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FUNCTIONS CHARACTERIZED BY IMAGES OF SETS

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For non-empty topological spaces X and Y and arbitrary families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ we put $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in Y^X : (\forall A \in \mathcal{A})(f[A] \in \mathcal{B})\}$. We examine which classes of functions $\mathcal{F} \subseteq Y^X$ can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. We are mainly interested in the case when $\mathcal{F} = \mathcal{C}(X,Y)$ is the class of all continuous functions from X into Y. We prove that for a non-discrete Tikhonov space X the class $\mathcal{F} = \mathcal{C}(X,\mathbb{R})$ is not equal to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$. Thus, $\mathcal{C}(X,\mathbb{R})$ cannot be characterized by images of sets. We also show that none of the following classes of real functions can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$: upper (lower) semicontinuous functions, derivatives, approximately continuous functions, Baire class 1 functions, Borel functions, and measurable functions.

1. Basic definitions and facts. Throughout the paper we use the standard definitions and notation. In particular, the family of all functions from a set X into Y is denoted by Y^X. The symbol $|X|$ stands for the cardinality of X and $\mathcal{P}(X)$ for the family of all subsets of X. For a cardinal number κ we write $[X]^\kappa$ to denote the family of all subsets Y of X with $|Y| = \kappa$. (In particular, $[X]^1$ stands for the set of all singletons in X and $[X]^2$ for the family of all doubletons in X.) Similarly we define $[X]^{<\kappa}$, $[X]^{\le\kappa}$ and $[X]^{\ge\kappa}$.

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We use the symbol $\mathrm{Const}_{X,Y}$ for the class of all constant functions from X into Y, and write just Const when the spaces X and Y are clear from the context. The identity map from X into X is denoted by id_X. For topological spaces X and Y the class of all continuous functions from X into Y is denoted by $\mathcal{C}(X,Y)$.

Following Engelking [4] we say that a space X is *totally disconnected* if all quasi-components of X are singletons. All topological spaces considered in this paper are at least T_0 (distinguish between points) and contain at least two points.

1.1. *Main results*. In order to announce our principal results we also need the following frequently used notation: for non-empty sets X, Y and families $\mathcal{A}\subseteq\mathcal{P}(X)$, $\mathcal{B}\subseteq\mathcal{P}(Y)$,

$$\mathcal{C}_{\mathcal{A},\mathcal{B}}=\{f\in Y^X:(\forall A\in\mathcal{A})(f[A]\in\mathcal{B})\}.$$

Some basic properties of $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ are outlined below in Facts 1.2 and 1.3.

This work is motivated by a paper of Velleman [8] in which it is proved that the class $\mathcal{F}=\mathcal{C}(\mathbb{R},\mathbb{R})$ is not equal to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A},\mathcal{B}\subseteq\mathcal{P}(\mathbb{R})$. Thus, $\mathcal{C}(\mathbb{R},\mathbb{R})$ cannot be characterized by images of sets. This stays in contrast with the fact that, by definition, the family $\mathcal{C}(X,Y)$ can be characterized by preimages of sets for every pair of topological spaces X, Y:

$$\mathcal{C}(X,Y)=\{f\in Y^X:f^{-1}(V)\text{ is open in }X\text{ for every open }V\subseteq Y\}.$$

This phenomenon justifies the following terminology.

Definition 1.1. Let X and Y be topological spaces. We say that:

- the pair $\langle X,Y\rangle$ of spaces has the *V-property* if there exist $\mathcal{A}\subseteq\mathcal{P}(X)$ and $\mathcal{B}\subseteq\mathcal{P}(Y)$ such that $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$;
- X is a *V-space* if $\langle X,X\rangle$ has the V-property.

In these terms Velleman's theorem says that $\mathbb{R}$ is not a V-space. Our aim is to generalize this result to a large class of pairs $\langle X,Y\rangle$ of topological spaces. In Section 3 we characterize the spaces X such that the pair $\langle X,\mathbb{R}\rangle$ has the V-property. These are the spaces X such that every connected component of X is open and admits only constant real-valued functions (Theorem 3.1). In particular, for a non-discrete functionally Hausdorff space (in particular, Tikhonov space) X the pair $\langle X,\mathbb{R}\rangle$ does not have the V-property (Corollary 3.6). The proof is, roughly speaking, based on:

(i) a reduction technique which permits us to consider only connected spaces X (Theorem 2.1);

(ii) a construction, for X such that $\langle X,\mathbb{R}\rangle$ has the V-property and $\mathcal{C}(X,\mathbb{R})\neq\mathrm{Const}$, of functions $h\in\mathcal{C}(X,\mathbb{R})$ that "detect non-closed sets", i.e., such that h^{-1} is not closed for some nowhere dense $S\subseteq\mathbb{R}$ (Lemma 3.8);

(iii) a construction, for X such that $\mathcal{C}(X,\mathbb{R}) \neq \mathrm{Const}$, of an appropriate discontinuous function $g \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ (Lemma 3.9).

Step (iii) also permits showing in Section 4 that no class of functions from $\mathbb{R}$ to $\mathbb{R}$ between $\mathcal{C}(\mathbb{R},\mathbb{R})$ and the class of measurable functions can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ (Corollary 4.2).

Properties of $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ are given in Section 1.2. In Section 1.3 we give the first examples of non-trivial V-spaces (Cook's continuum) and their permanence properties. More precisely, if the pair $\langle X,Y\rangle$ has the V-property, then so does every pair $\langle X',Y'\rangle$ where X' is a retract of X and Y' is a subspace of Y (Proposition 1.8 and Corollary 1.10). In Section 5.1 step (i) is elaborated further in Theorem 5.1 which permits one to describe the behavior of V-spaces under topological sums (Corollary 5.2). This gives new examples of V-spaces (Corollary 5.6 and Proposition 5.7).

In our main result, Corollary 3.6, $\mathbb{R}$ can be replaced by Sierpiński's dyad S: for a T_0-space X the pair $\langle X,S\rangle$ has the V-property if and only if X is discrete. (See also open question 5.16.) Consequently, if a pair $\langle X,Y\rangle$ has the V-property for T_0-spaces X and Y with $\mathcal{C}(X,Y)\neq Y^X$, then Y is necessarily T_1. Hence among T_0-spaces the finite V-spaces are precisely the discrete ones (Example 5.8(I)). Here we discuss also another class of V-spaces—the spaces with the co-finite (more generally, co-α) topology (Example 5.8(II)).

In Section 5.2 we study stability of the V-property under cartesian products (Proposition 5.9, Corollaries 5.10 and 5.11). We also show that all finite powers of a Cook continuum are V-spaces (Corollary 5.12). We finish Section 5.2 with further examples of V-spaces based on another natural topological construction carried out on Cook's continuum (Example 5.17, Remark 5.18).

1.2. *Properties of* $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. First, note that $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ can be the empty family. This happens, for example, if $\emptyset\in\mathcal{A}$ and $\emptyset\notin\mathcal{B}$. Since this is a trivial case, in what follows we always assume that all classes of functions we consider are non-empty.

Now, if $\mathcal{C}_{\mathcal{A},\mathcal{B}}\neq\emptyset$ it is easy to see that $\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}_{\mathcal{A}\setminus\{\emptyset\},\mathcal{B}\setminus\{\emptyset\}}$. Thus, for the remainder of this paper we assume that $\emptyset\notin\mathcal{A}$.

Note also that if $\mathcal{A}=\emptyset$ then $\mathcal{C}_{\mathcal{A},\mathcal{B}}=Y^X$. However, we also have $Y^X=\mathcal{C}_{\mathcal{P}(X),\mathcal{P}(Y)}=\mathcal{C}_{\mathcal{P}(X)\setminus\{\emptyset\},\mathcal{P}(Y)\setminus\{\emptyset\}}$. Thus, we always assume that $\mathcal{A}$ contains a non-empty set.

With this agreement in place we can state the first basic observation that is similar in flavor to that from [2, Thm. 1].

FACT 1.2. (i) *If* $\mathcal{A}^*=\{f[A] : A\in\mathcal{A}\ \&\ f\in\mathcal{C}_{\mathcal{A},\mathcal{B}}\}\subseteq\mathcal{B}$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}_{\mathcal{A},\mathcal{A}^*}$.

(ii) $\mathrm{Const}\subseteq\mathcal{C}_{\mathcal{A},\mathcal{B}}$ *if and only if* $[Y]^1\subseteq\mathcal{B}$.

(iii) *If* $[Y]^1\subseteq\mathcal{B}$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}_{\mathcal{A}\setminus[X]^1,\mathcal{B}}=\mathcal{C}_{\mathcal{A}\cup[X]^1,\mathcal{B}}$.

(iv) *If* $[Y]^1 \subseteq \mathcal{B}$ *and there exists* $B \in \mathcal{B} \cap [Y]^2$ *then* $B^X \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

(v) *If* $X = Y$ *then* $\mathrm{id}_X \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *if and only if* $\mathcal{A} \subseteq \mathcal{B}$.

(vi) *If* $X = Y$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ *forms a semigroup with respect to composition iff* $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, *where* $\mathcal{A}^*$ *is as in* (i).

