topology proceedings
Volume 22, 1997

ON SIERPIŃSKI-ZYGMUND BIJECTIONS AND THEIR INVERSES

Krzysztof Ciesielski and Tomasz Natkaniec

Abstract

In the paper we will examine when the inverses of one-to-one Sierpiński-Zygmund partial functions from $\mathbb{R}$ to $\mathbb{R}$ are also of Sierpiński-Zygmund type. We show that the existence of a partial Sierpiński-Zygmund function f with f^{-1} being also Sierpiński-Zygmund is independent of ZFC axioms of set theory. However,

1991 *Mathematics Subject Classification:* Primary 26A15; Secondary 03E35, 03E75.

Keywords and phrases: Sierpiński-Zygmund function; Blumberg Theorem; inverse function; covering of category.

*This work was partially supported by NSF Cooperative Research Grant INT-9600548 with its Polish part being financed by Polish Academy of Science PAN. The results were obtained during a visit of the second author at West Virginia University.

there exists a one-to-one Sierpiński-Zygmund injection $f: \mathbb{R} \to \mathbb{R}$ such that f^{-1} is not Sierpiński-Zygmund. This work is related to the investigation of algebraic properties of the Sierpiński-Zygmund functions discussed in [5].

1 Preliminaries

We will use the standard terminology and notation as in [3]. In particular, the functions will be identified with their graphs. The family of all functions from a set X into Y will be denoted by Y^X. Ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. Symbol $|X|$ will stand for the cardinality of a set X. For a cardinal number κ we will write κ^+ for its cardinal successor while symbol $[X]^\kappa$ will denote the family of all subsets Y of X with $|Y| = \kappa$. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathbf{c}$.

For $X \subset \mathbb{R}$ we say that a function $f: X \to \mathbb{R}$ is of *Sierpiński-Zygmund type* (shortly, an SZ-function) if its restriction $f|M$ is discontinuous for any set $M \in [X]^{\mathbf{c}}$. The class of Sierpiński-Zygmund functions was introduced in [7] in connection with a discussion around the Blumberg theorem which says that for every function $f: \mathbb{R} \to \mathbb{R}$ there exists a dense set $D \subset \mathbb{R}$ such that $f|D$ is continuous. (For more information on this discussion see survey [4, Sec. 2].) The family of all SZ-functions from a set $X \in [\mathbb{R}]^{\mathbf{c}}$ into $\mathbb{R}$ will be denoted by SZ. The symbol $\mathcal{C}$ will stand for the family of all continuous functions $f: \mathbb{R} \to \mathbb{R}$ and $\mathcal{C}_{G_\delta}$ for the family of all continuous functions defined on G_δ subsets of $\mathbb{R}$.

Recall that a function for $X \in [\mathbb{R}]^{\mathbf{c}}$ a function $f: X \to \mathbb{R}$ is an SZ-function if and only if $|f \cap g| < \mathbf{c}$ for every $g \in \mathcal{C}_{G_\delta}$. (See e.g. [7].) This is due to the fact that every continuous function from a subset of a Polish space into a Polish space can be extended continuously so that its domain is a G_δ set. We will also use the following easy characterization of one-to-one SZ-functions, where symbol $\mathcal{C}^\star_{G_\delta}$ stands for the class of nowhere constant functions $g \in \mathcal{C}_{G_\delta}$.

Lemma 1 ([5, Lemma 4.24]) *Assume that $f\colon X \to \mathbb{R}$ is one-to-one, where $X \in [\mathbb{R}]^{\mathbf{c}}$. Then $f \in \mathrm{SZ}$ if and only if $|f \cap g| < \mathbf{c}$ for every $g \in \mathcal{C}^\star_{G_\delta}$.*

Lemma 1 is a consequence of a fact that for every continuous function $f\colon G \to \mathbb{R}$ with $G \subset \mathbb{R}$ there exists an open set $U \subset \mathbb{R}$ such that $f|(G \cap U)$ is locally constant while $f|(G \setminus U)$ is nowhere constant.

2 The results

It is easy to construct a one-to-one SZ-function $f\colon \mathbb{R} \to \mathbb{R}$. On the other hand the existence of an SZ-bijection $f\colon \mathbb{R} \to \mathbb{R}$ is not provable in ZFC, since in the iterated perfect set model there is no SZ-function from $\mathbb{R}$ *onto* $\mathbb{R}$. (See [1].) In particular in this model there are no bijections $f\colon \mathbb{R} \to \mathbb{R}$ such that f and f^{-1} are both of an SZ-type. Therefore in our investigation which one-to-one SZ-functions $f\colon \mathbb{R} \to \mathbb{R}$ have SZ inverses we are forced either to consider partial functions or to work with some additional set theoretical assumptions. Below we will use both of these approaches.

