implication, let $\{s,t\} \cup X$ be a linearly independent subset of V of cardinality ω_1 such that $s, t \notin X$ and $s \neq t$. Define

$$S = \{s - x : x \in X\} \cup \{x - t : x \in X\}.$$

To see that S is 3 difference free take $a \neq 0$. Consider all possible solutions of $x - y = a$ with $x, y \in S$. They must be of one of the following forms:

(1) $a = (s - x_1) - (y_1 - t) = -x_1 - y_1 + s + t$ for some $x_1, y_1 \in X$;
(2) $a = (y_2 - t) - (s - x_2) = x_2 + y_2 - s - t$ for some $x_2, y_2 \in X$;
(3) $a = (s - x_3) - (s - y_3) = -x_3 + y_3$ for some $x_3, y_3 \in X$;
(4) $a = (x_4 - t) - (y_4 - t) = x_4 - y_4$ for some $x_4, y_4 \in X$.

By linear independence of $\{s,t\} \cup X$, if a can be represented in the form (1) or (2) than it cannot be represented in any other form. Moreover, such an a can be obtained in at most two ways: by exchanging x_i with y_i. If a is in the form (3) or (4) then its representation is unique in each form. Thus, such an a can be represented in at most two ways: one in the form (3) and one in the form (4). So, S is 3 difference free.

To see that there is no countable 3 difference free partition of S let $f : S \to \omega$. Define $F : X \to \omega \times \omega$ by $F(x) = \langle f(s - x), f(x - t)\rangle$ and let X_0 be an uncountable subset of X on which F is constant. Thus, for different $a, b \in X_0$ we have $f(s - a) = f(s - b)$ and $f(a - t) = f(b - t)$. However,

$$(a - t) - (b - t) = a - b = (s - b) - (s - a).$$

Therefore the equation $x - y = a - b$ has two different solutions $\langle a - t, b - t\rangle$ and $\langle s - b, s - a\rangle$ that agree with f. ∎

PROBLEM 5.1. *Find conditions analogous to Theorems* 5.2 *and* 5.6 *for* $n+1$ *difference free subsets of* V *without any* n *difference free partition*, *or a partition into* n *difference free sets.*

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