Proof. The properties (i)–(v) are obvious, as is the implication "$\Leftarrow$" in (vi). To see the other implication of (vi) notice that, by (i), $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, since, by (v), $\mathcal{A} \subseteq \mathcal{A}^*$. On the other hand, $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} \subseteq \mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*}$, as $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is closed under composition. ∎

In the case when a pair $\langle X, Y\rangle$ has the V-property we can extend the remarks of Fact 1.2 as follows. Note first that if X is discrete (or Y is indiscrete) then $\mathcal{C}(X,Y) = Y^X$, and so $\langle X, Y\rangle$ has the V-property. In fact, any discrete space (and any indiscrete space) is a V-space. Thus, to avoid this trivial case we will try to stay away from the situation when X is discrete.

FACT 1.3. *Let* X *be a non-discrete topological space and* $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}(X,Y)$. *Then*

(i) $\mathrm{Const} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *and* $[Y]^1 \subseteq \mathcal{B}$;

(ii) $\mathcal{B} \cap [Y]^2 = \emptyset$;

(iii) $\mathcal{A} \subseteq \mathcal{P}(W) \cup \mathcal{P}(X \setminus W)$ *for every clopen subset* W *of* X;

(iv) *each* $A \in \mathcal{A}$ *is contained in some quasi-component of* X;

(v) X *is not a totally disconnected space*;

(vi) $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}(X,Y)$ *with* $\mathcal{A}^* = \{f[A] : A \in \mathcal{A}\ \&\ f \in \mathcal{C}(X,Y)\} \subseteq \mathcal{B}$;

(vii) *if* $X = Y$ *then* $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} = \mathcal{C}(X,X)$ *where* $\mathcal{A}^*$ *is as in* (vi).

Proof. (i) follows from Fact 1.2(ii).

(ii) follows from (i) and Fact 1.2(iv) since X is not discrete and Y is T_0.

(iii) follows from (ii) since for every $A \in \mathcal{A} \setminus (\mathcal{P}(W) \cup \mathcal{P}(X \setminus W))$ and any distinct $b_0, b_1 \in Y$ the characteristic function $f : X \to Y$ equal to b_1 on W and b_0 on $X \setminus W$ belongs to $\mathcal{C}(X,Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, and so $\{b_0, b_1\} = f[A] \in \mathcal{B}$.

(iv) follows immediately from (iii).

To see (v) note that if X were totally disconnected then, by (iv), $\mathcal{A} \subseteq [X]^1$ and, by Fact 1.2(iii), $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}_{\emptyset,\mathcal{B}} = Y^X$, implying that X is discrete. (vi) and (vii) follow immediately from Fact 1.2(i) and (vi), respectively. ∎

Note that by Facts 1.2(ii) and 1.3(i) we can assume that $[X]^1 \subseteq \mathcal{A}$ and

$$X = \bigcup \mathcal{A} \tag{1}$$

when considering the problem whether $\langle X, Y\rangle$ has the V-property. Notice also that Fact 1.3(v) implies, in particular, that no non-discrete zero-dimensional space is a V-space.

According to Fact 1.3(vi) if $\langle X, Y\rangle$ is a pair with the V-property for some $\mathcal{A}$ and $\mathcal{B}$, then it is so for $\mathcal{A}$ and $\mathcal{A}^*$, where $\mathcal{A}^*$ consists of all *continuous* images of sets of $\mathcal{A}$. In other words, *the class $\mathcal{B}$ is not relevant* once we know that the V-property is available. In particular, for a V-space X we have $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*}$ for some family $\mathcal{A}^*\subseteq\mathcal{P}(X)$.

1.3. *When the V-property is available*. Below we give some easy examples of pairs with the V-property. The case $\mathcal{C}(X,Y)=Y^X$ was already discussed above. Now we consider the opposite case, i.e., when $\mathcal{C}(X,Y)=\mathrm{Const}$.

LEMMA 1.4. *If $\mathcal{C}(X,Y)=\mathrm{Const}$, then the pair $\langle X,Y\rangle$ has the V-property.*

Proof. It suffices to note that $\mathcal{C}(X,Y)=\mathcal{C}_{\{X\},[Y]^1}$. ■

A large number of examples of pairs $\langle X,Y\rangle$ with the V-property can be found with the help of the above proposition. We recall that a space X is *irreducible* if every non-empty open subset of X is dense in X (or, equivalently, every open subspace of X is connected).

COROLLARY 1.5. *The pair $\langle X,Y\rangle$ has the V-property in either of the following cases.*

- *X is arcwise connected and Y does not contain any arc.*
- *X is connected and Y is totally disconnected.*
- *X is irreducible and Y is Hausdorff.*

Proof. This follows from the fact that in all these cases $\mathcal{C}(X,Y)=\mathrm{Const}$. ■

This idea cannot help to get non-trivial V-spaces X as $\mathrm{id}_X\in\mathcal{C}(X,X)$. To this end we have to take larger $\mathcal{C}(X,X)$.

PROPOSITION 1.6. *If X is a compact topological space such that every continuous function $f:X\to X$ is either constant or a homeomorphism then X is a V-space.*

Proof. Let $\mathcal{A}$ be the family of all closed subsets of X that do not have precisely two elements. We claim that $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A},\mathcal{A}}$.

Clearly $\mathcal{C}(X,X)\subset\mathcal{C}_{\mathcal{A},\mathcal{A}}$. To see the other inclusion take $f\in\mathcal{C}_{\mathcal{A},\mathcal{A}}\setminus\mathrm{Const}$. Then f is one-to-one, since otherwise there is a three-element set A (which belongs to $\mathcal{A}$) such that $f[A]$ has two elements, i.e., does not belong to $\mathcal{A}$. Thus, f is continuous, being a closed mapping which is one-to-one and defined on a compact space. ■

In [3] Cook constructed a continuum K such that $\mathcal{C}(K,K)=\mathrm{Const}\cup\{\mathrm{id}_K\}$.

COROLLARY 1.7. *There exists a continuum K (Cook's continuum) which is a V-space.*

Proof. Follows from Proposition 1.6. ■

We finish this section with the following easy but fundamental facts.

PROPOSITION 1.8. *If $\langle X,Z\rangle$ has the V-property and Y is a subspace of Z then $\langle X,Y\rangle$ also has the V-property.*

Proof. Let $\mathcal{A}\subseteq\mathcal{P}(X)$ and $\mathcal{B}\subseteq\mathcal{P}(Z)$ be such that $\mathcal{C}(X,Z)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ and let $\mathcal{B}'=\mathcal{B}\cap\mathcal{P}(Y)$. It is enough to notice that $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}'}$.

To see this, let $f:X\to Y$. If $f\in\mathcal{C}(X,Y)\subseteq\mathcal{C}(X,Z)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ then $f[A]\in\mathcal{B}\cap\mathcal{P}(Y)=\mathcal{B}'$ for every $A\in\mathcal{A}$, i.e., $f\in\mathcal{C}_{\mathcal{A},\mathcal{B}'}$. Conversely, if $f\in\mathcal{C}_{\mathcal{A},\mathcal{B}'}\subseteq\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}(X,Z)$ then $f\in\mathcal{C}(X,Y)$. ∎

Notice that the domain counterpart of Proposition 1.8 strongly fails, in the sense that the V-property of a pair $\langle X,Y\rangle$ is not necessarily inherited even by closed compact subsets of X. (Compare with Corollaries 1.10 and 1.11.) To see this, let K be a continuum which is a V-space (e.g., a Cook continuum) and let S be a converging sequence in K together with its limit point. Then $\langle K,K\rangle$ has the V-property. However, by Fact 1.3(v), $\langle S,K\rangle$ does not have the V-property since S is non-discrete totally disconnected.

Note also that the pair $\langle K,S\rangle$ has the V-property since K is connected and S is totally disconnected (Corollary 1.5). In particular, the property "$\langle X,Y\rangle$ has the V-property" is not symmetric in the sense that there are examples of pairs $\langle X,Y\rangle$ with the V-property such that $\langle Y,Y\rangle$ does not have the V-property. Another example of a "non-symmetric pair" is given by the pairs $\langle\mathbb{R},K\rangle$ and $\langle K,\mathbb{R}\rangle$. The pair $\langle\mathbb{R},K\rangle$ has the V-property again by Corollary 1.5 (Cook's continuum K does not contain any arc), while the second pair does not have the V-property by Theorem 3.1.

LEMMA 1.9. *If $\langle X,Y\rangle$ has the V-property and $f:X\to Z$ is a continuous quotient map, then $\langle Z,Y\rangle$ has the V-property.*

Proof. Let $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ with $\mathcal{A}\subseteq\mathcal{P}(X)$ and $\mathcal{B}\subseteq\mathcal{P}(Y)$. Then $\mathcal{C}(Z,Y)=\mathcal{C}_{f[\mathcal{A}],\mathcal{B}}$ with $f[\mathcal{A}]=\{f[A]:A\in\mathcal{A}\}$. The inclusion $\mathcal{C}(Z,Y)\subseteq\mathcal{C}_{f[\mathcal{A}],\mathcal{B}}$ is obvious. The other inclusion follows easily from our assumption that f is a quotient map. ∎

Note that in this lemma f being just "continuous surjective" does not suffice. To see this, take any pair $\langle Z,Y\rangle$ that does not have the V-property and take as X the underlying set of Z equipped with the discrete topology. Then $\langle X,Y\rangle$ has the V-property and $\mathrm{id}_Z:X\to Z$ is a continuous bijection.

The above lemma gives a partial domain counterpart of Proposition 1.8.

COROLLARY 1.10. *If $\langle X,Y\rangle$ has the V-property and Z is a retract of X, then also $\langle Z,Y\rangle$ has the V-property. In particular, any retract of a V-space is again a V-space.* ∎

For further use we also give the following particular cases.

COROLLARY 1.11. *If $\langle X, Y\rangle$ has the V-property and Z is a clopen subset of X then $\langle Z, Y\rangle$ also has the V-property. In particular, a clopen subset of a V-space is a V-space.*

Proof. Every clopen subset of a space is its retract. ∎

COROLLARY 1.12. *If $\langle X\times Z, Y\rangle$ has the V-property, then $\langle Z, Y\rangle$ also has the V-property. In particular, if $X\times Z$ is a V-space then so are X and Z.* ∎

2. A reduction theorem. The main goal of this section is to prove the next theorem which partially reduces the question of when the pair $\langle X, Y\rangle$ has the V-property to the case when X is connected. It is a particular case of Theorem 5.1.

THEOREM 2.1. *The pair $\langle X, Y\rangle$ has the V-property if and only if there exists $\mathcal{B}\subseteq\mathcal{P}(Y)$ such that for every component C of X,*

(a) *C is open in X;*
(b) *there exists $\mathcal{A}_C\subseteq\mathcal{P}(C)$ such that $\mathcal{C}(C, Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$.*

In particular, all pairs $\langle C, Y\rangle$ have the V-property.

In the proof we use the following easy fact.

LEMMA 2.2. *If $\langle X, Y\rangle$ has the V-property then every quasi-component of X is open and connected.*

Proof. Let $\mathcal{C}(X, Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ and Q be a quasi-component of X. Choose $a\neq b$ in Y and consider the characteristic function $f: X\to\{a, b\}\subseteq Y$ of Q. By Fact 1.3(iii) each $A\in\mathcal{A}$ is either contained in Q or disjoint from Q. In either case $f[A]$ is a singleton, so $f[A]\in\mathcal{B}$ and $f\in\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}(X, Y)$. This yields that Q is clopen. In particular, Q cannot contain proper clopen subsets, hence Q is connected. ∎

Proof of Theorem 2.1. Let $\mathcal{C}(X, Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$. The necessity of (a) follows from Lemma 2.2. To see (b) let C be a component of X and $\mathcal{A}_C=\mathcal{A}\cap\mathcal{P}(C)$. We claim that $\mathcal{C}(C, Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$.

The inclusion $\mathcal{C}(C, Y)\subset\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ follows from the fact that, by (a), any continuous $f: C\to Y$ can be extended to a continuous function $\widetilde{f}: X\to Y$ and any such function sends sets from $\mathcal{A}_C=\mathcal{A}\cap\mathcal{P}(C)$ into $\mathcal{B}$.

To see the other inclusion take $f: C\to Y$ from $\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and extend it to $\widetilde{f}: X\to Y$ assigning a constant value on $X\setminus C$. Then, by (a) and Fact 1.3(iii), any $A\in\mathcal{A}$ is either in $\mathcal{A}_C$ or is disjoint from C. In any case $\widetilde{f}[A]\in\mathcal{B}$, i.e., $\widetilde{f}\in\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}(X, Y)$. So $f\in\mathcal{C}(C, Y)$.

To see that the conditions (a) and (b) are sufficient, for every component C of X let $\mathcal{A}_C\subset\mathcal{P}(C)$ be such that $\mathcal{C}(C, Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and define $\mathcal{A}$ as the union of all families $\mathcal{A}_C$. We claim that $\mathcal{C}(X, Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Let $f \in \mathcal{C}(X,Y)$ and $A \in \mathcal{A}$. Then there exists a component C of X such that $A \in \mathcal{A}_C$. So, $f[A] = f|_C[A] \in \mathcal{B}$, since $f|_C \in \mathcal{C}(C,Y) = \mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$. Thus, $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

To see the other inclusion take $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Then for every component C of X we have $f \in \mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and $f|_C \in \mathcal{C}_{\mathcal{A}_C,\mathcal{B}} = \mathcal{C}(C,Y)$. But all sets C are clopen. So, f is continuous. ∎

Note that according to Theorem 2.1(a), for every connected space C and every space Y the pair $\langle \mathbb{Q} \times C, Y\rangle$ fails to have the V-property. (Here, as elsewhere in the paper, we assume that Y is not indiscrete and $\mathbb{Q}$ denotes the rationals.)

Theorem 2.1 also gives a new proof of Corollary 1.11: if $\langle X,Y\rangle$ has the V-property and Z is a clopen subset of X then $\langle Z,Y\rangle$ also has the V-property. Indeed, let $\mathcal{B} \subset \mathcal{P}(Y)$ be a family satisfying (a) and (b) of Theorem 2.1 for $\langle X,Y\rangle$. Then $\mathcal{B}$ and the same families $\mathcal{A}_C$ satisfy (a) and (b) for $\langle Z,Y\rangle$ since Z is clopen in X.

3. When the pair $\langle X,\mathbb{R}\rangle$ has the V-property. The main goal of this section is to prove the following generalization of Velleman's theorem.

THEOREM 3.1. *Let X be a topological space. The pair $\langle X,\mathbb{R}\rangle$ has the V-property if and only if for every component C of X,*

(i) *C is open in X; and*
(ii) $\mathcal{C}(C,\mathbb{R}) = \mathrm{Const}$.

Before we prove it, let us notice the following corollaries.

COROLLARY 3.2. *Let X be a topological space for which there exists a component C of X such that either C is not open or $\mathcal{C}(C,\mathbb{R}) \neq \mathrm{Const}$. If Y contains an arc then $\langle X,Y\rangle$ does not have the V-property.*

Proof. Follows from Theorem 3.1 and Proposition 1.8. ∎

COROLLARY 3.3. *Let C be a connected topological space. Then the pair $\langle C,\mathbb{R}\rangle$ has the V-property if and only if $\mathcal{C}(C,\mathbb{R}) = \mathrm{Const}$.* ∎

Before we give further corollaries let us see that one can have regular connected topological spaces with the property (ii).

EXAMPLE 3.4. There exists a regular topological space X with $\mathcal{C}(X,\mathbb{R}) = \mathrm{Const}$. (See [4, Sect. 1.5 and Exerc. 2.R] or [5].) In particular, such an X is connected and $\langle X,\mathbb{R}\rangle$ has the V-property.

A topological space X is *functionally Hausdorff* if the functions $f \in \mathcal{C}(X,\mathbb{R})$ separate the points of X. Note that every completely regular space is functionally Hausdorff.

Corollary 3.5. *Let X be a non-discrete functionally Hausdorff space. If Y contains an arc then $\langle X, Y\rangle$ does not have the V-property.* ∎

Corollary 3.6. *Let X be a functionally Hausdorff space. The pair $\langle X, \mathbb{R}\rangle$ has the V-property if and only if X is discrete.* ∎

We split the proof of Theorem 3.1 into a sequence of steps. The first one, based on the reduction theorem, reduces the proof to the case of a connected space, i.e., to Corollary 3.3.

Proof of Theorem 3.1. Assume that (i) and (ii) are fulfilled. Then $\mathcal{C}(C,\mathbb{R}) = \mathcal{C}_{\mathcal{P}(C),[\mathbb{R}]^1}$ for every component C of X. So, by Theorem 2.1, the pair $\langle X, \mathbb{R}\rangle$ has the V-property.

On the other hand, assume that $\langle X, \mathbb{R}\rangle$ has the V-property. By Theorem 2.1 every component C of X is open in X, and $\langle C, \mathbb{R}\rangle$ has the V-property. So, Corollary 3.3 yields $\mathcal{C}(C,\mathbb{R}) = \mathrm{Const}$. ∎

The proof of Corollary 3.3 is split into the following two steps.

Proposition 3.7. *If X is a topological space for which there exists a continuous function $h : X \to \mathbb{R}$ such that*

$$h^{-1}(S) \text{ is not closed in } X \text{ for some nowhere dense } S \subseteq \mathbb{R} \tag{2}$$

then $\langle X, \mathbb{R}\rangle$ does not have the V-property.

The next lemma ensures the validity of (2) for connected topological spaces with non-constant continuous real-valued functions. The proof of Proposition 3.7 will be given later in this section.

Lemma 3.8. *Let X be a connected topological space with $\mathcal{C}(X,\mathbb{R}) \neq \mathrm{Const}$. Then there exists a function as in* (2).

Proof. Let $f : X \to \mathbb{R}$ be a non-constant continuous function. We prove first that

$$\text{there exists } T \subseteq \mathbb{R} \text{ such that } f^{-1}(T) \text{ is not closed in } X.$$

Assume otherwise. Since f is non-constant there exists $a \in \mathbb{R}$ such that both $T = [a, \infty)$ and $\mathbb{R} \setminus T$ intersect $f[X]$. This produces a non-trivial partition $f^{-1}(T) \cup f^{-1}(\mathbb{R} \setminus T)$ of X into closed sets, a contradiction. This proves our claim.

Now fix a $T \subseteq \mathbb{R}$ such that $f^{-1}(T)$ is not closed in X. Pick an $x \in \mathrm{cl}(f^{-1}(T)) \setminus f^{-1}(T)$ and define $T^+ = T \cap [f(x), \infty)$ and $T^- = T \cap (-\infty, f(x)]$. Since obviously at least one of the two possibilities

$$x \in \mathrm{cl}(f^{-1}(T^+)) \setminus f^{-1}(T^+) \quad \text{or} \quad x \in \mathrm{cl}(f^{-1}(T^-)) \setminus f^{-1}(T^-)$$

occurs, we can assume without loss of generality that $T = T^-$. Since $f(x) \notin T$, we have $T = T^- \subseteq (-\infty, f(x))$. Next we note that it is not restrictive to assume $T = (-\infty, f(x))$ as $x \in \mathrm{cl}(f^{-1}(-\infty, f(x))) \setminus f^{-1}(-\infty, f(x))$.

Now fix a strictly increasing sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converging to $f(x)$ and set

$$A = \bigcup_{n=0}^{\infty} f^{-1}(a_{2n}, a_{2n+1}], \qquad B = \bigcup_{n=0}^{\infty} f^{-1}(a_{2n+1}, a_{2n+2}]$$

with $a_0 = -\infty$. Clearly $f^{-1}(T) = f^{-1}(A \cup B)$, so either $x \in \mathrm{cl}(f^{-1}(A))$ or $x \in \mathrm{cl}(f^{-1}(B))$. Since the proof is similar in both cases assume the first of these. Now define a continuous map $j : \mathbb{R} \to \mathbb{R}$ such that $j(f(x)) = 0$ and $j[(a_{2n}, a_{2n+1}]] = 1/(n+1)$. Consider the continuous map $h = j \circ f$ and let S be the set $\{1/n : n \in \omega\}$. Note that $h^{-1}(S)$ contains $f^{-1}(A)$ which has x in its closure but $x \notin h^{-1}(S)$ since $h(x) = j(f(x)) = 0$. So, h and S satisfy (2). ■

In the proof of Proposition 3.7 we will use the following lemma. (The "moreover" part will also be used in the next section.)

LEMMA 3.9. *Let X, h and S be as in Proposition* 3.7, $[\mathbb{R}]^1 \subseteq \mathcal{B} \subseteq \mathcal{P}(X)$, *$B \in \mathcal{B}$ be infinite, and $\mathcal{A} \subseteq \mathcal{P}(X)$ be such that*

(3) $\quad \mathrm{cl}(h[A])$ *is an interval for every $A \in \mathcal{A}$.*

Assume that there exists a family $\mathcal{J}$ of pairwise disjoint closed subsets of $\mathbb{R} \setminus \mathrm{cl}(S)$ with the property that for every $x < y$,

(4) $\quad$ *either $[x, y] \subseteq J$ for some $J \in \mathcal{J}$ or $|\{J \in \mathcal{J} : J \subset (x, y)\}| \geq |B|$*

and

(5) $\quad h[A] \cap J \neq \emptyset$ *for every $A \in \mathcal{A}$ and $J \in \mathcal{J}$ with $J \subset \mathrm{cl}(h[A])$.*

Then there exists $g : \mathbb{R} \to B$ such that $f = g \circ h \in \mathcal{C}_{\mathcal{A},\mathcal{B}} \setminus \mathcal{C}(X, \mathbb{R})$. Moreover, if $\mathrm{cl}(S)$ has positive Lebesgue measure, then g can be chosen non-measurable.

Proof. Let $\mathcal{I}$ be the family of all non-empty open intervals with rational endpoints and let $\langle \langle I_\xi, b_\xi \rangle : \xi < |B| \rangle$ be an enumeration of $\mathcal{I} \times B$. By induction on $\xi < |B|$ choose a one-to-one sequence $\langle J_\xi \in \mathcal{J} : \xi < |B| \rangle$ such that

(6) $\quad J_\xi \subseteq I_\xi$ provided $|\{J \in \mathcal{J} : J \subset I_\xi\}| \geq |B|$.

Fix distinct $a, c \in B$ and define $g : \mathbb{R} \to B$ by

$$g(x) = \begin{cases} b_\xi & \text{if } x \in J_\xi \text{ for some } \xi < |B|, \\ a & \text{if } x \in S, \\ c & \text{otherwise.} \end{cases}$$

To see that $f = g \circ h$ belongs to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ take $A \in \mathcal{A}$. We now show that $f[A] = g[h[A]] \in [\mathbb{R}]^1 \cup \{B\} \subseteq \mathcal{B}$.

If $\mathrm{cl}(h[A])$ is a singleton, then so are $h[A]$ and $f[A] = g[h[A]]$. In particular, $f[A] \in [\mathbb{R}]^1 \subseteq \mathcal{B}$. So, assume that $\mathrm{cl}(h[A])$ is not a singleton. Then, by (3), there are $x < y$ such that $(x, y) \subseteq \mathrm{cl}(h[A]) \subseteq [x, y]$. Consider two cases.

CASE 1: There exists $I \in \mathcal{I}$ such that $I \subseteq (x,y)$ and $|\{J \in \mathcal{J} : J \subset I\}| \geq |B|$. Take $b \in B$. Then there exists $\xi < |B|$ such that $\langle I, b\rangle = \langle I_\xi, b_\xi\rangle$ and, by (6), $J_\xi \subseteq I_\xi = I \subseteq (x,y) \subseteq \mathrm{cl}(h[A])$. In particular, by (5), $h[A] \cap J_\xi \neq \emptyset$ and so $\emptyset \neq g[h[A] \cap J_\xi] \subseteq g[J_\xi] = \{b_\xi\} = \{b\}$. Thus, $b \in g[h[A]]$. Since $b \in B$ was arbitrary, we conclude that $B \subseteq g[h[A]]$. So, $g[h[A]] = B \in \mathcal{B}$.

CASE 2: For every $I \in \mathcal{I}$ if $I \subseteq (x,y)$ then $|\{J \in \mathcal{J} : J \subset I\}| < |B|$. Then, by (4), for every $I \in \mathcal{I}$ with $I \subseteq (x,y)$ there exists $J_I \in \mathcal{J}$ such that $I \subseteq J_I$. Since elements of $\mathcal{J}$ are pairwise disjoint, all J_I must be equal to the same $J_0 \in \mathcal{J}$ and $(x,y) = \bigcup\{I \in \mathcal{I} : I \subseteq (x,y)\} \subseteq J_0$. So, $h[A] \subseteq [x,y] \subseteq \mathrm{cl}(J_0) = J_0$. But g is constant on every $J \in \mathcal{J}$. Thus, $g[J_0]$ is a singleton, implying that $g[h[A]] \in [\mathbb{R}]^1 \subseteq \mathcal{B}$.

To see that $f \notin \mathcal{C}(X,\mathbb{R})$ let $V = h^{-1}(S)$ and $x \in \mathrm{cl}(V) \setminus V$, existing by (2). Then $h(x) \in h[\mathrm{cl}(V)] \subseteq \mathrm{cl}(h[V]) \subset \mathrm{cl}(S)$, while $x \notin V = h^{-1}(S)$, i.e., $h(x) \in \mathrm{cl}(S) \setminus S$. So,

$$c = g(h(x)) = f(x) \in f[\mathrm{cl}(V)]$$

while $c \notin \{a\} = \mathrm{cl}(g[S]) = \mathrm{cl}(g[h[V]]) = \mathrm{cl}(f[V])$, proving that f is discontinuous.

To prove the "moreover" part, take a non-measurable set $E \subseteq \mathrm{cl}(S)$, fix distinct $a, a', c, c' \in B$ and redefine $g : \mathbb{R} \to B$ by

$$g(x) = \begin{cases} b_\xi & \text{if } x \in J_\xi \text{ for some } \xi < |B|, \\ a & \text{if } x \in S \cap E, \\ a' & \text{if } x \in S \setminus E, \\ c & \text{if } x \in E \setminus S, \\ c' & \text{otherwise.} \end{cases}$$

Then $g^{-1}(\{a,c\}) = E$ is non-measurable, so g is not measurable. It is easy to see that for this modification of our original g we still have $f = g \circ h \in \mathcal{C}_{\mathcal{A},\mathcal{B}} \setminus \mathcal{C}(X,\mathbb{R})$. ∎

Proof of Proposition 3.7. By way of contradiction assume that there exist $\mathcal{A} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ such that $\mathcal{C}(X,\mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Note that, by (2), X is not discrete. So, by Fact 1.3, $\mathcal{B}$ contains all singletons and does not contain any doubleton. Moreover, we can assume that

$$\mathcal{B} = \mathcal{A}^* = \{f[A] : A \in \mathcal{A} \ \&\ f \in \mathcal{C}(X,\mathbb{R})\}.$$

Next notice that

(7) $\quad \mathrm{cl}(f[A])$ is an interval for every $A \in \mathcal{A}$ and $f \in \mathcal{C}(X,\mathbb{R})$.

Indeed, otherwise $\mathrm{cl}(f[A])$ is disconnected, so there are two disjoint non-empty closed subsets F_0 and F_1 of $\mathrm{cl}(f[A])$. Then, by normality of $\mathbb{R}$, there exists a continuous function $g : \mathbb{R} \to [0,1]$ with $g[F_0] = \{0\}$ and $g[F_1] = \{1\}$.

Therefore $\bar{f} = g \circ f \in \mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and $\{0,1\} = \bar{f}[A] \in \mathcal{B}$, contradicting $\mathcal{B} \cap [\mathbb{R}]^2 = \emptyset$.

Now, $\mathcal{B} \not\subseteq [\mathbb{R}]^1$, since $h \in \mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is not constant. Hence, by (7),

(8) $\mathcal{B}$ contains an infinite set.

Next note that

(9) $\mathcal{B}$ does not contain any infinite countable set.

We apply Lemma 3.9 to show this. So, by way of contradiction assume that there exists a countable infinite $B \in \mathcal{B}$. Note that (7) implies (3). Let $\mathcal{J}$ be a family of non-trivial pairwise disjoint closed subintervals of $\mathbb{R} \setminus \mathrm{cl}(S)$ with the property that between any two distinct intervals from $\mathcal{J}$ there is another interval $J \in \mathcal{J}$, and $\bigcup \mathcal{J}$ is dense in $\mathbb{R}$. It is easy to see that such a $\mathcal{J}$ satisfies (4) and (5). So, Lemma 3.9 leads to a contradiction with our assumption that $\mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Next note that for every $A \in \mathcal{A}$,

(10) $h[A] \cap P \neq \emptyset$ for every perfect set $P \subset \mathrm{cl}(h[A])$.

Indeed, otherwise there is a continuous "Cantor-like" function g from $\mathbb{R}$ onto $[0,1]$ with $g[\mathrm{cl}(h[A]) \setminus P]$ being countable infinite. Now $g \circ h : X \to \mathbb{R}$ is continuous and $(g \circ h)[A] \subseteq g[\mathrm{cl}(h[A]) \setminus P]$ is infinite countable, contradicting (9).

To finish the proof, take an arbitrary infinite $B \in \mathcal{B}$, which exists by (8), and let $\mathcal{J}$ be a family of pairwise disjoint perfect subsets of $\mathbb{R} \setminus \mathrm{cl}(S)$ such that continuum many of them lie inside any non-degenerate subinterval of $\mathbb{R}$. Then conditions (3)–(5) of Lemma 3.9 are satisfied, implying that $\mathcal{C}(X, \mathbb{R}) \neq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. ■

4. Families of real functions. Notice that there are non-trivial classes of real functions that are equal to $\mathcal{C}_{\mathcal{A},\mathcal{A}}$ for some $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$. For example the class $\mathcal{D}$ of all Darboux functions is defined as the class of functions for which the images of connected sets are connected. Thus, $\mathcal{D} = \mathcal{C}_{\mathcal{A},\mathcal{A}}$, where $\mathcal{A}$ is the family of all connected subsets of $\mathbb{R}$.

The next theorem is a generalization of Theorem 3.1 in the case $X = \mathbb{R}$ and it implies that many classes of real functions cannot be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$.

THEOREM 4.1. *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ are such that $\mathcal{C}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ then there is a non-measurable function $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.*

Proof. The proof is very similar to that of Theorem 3.1. We will use here the identity function id as an h for which any non-closed nowhere dense $S \subseteq \mathbb{R}$ will satisfy (2). We will choose such an S with $\mathrm{cl}(S)$ having positive Lebesgue measure.

Take $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ such that $\mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. By Fact 1.2(ii) the family $\mathcal{B}$ contains all singletons. Also, by Fact 1.2(iv), if $\mathcal{B}$ contains a doubleton B then $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ contains the characteristic function $h : \mathbb{R} \to B$ of a non-measurable set, i.e., a non-measurable function. So, without loss of generality we can assume that $\mathcal{B}$ does not contain any doubleton. By Fact 1.2(i) we can also assume that

$$\mathcal{B} = \mathcal{A}^* = \{f[A] : A \in \mathcal{A} \ \&\ f \in \mathcal{C}(\mathbb{R},\mathbb{R})\}.$$

Next note that

$$\mathrm{cl}(f[A]) \text{ is an interval for every } A \in \mathcal{A} \text{ and } f \in \mathcal{C}(\mathbb{R},\mathbb{R}), \tag{11}$$

the argument being identical to that for the condition (7) of Theorem 3.1.

Now, $\mathcal{B} \not\subseteq [\mathbb{R}]^1$, since $\mathrm{id} \in \mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Hence, by (11),

$$\mathcal{B} \text{ contains an infinite set.} \tag{12}$$

If $\mathcal{B}$ contains a countable infinite set B then we can apply Lemma 3.9 to the family $\mathcal{J}$ of intervals used to prove the condition (9) of Theorem 3.1, and conclude that there exists a non-measurable function in $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. So assume that $\mathcal{B}$ does not contain a countable infinite subset. Then, as in the case of the proof of condition (10) of Theorem 3.1, we see that

$$A \cap P \neq \emptyset \quad \text{for every } A \in \mathcal{A} \text{ and every perfect set } P \subset \mathrm{cl}(A).$$

To finish the proof, it is enough to apply Lemma 3.9 to the family $\mathcal{J}$ of pairwise disjoint perfect subsets of $\mathbb{R} \setminus \mathrm{cl}(S)$ such that continuum many of them lie inside any non-degenerate subinterval of $\mathbb{R}$. ■

COROLLARY 4.2. *Neither of the following classes of functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *can be represented as* $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ *for any* $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$:

- *the class of upper or lower semicontinuous functions*;
- *the class of derivatives*;
- *the class of approximately continuous functions*;
- *the class of Baire class* 1 *functions*;
- *the class of Borel functions*;
- *the class of measurable functions.*

Proof. If $\mathcal{F}$ is any of the above classes then $\mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{F}$ and every function in $\mathcal{F}$ is measurable. ■

PROBLEM 4.3. Can the class of all functions $f : \mathbb{R} \to \mathbb{R}$ with the Baire property be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$?

As far as smaller classes of functions are concerned we have the following questions.

PROBLEM 4.4. Can any of the following classes of real functions be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$?

- The class of all linear functions $f(x) = ax + b$.
- The class of all polynomials.
- The class of all real-analytic functions.
- The class C^∞ of infinitely many times differentiable functions.
- The class D^n of n-times differentiable functions, with $1 \le n < \omega$.

5. Further remarks and examples

5.1. *Second reduction theorem.* The next theorem can be considered as a generalization of Theorem 2.1.

THEOREM 5.1. *Let $X = \bigcup_{\alpha\in I} C_\alpha$ and $Y = \bigcup_{\gamma\in J} K_\gamma$ be the partitions of the topological spaces X and Y into connected components. Then $\langle X, Y\rangle$ has the V-property if and only if*

(A) *each C_α is clopen in X; and*

(B) *for every $\alpha \in I$ and $\gamma \in J$ there exist families $\mathcal{A}_\alpha \subseteq \mathcal{P}(C_\alpha)$ and $\mathcal{B}_\gamma \subseteq \mathcal{P}(K_\gamma)$ with the property that*

$$\mathcal{C}(C_\alpha, K_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma} \quad \textit{for every } \alpha \in I \textit{ and } \gamma \in J.$$

Proof. Assume first that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ with $\mathcal{B} = \mathcal{A}^*$. Then condition (A) follows from Lemma 2.2.

To see (B) define $\mathcal{A}_\alpha = \mathcal{A} \cap \mathcal{P}(C_\alpha)$ and $\mathcal{B}_\gamma = \mathcal{B} \cap \mathcal{P}(K_\gamma)$. First notice that $\mathcal{A} = \bigcup_{\alpha\in I} \mathcal{A}_\alpha$ follows from Fact 1.3(iv). Also $\mathcal{B} = \bigcup_{\gamma\in J} \mathcal{B}_\gamma$ since continuous functions send connected sets to connected sets. In order to prove that $\mathcal{C}(C_\alpha, K_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$ take a continuous map $f : C_\alpha \to K_\gamma$. Extend f to a continuous map $\widetilde{f} : X \to Y$ by choosing an arbitrary point $b \in Y$ and assigning value b to any $x \in X \setminus C_\alpha$. Then $\widetilde{f} \in \mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Hence for every $A \in \mathcal{A}_\alpha$ we have $f[A] = \widetilde{f}[A] \in \mathcal{B}$ and $f[A] \in \mathcal{B}_\gamma$ as $f[A] \subseteq K_\gamma$. Thus $f \in \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$. The proof of the other inclusion is similar to that for Theorem 2.1.

To prove the other implication first notice that it is true for Y being discrete since we can take $\mathcal{A} = \{C_\alpha : \alpha \in I\}$ and $\mathcal{B} = [Y]^1$. Thus we assume that Y is not discrete.

Define $\mathcal{A} = \bigcup_{\alpha\in I} \mathcal{A}_\alpha$ and $\mathcal{B} = \bigcup_{\gamma\in J} \mathcal{B}_\gamma$. We now prove that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

First note that each function $f : X \to Y$ which is either continuous or in $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ defines a map $\theta : I \to J$ such that

$$f[C_\alpha] \subseteq K_{\theta(\alpha)}. \tag{13}$$

For continuous f this is obvious. So, let $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and fix $\alpha \in I$ and $x \in C_\alpha$. Let $\mathrm{St}^\omega(x, \mathcal{A}) = \bigcup_n \mathrm{St}^n(x, \mathcal{A})$, where $\mathrm{St}^n(x, \mathcal{A})$ denotes the nth iterated star of the point x with respect to the cover $\mathcal{A}$ of X. (See (1).) It is easy to see that $\mathrm{St}^\omega(x, \mathcal{A}) \subseteq C_\alpha$, and $f[\mathrm{St}^\omega(x, \mathcal{A})]$ is a subset of precisely one K_γ. Thus, it is enough to show that $\mathrm{St}^\omega(x,\mathcal{A}) = C_\alpha$. To this end take a component K_γ of Y with more than one point and consider the characteristic function $f : C_\alpha \to K_\gamma$ of $\mathrm{St}^\omega(x,\mathcal{A})$. It belongs to $\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma} = \mathcal{C}(C_\alpha, K_\gamma)$, so f is continuous. Hence $\mathrm{St}^\omega(x,\mathcal{A})$ is clopen in C_α. As C_α is connected we conclude $\mathrm{St}^\omega(x,\mathcal{A}) = C_\alpha$.

Now to prove $\mathcal{C}(X,Y) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ notice that if $f : X \to Y$ is continuous and θ is as in (13) then $f[A] \in \mathcal{B}_{\theta(\alpha)}$ for every $\alpha \in I$ and $A \in \mathcal{A}_\alpha$. So, $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

To see the other inclusion let $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and let θ be as in (13). Since the components of X are clopen, it suffices to prove that each restriction $f_\alpha = f|_{C_\alpha}$ is continuous. By the formula (13) we can factorize f_α as the composition of $g_\alpha : C_\alpha \to K_{\theta(\alpha)}$ and the inclusion $K_{\theta(\alpha)} \hookrightarrow Y$, so that the continuity of f_α follows from the continuity of $g_\alpha \in \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_{\theta(\alpha)}} = \mathcal{C}(C_\alpha, K_{\theta(\alpha)})$. ∎

COROLLARY 5.2. *Let $X = \bigoplus_\alpha X_\alpha$ be the topological direct sum of the spaces X_α. Then $\langle X,Y\rangle$ has the V-property if and only if all pairs $\langle X_\alpha,Y\rangle$ have the V-property witnessed by the same $\mathcal{B} \subseteq \mathcal{P}(Y)$.* ∎

COROLLARY 5.3. *Let $X = \bigcup_{\alpha\in I} C_\alpha$ be the partition of X into connected components. Then X is a V-space if and only if*

(A) *each C_α is clopen in X; and*

(B) *for each $\alpha \in I$ there exists a family $\mathcal{A}_\alpha \subseteq \mathcal{P}(C_\alpha)$ such that $\mathcal{C}(C_\alpha, C_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\gamma}$ for every $\alpha,\gamma \in I$.*

P r o o f. From the formulation of Theorem 5.1 it follows immediately that for each $\alpha \in I$ there exist families $\mathcal{A}_\alpha, \mathcal{B}_\alpha \subseteq \mathcal{P}(C_\alpha)$ such that $\mathcal{C}(C_\alpha, C_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$ for every $\alpha,\gamma \in I$. To see that the families $\mathcal{A}_\alpha$ and B_α can be chosen equal it is enough to notice that for a V-space X we can choose $\mathcal{B} = \mathcal{A}$, and then check the definition of $\mathcal{A}_\alpha$ and $\mathcal{B}_\gamma$ in the proof of Theorem 5.1. ∎

COROLLARY 5.4. *Let D be a discrete space. Then $\langle X,D\rangle$ has the V-property if and only if each connected component of X is clopen in X.* ∎

COROLLARY 5.5. *Let D be a discrete space. Then $X \times D$ is a V-space if and only if X is a V-space.*

P r o o f. The product $X \times D$ is a topological direct sum of $|D|$-many copies of the space X. ∎

COROLLARY 5.6. *If K is a Cook's continuum and D is a discrete space then $X \times D$ is a V-space.* ∎

A family (possibly a proper class) $\{X_\alpha\}_\alpha$ of spaces is *strongly rigid* if the only non-constant maps $X_\alpha \to X_\beta$ are the identities $X_\alpha \to X_\alpha$. A space X is *strongly rigid* if the family $\{X\}$ is strongly rigid. (See [1], [6], [7] for the existence of strongly rigid spaces and families.) Obviously every strongly rigid pair $\{X,Y\}$ of distinct spaces gives rise to two pairs $\langle X,Y\rangle$ and $\langle Y,X\rangle$ having the V-property.

PROPOSITION 5.7. *Let $\{C_\alpha\}_{\alpha\in I}$ be a strongly rigid family of continua. Then the topological direct sum $X=\bigoplus_{\alpha\in I} C_\alpha$ is a V-space.*

Proof. For every $\alpha\in I$ let $\mathcal{A}_\alpha$ be the family of closed subsets of C_α which are not doubletons. Set $\mathcal{A}=\bigcup_{\alpha\in I}\mathcal{A}_\alpha$. We prove that $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A},\mathcal{A}}$ by using Corollary 5.3. To this end we must check that $\mathcal{C}(C_\alpha,C_\beta)=\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\beta}$ for every $\alpha,\beta\in I$.

The case $\alpha=\beta$ was already established in Proposition 1.6. So, assume that $\alpha\neq\beta$. Then $\mathcal{C}(C_\alpha,C_\beta)$ has only constant maps. Suppose $f\in\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\beta}$ is non-constant. By the choice of $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ the map f is injective. Since $C_\alpha\in\mathcal{A}_\alpha$, it follows that $Z=f[C_\alpha]$ is a closed, hence compact, subset of C_β. Moreover, every closed subset of C_α is mapped onto a closed subset of Z. Therefore $f:C_\alpha\to C_\beta$ is a non-constant continuous map, a contradiction. ■

EXAMPLE 5.8. (I) In analogy with our main result in Section 3 we discuss here when the pair $\langle X,S\rangle$ has the V-property, where S denotes the Sierpiński dyad. It is easy to see (using Fact 1.2) that for a T_0-space X the pair $\langle X,S\rangle$ has the V-property if and only if X is discrete. Further, using this fact and Proposition 1.8 one can conclude that for T_0-spaces X and Y with $\mathcal{C}(X,Y)\neq Y^X$ (i.e., Y is not indiscrete and X is not discrete) the pair $\langle X,Y\rangle$ may have the V-property only if Y is T_1. Consequently, a finite T_0-space is a V-space if and only if it is discrete.

(II) Now we give examples of V-spaces of arbitrary infinite cardinality which need not be locally compact. (Note that all examples given above were locally compact.) These are non-Hausdorff T_1-spaces. Let X be a set and $\alpha\le|X|$ be a regular cardinal. Consider the co-α topology τ_α on X (having as closed sets: X and all subsets $Y\subseteq X$ with $|Y|<\alpha$). It is easy to see that $f\in\mathcal{C}(X,X)\setminus\mathrm{Const}$ if and only if f has *small fibers* (i.e., $|f^{-1}(x)|<\alpha$ for every $x\in X$). Now with $\mathcal{A}=[X]^1\cup[X]^{\ge\alpha}$ we have $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A},\mathcal{A}}$, so that X is a V-space. Note that X is always connected, while τ_α is (locally) compact precisely for $\alpha=\omega$.

5.2. *Behavior under products*. Next we examine when the V-property of a pair $\langle X,Y\rangle$ is preserved under product operations.

Now we prove the counterpart of Corollary 5.2 in the case of products.

PROPOSITION 5.9. *Let X be a space, let $\{Y_\alpha\}_{\alpha\in I}$ be a family of spaces and let $Y=\prod_{\alpha\in I}Y_\alpha$. Then $\langle X,Y\rangle$ has the V-property if and only if all pairs $\langle X,Y_\alpha\rangle$ have the V-property witnessed by the same family $\mathcal{A}\subseteq\mathcal{P}(X)$.*

Proof. The necessity follows from Proposition 1.8. Now assume that all pairs $\langle X,Y_\alpha\rangle$ have the V-property witnessed by the same family $\mathcal{A}\subseteq\mathcal{P}(X)$. According to Fact 1.3(vi),

$$\mathcal{C}(X,Y_\alpha)=\mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha}\quad\text{for all }\alpha\in I,$$

where $\mathcal{B}_\alpha = \{f_\alpha[A] : A \in \mathcal{A}\ \&\ f_\alpha \in \mathcal{C}(X, Y_\alpha)\}$. For a family $\{f_\alpha : X \to Y_\alpha\}_{\alpha\in I}$ of functions, we denote by $\langle f_\alpha\rangle$ the diagonal map $X \to Y$. We will use the fact that every continuous function $f : X \to Y$ has the form $f = \langle f_\alpha\rangle$, where each $f_\alpha : X \to Y_\alpha$ is continuous. Let $\mathcal{B} = \{\langle f_\alpha\rangle[A] : A \in \mathcal{A}\ \&\ \langle f_\alpha\rangle \in \mathcal{C}(X, Y)\}$. We now show that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

So, let $f = \langle f_\alpha\rangle \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$. To prove that $f \in \mathcal{C}(X, Y)$ it is enough to show that $f_\alpha \in \mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha} = \mathcal{C}(X, Y_\alpha)$ for every $\alpha \in I$. So, take $A \in \mathcal{A}$. Then $f[A] \in \mathcal{B}$, i.e., $f[A] = g[A']$ for some $g \in \mathcal{C}(X, Y)$ and $A' \in \mathcal{A}$. Applying the canonical projection $p_\alpha : Y \to Y_\alpha$ to both sides of this equality we get $f_\alpha[A] = g_\alpha[A'] \in \mathcal{B}_\alpha$. So, $f[A] \in \mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha}$.

The inclusion $\mathcal{C}(X, Y) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is a trivial consequence of the definition of $\mathcal{B}$. ∎

Corollary 5.10. *Let $\{Y_\alpha\}_{\alpha\in I}$ be a family of spaces. Then $Y = \prod_{\alpha\in I} Y_\alpha$ is a V-space if and only if all pairs $\langle Y, Y_\alpha\rangle$ have the V-property witnessed by the same family $\mathcal{A} \subseteq \mathcal{P}(Y)$.* ∎

In particular, according to Corollary 1.12 every Y_α is a V-space when $\prod_{\alpha\in I} Y_\alpha$ is a V-space.

Corollary 5.11. *Let X be a topological space and let α be a cardinal.*

(i) *X is a V-space if and only if $\langle X, X^\alpha\rangle$ has the V-property.*
(ii) *$\langle X^\alpha, X\rangle$ has the V-property if and only if X^α is a V-space.* ∎

Note that by Corollary 1.12 if X^α is a V-space then X is a V-space.

Corollary 5.12. *Let K be a Cook continuum and let $n > 0$ be a natural number. Then K^n is a V-space.*

Proof. According to the above corollary it suffices to check that $\langle K^n, K\rangle$ has the V-property.

Let $p_k : K^n \to K$, $1 \le k \le n$, denote the kth projection. We prove by induction on n the following claim:

(I) every non-constant continuous map $f : K^n \to K$ coincides with some projection p_k.

The case $n = 1$ is trivial. Assume that $n > 1$ and that the statement is true for $n - 1$. Fix $a \in K^{n-1}$ and consider a continuous function $f : K^n = K \times K^{n-1} \to K$. Then the function $h_a : K \to X$ defined by $h_a(x) = f(x, a)$ is continuous. Hence, either $h_a = \mathrm{id}_K$, or $h_a \in$ Const. Let $g(a) \in K$ be the value of that constant function in the second case. Put $F = \{a \in K^{n-1} : h_a \equiv g(a)\}$ and $G = \{a \in K^{n-1} : h_a = \mathrm{id}_K\}$. These are disjoint closed subsets of K^{n-1} with $K^{n-1} = F \cup G$. By the connectedness of K^{n-1} we have either $F = K^{n-1}$, or $K^{n-1} = G$.

In the first case we have $h_a \equiv g(a)$ for all $a \in K^{n-1}$. The function $g : K^{n-1} \to K$ obtained in this way is continuous. So, by our inductive hypothesis, g is a projection. (Note that g cannot be constant since f is non-constant and each h_a is constant.) In the second case $h_a = \mathrm{id}_K$ for every a, hence $f = p_1$ is again a projection. This proves our claim.

For a non-empty subset $D \subseteq F = \{1, \ldots, n\}$ denote by $\Delta_D : K \to K^D$ the diagonal map defined by $\Delta_D(x) = \langle x, \ldots, x\rangle \in K^D$. Then it is easy to see that for every continuous map $\varphi : K \to K^n$, $\varphi \neq \Delta_F$, there exists a subset $D \subset F = \{1, \ldots, n\}$ and an element $a \in K^{F\setminus D}$ such that $\varphi : K \to K^n = K^D \times K^{F\setminus D}$ coincides with the map $\langle \Delta_D, g_a\rangle$, where $g_a \in \mathrm{Const}$ is the constant map with value a. Since φ is completely determined by the pair $\langle D, a\rangle \in \mathcal{P}(F) \times K^{F\setminus D}$, we denote this map by $\varphi_{D,a}$.

Now fix $\mathcal{A}$ to be the family of all closed subsets of K which are not doubletons. It follows from the proof of Proposition 1.6 that $\mathcal{C}_{\mathcal{A},\mathcal{A}} = \mathcal{C}_{K,K} = \mathrm{Const} \cup \{\mathrm{id}_K\}$. Set $\mathcal{B} = \{\varphi[A] : \varphi \in \mathcal{C}(K, K^n)\ \&\ A \in \mathcal{A}\}$.

We show that $\mathcal{C}(K^n, K) = \mathcal{C}_{\mathcal{B},\mathcal{A}}$. The inclusion $\mathcal{C}(K^n, K) \subseteq \mathcal{C}_{\mathcal{B},\mathcal{A}}$ is obvious. Assume $f \in C_{\mathcal{B},\mathcal{A}}$. Note that for every $a \in K$ the composition

$$h_a = f \circ \varphi_{\{1,\ldots,n-1\},a} \tag{14}$$

belongs to $\mathcal{C}_{\mathcal{A},\mathcal{A}}$, hence

$$h_a \in \mathrm{Const} \quad \text{or} \quad h_a = \mathrm{id}_K \tag{15}$$

by virtue of the equation $\mathcal{C}_{\mathcal{A},\mathcal{A}} = \mathcal{C}_{K,K}$ and (I). Consider the restriction $d_n = f \circ \Delta_F$ of f to the diagonal of K^n, i.e., $d_n(x) = f(x, \ldots, x)$. The proof of the corollary follows immediately from the next claim which we prove by induction on n.

Claim. (1_n) *If* $d_n \in \mathrm{Const}$ *then* $f \in \mathrm{Const}$.

(2_n) *If* $d_n = \mathrm{id}_K$ *then* $f = p_i$ *for some* $i \in \{1, \ldots, n\}$.

Proof. The case $n = 1$ trivially follows from the equalities $\Delta_F = \mathrm{id}_K$ and $d_n = f$, which are valid for $n = 1$. Assume that $n > 1$ and that the claim is true for $n - 1$.

Case 1: Let $d_n(x) = b \in K$ for every $x \in K$. Fix an arbitrary $a \in K\setminus\{b\}$ and consider h_a as in (14). Then $h_a(a) \neq \mathrm{id}_K$ since $h_a(a) = b \neq a$. Now (15) yields $h_a \in \mathrm{Const}$. Consider the function $f_a : K^{n-1} \to K$ defined by

$$f_a(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, a). \tag{16}$$

Then $f_a \circ \Delta_{\{1,\ldots,n-1\}} = h_a \in \mathrm{Const}$, so that the inductive hypothesis (1_{n-1}) holds for f_a. Hence $f(x_1, \ldots, x_{n-1}, a) = b$ for every $\langle x_1, \ldots, x_{n-1}\rangle \in K^{n-1}$ and $a \in K\setminus\{b\}$. Assume $f \notin \mathrm{Const}$. Then there exists $\langle c_1, \ldots, c_{n-1}\rangle \in K^{n-1}$ such that $f(c_1, \ldots, c_{n-1}, b) \neq b$. Now $B = \{\langle c_1, \ldots, c_{n-1}\rangle\} \times K \in \mathcal{B}$ and $|f[B]| = 2$, so that $f[B] \notin \mathcal{A}$, a contradiction. This proves that $f \in \mathrm{Const}$.

Case 2: Let $d_n = \mathrm{id}_K$. For $a \in K$ consider the functions $h_a : K \to K$ as in (14). According to (15) we have two cases.

CASE 2.1: There exists $a \in K$ such that $h_a \in \mathrm{Const}$. From $h_a(a) = d_n(a) = \mathrm{id}_K(a) = a$ we get $h_a(x) = a$ for every $x \in K$. For the function f_a defined as in (16) we have $f_a \circ \Delta_{\{1,\ldots,n-1\}} = h_a \in \mathrm{Const}$, so that the inductive hypothesis (1_{n-1}) holds for f_a. Hence $f_a \in \mathrm{Const}$. This yields $f = p_n$.

CASE 2.2: $h_a = \mathrm{id}_K$ for all $a \in K$. Now for every $a \in K$ the function f_a defined as in (16) satisfies the inductive hypothesis (2_{n-1}), hence there exists $i_a \in \{1, \ldots, n-1\}$ such that $f_a = p_{i_a}$. The proof will be finished if we show that the function $K \to \{1, \ldots, n-1\}$ defined by $a \mapsto i_a$ is constant. Assume the contrary. Then $i_a \neq i_{a'}$ for some $a \neq a'$ from K. Fix $\langle x_1, \ldots, x_{n-1}\rangle \in K^{n-1}$ such that $x_{i_a} \neq x_{i_{a'}}$ and $x_k \in \{x_{i_a}, x_{i_{a'}}\}$ for $k \in \{1, \ldots, n-1\}$. (This is possible since our assumption entails $n > 2$.) Then for the set $B = \{\langle x_1, \ldots, x_{n-1}\rangle\} \times K \in \mathcal{B}$ we have $|f[B]| = 2$, so that $f[B] \notin \mathcal{A}$, a contradiction. ■

We do not know if this result can be extended to all V-spaces:

PROBLEM 5.13. Are finite powers of V-spaces again V-spaces?

In particular, we do not know whether finite powers of the V-spaces defined in Example 5.8(II) are V-spaces. On the other hand, note that infinite powers of a V-space need not be V-spaces. (For example, take any finite discrete non-singleton space.)

In analogy with Proposition 5.7, one could try to extend the validity of Corollary 5.12 to the product of any (finite) strongly rigid family of continua. We offer a partial result here.

PROPOSITION 5.14. *Let $\{X_\alpha\}_{\alpha\in I}$ be a strongly rigid family of continua. Then all pairs $\langle \prod_{\beta\in I} X_\beta, X_\alpha\rangle$ have the V-property.*

Proof. Let $X = \prod_{\beta\in I} X_\beta$. We prove first that $\mathcal{C}(X, X_\alpha) = \mathrm{Const} \cup \{p_\alpha\}$ where $p_\alpha : X \to X_\alpha$ is the canonical projection for $\alpha \in I$.

Fix $\alpha \in I$ and let $X' = \prod\{X_\beta : \beta \in I,\ \beta \neq \alpha\}$. We identify X with $X_\alpha \times X'$.

We show first that $\mathcal{C}(X', X_\alpha) = \mathrm{Const}$. Fix $y = \langle y_\beta\rangle \in X'$ and let

$$X'' = \{x = \langle x_\beta\rangle \in X' : x_\beta \neq y_\beta \text{ for only finitely many } \beta \in I\}.$$

Now fix $f \in \mathcal{C}(X', X_\alpha)$ and set $b = f(y)$. It is easy to see that f takes constant value b on X''. (For $x = \langle x_\beta\rangle \in X''$ argue by induction on the number of $\beta \in I$ with $x_\beta \neq y_\beta$.) Since X_α is Hausdorff and X'' is dense in X' we conclude that f is constant on X'.

Now take $f \in \mathcal{C}(X, X_\alpha)$ and for every $x \in X_\alpha$ consider the restriction of f on $Z = \{x\} \times X'$. By the above claim f has a constant value $\widetilde{f}(x) \in X_\alpha$ on Z. The mapping $x \mapsto \widetilde{f}(x)$ is a continuous function from X_α into X_α.

Hence it is either constant or the identity. Thus f is either constant or equal to p_α.

Let $\mathcal{B}$ be the family of all closed subsets of X_α that are not doubletons and let $\mathcal{A} = \{B \times D \subseteq X : B \in \mathcal{B} \ \&\ D \in [X']^2\}$. Then $\mathcal{C}(X, X_\alpha) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. ∎

We do not know if it is possible to find a single $\mathcal{A}$ witnessing the property V for all pairs $\langle \prod_{\beta\in I} X_\beta, X_\alpha\rangle$ simultaneously. If this were true, then applying Corollary 5.10 we could conclude that $X = \prod_{\alpha\in I} X_\alpha$ is a V-space.

The above results also leave open the following question regarding subspaces of products. Putting the comment following Corollary 5.11 in negative form we get: *if X is not a V-space, then none of the powers X^α is a V-space*. Hence, by Corollary 5.11, the pair $\langle X^\alpha, X\rangle$ does not have the V-property.

Problem 5.15. Suppose X is not a V-space. Is it true that no pair $\langle Y, X\rangle$ has the V-property where Y is a non-discrete subspace of X^α for some α?

This is true for X equal to $\mathbb{R}$, the Sierpiński dyad S, and the discrete doubleton $\{0, 1\}$. Actually, in these cases the V-property fails for all pairs $\langle Y, X\rangle$ where Y belongs to the larger class $\mathbf{S}(X)$ of spaces that admit a continuous injection into a power of X. (See Corollary 3.5, Example 5.8(I), and Fact 1.3(v). Note that $\mathbf{S}(\mathbb{R})$ are the functionally Hausdorff spaces, $\mathbf{S}(S)$ are the T_0-spaces and $\mathbf{S}(\{0, 1\})$ are the totally disconnected spaces.) We propose the question also in its stronger form:

Problem 5.16. Suppose X is not a V-space. Is it true that for a space $Y \in \mathbf{S}(X)$ the pair $\langle Y, X\rangle$ has the V-property if and only if Y is discrete?

In the semigroup $\mathcal{C}(X, X)$ the largest subgroup $\mathcal{H}(X)$ of all autohomeomorphisms of X has as its smallest natural extension the subsemigroup $\mathcal{H}(X) \cup \mathrm{Const}$. Most of the examples of Hausdorff connected V-spaces we have seen till this point have the property $\mathcal{C}(X, X) = \mathcal{H}(X) \cup \mathrm{Const}$. This suggests the question: does there exist a Hausdorff connected V-space X such that $\mathcal{C}(X, X)$ has non-constant non-injective maps? The powers of Cook's continuum have this property by Corollary 5.12. Here is another example of a V-space with this property.

Example 5.17. Let K be a strongly rigid continuum and $a \in K$. Then a is not a cut point of C [6, Theorem 2.2.1]. Let $X = K \vee_a K$ be the adjunction space obtained by gluing two copies of K along the set $\{a\}$. Let $j_i : K \hookrightarrow X$, $i = 1, 2$, be the canonical embeddings of K into X. Then every point of X has the form $j_i(x)$ for some $x \in K$ and $i = 1, 2$. The canonical projection $p : X \to K$ is defined by $p \circ j_1 = p \circ j_2 = \mathrm{id}_K$. The symmetry $s : X \to X$ is defined by $s \circ j_1 = j_2$ and $s \circ j_2 = j_1$. We also have the map $h_1 : X \to X$ with $h_1 \circ j_1 = \mathrm{id}_K$ and $h_1 \circ j_2 : K \to K$ the constant function with value a. The map h_2 is defined analogously. It is easy to see that $\mathcal{C}(X, X) = \mathrm{Const} \cup \{1_X, s, h_1, h_2\}$.

Let $\mathcal{A}$ be the family of closed subsets of K which are not doubletons and $\widetilde{\mathcal{A}} = \{j_i[A] : A \in \mathcal{A}, i = 1,2\}$. Then $\mathcal{C}(X,X) = \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$. The inclusion $\mathcal{C}(X,X) \subseteq \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$ is obvious. If $f \in \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$, then for $i = 1,2$ the restriction of f on $j_i[K]$ is continuous, so that f is continuous as well since $j_1[K]$ and $j_2[K]$ are closed in X.

An alternative proof that $X = K \vee_a K$ is a V-space is given in the following remark.

Remark 5.18. The above example hides several more general facts which we isolate now. For a space Y and a subspace M of Y the adjunction space $X = Y \vee_M Y$ is obtained as above by gluing two copies of Y along M. The maps $j_i : Y \hookrightarrow X$, $i = 1,2$, $s : X \to X$ and $p : X \to Y$ are defined as above. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is *symmetric* if $s(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$.

(a) If $\langle Y, Z\rangle$ has the V-property, then also $\langle X, Z\rangle$ has the V-property witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$. In particular, if Y is a V-space, then $\langle X, Y\rangle$ has the V-property. (If $\mathcal{C}(Y,Z) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, then $\widetilde{\mathcal{A}}$ defined as in Example 5.17 is symmetric and $\mathcal{C}(X,Z) = \mathcal{C}_{\tilde{\mathcal{A}},\mathcal{B}}$.)

(b) If $\langle X, Z\rangle$ has the V-property, then it can be witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$. (Exploit the symmetry s of X.)

(c) If $\langle X, Z\rangle$ has the V-property witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$ then also $\langle Y, Z\rangle$ has the V-property. In particular, Y is a V-space if and only if $\langle X, Y\rangle$ has the V-property. (Note that Y can be considered as a retract of X via the embeddings j_i.)

(d) If Y is a strongly rigid V-space and M does not cut Y (i.e., $Y \setminus M$ is connected), then X is also a V-space. (It suffices to see that $\langle Y, X\rangle$ has the V-property. If $\mathcal{C}(Y,Y) = \mathcal{C}_{\mathcal{A},\mathcal{A}}$ define $\widetilde{\mathcal{A}}$ as before. To see that $\mathcal{C}(Y,X) \subseteq \mathcal{C}_{\mathcal{A},\tilde{\mathcal{A}}}$ it suffices to note that every $f \in \mathcal{C}(X,Z)$ factorizes either through j_1 or through j_2. For the inverse inclusion one has to prove first that $\mathcal{C}(Y,Y) = \mathcal{C}_{\mathcal{A},\mathcal{A}}$ yields that for the family $\mathcal{A}$ and every $x \in Y$, $Y = \mathrm{St}^{\omega}(x, \mathcal{A})$ as in the proof of Theorem 5.1. This forces the functions of $\mathcal{C}_{\mathcal{A},\tilde{\mathcal{A}}}$ to factorize through either j_1 or j_2.)

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