We will start with noticing that there are ZFC examples of SZ injections $f\colon \mathbb{R} \to \mathbb{R}$ for which $f^{-1} \notin \mathrm{SZ}$.

Theorem 2 *There exists a one-to-one* SZ*-function $f\colon [0,1] \to [0,1]$ such that $f^{-1}\colon f[0,1] \to \mathbb{R}$ is continuous. Moreover, if $C \subset [0,1]$ is nowhere dense then so is $f[C]$.*

Proof: Let $g\colon [0,1] \to [0,1]$ be a continuous nowhere constant function with $|g^{-1}(y)| = \mathbf{c}$ for every $y \in [0,1]$. (For a construction of such a function see e.g. [2, pp. 148–150].) Fix enumerations $\{g_\alpha : \alpha < \mathbf{c}\}$ of $\mathcal{C}^\star_{G_\delta}$ and $\{x_\alpha : \alpha < \mathbf{c}\}$ of $[0,1]$. By transfinite induction for every $\alpha < \mathbf{c}$ choose

$$y_\alpha \in g^{-1}(x_\alpha) \setminus (\{y_\beta : \beta < \alpha\} \cup \{g_\beta(x_\alpha) : \beta \le \alpha\})$$

and put $f(x_\alpha) = y_\alpha$. Then f is one-to-one and, by Lemma 1, it is SZ. Moreover, $f^{-1} \subset g$. Thus f^{-1} is continuous.

To see the the additional property suppose that $C \subset [0,1]$ is nowhere dense. Then $g^{-1}(C)$ is a closed nowhere dense subset of $[0,1]$, since continuous functions map connected sets into connected sets and g is nowhere constant. Thus $f[C] \subset g^{-1}(C)$ is nowhere dense. $\Box$

Corollary 3 *There exists a one-to-one* SZ*-function* $g : \mathbb{R} \to \mathbb{R}$ *such that* $g^{-1} \notin \mathrm{SZ}$.

Proof: Let h be a homeomorphism between $\mathbb{R}$ and $(0,1)$ and f be from Theorem 2. Then $g = f \circ h$ has the desired properties. $\Box$

Remark 4 *If* $f : \mathbb{R} \to \mathbb{R}$ *is an* SZ *injection then* f^{-1} *is not continuous on any perfect set* $P \subset f[\mathbb{R}]$. *In particular, if* f *is the function from Theorem 2 then* $f[0,1]$ *does not contain any perfect set.*

Proof: If f^{-1} is continuous on a perfect set $P \subset f[\mathbb{R}]$ then it is a homeomorphism between P and a perfect set $T = f^{-1}(P)$. Then $f|T$ is continuous, contrary to $f \in \mathrm{SZ}$. $\Box$

Next we will show that the function f from Corollary 3 can be a bijection under appropriate set theoretical assumption guaranteeing existence of SZ-surjection $f : \mathbb{R} \to \mathbb{R}$. For this we will use the following lemma.

Lemma 5 *Assume that* $\mathbb{R}$ *cannot be covered by less than* $\mathbf{c}$ *many meager sets. Then for every residual set* $G \subset \mathbb{R}$ *and every* $Y \in [\mathbb{R}]^{\mathbf{c}}$ *there exists an* SZ*-bijection* $g\colon G \to Y$*.*

Proof: Let $\{g_\alpha\colon \alpha < \mathbf{c}\}$, $\{y_\alpha\colon \alpha < \mathbf{c}\}$, and $\{x_\alpha\colon \alpha < \mathbf{c}\}$ be the enumerations of $\mathcal{C}^\star_{G_\delta}$, Y and G, respectively. By induction on $\alpha < \mathbf{c}$ we will construct the sequences $\{\{a_\alpha, b_\alpha\} \in [G]^2\colon \alpha < \mathbf{c}\}$ and $\{\{c_\alpha, d_\alpha\} \in [Y]^2\colon \alpha < \mathbf{c}\}$ aiming for defining g by $g(a_\alpha) = c_\alpha$ and $g(b_\alpha) = d_\alpha$. The construction is done maintaining the following inductive conditions for every $\alpha < \mathbf{c}$.

(i) $a_\alpha = x_\alpha$ if $x_\alpha \notin \bigcup_{\xi<\alpha}\{a_\xi, b_\xi\}$. Otherwise a_α is an arbitrary element of $G \setminus \bigcup_{\xi<\alpha}\{a_\xi, b_\xi\}$.

(ii) $d_\alpha = y_\alpha$ if $y_\alpha \notin \bigcup_{\xi<\alpha}\{c_\xi, d_\xi\}$. Otherwise d_α is an arbitrary element of $Y \setminus \bigcup_{\xi<\alpha}\{c_\xi, d_\xi\}$.

(iii) $b_\alpha \in G \setminus \left(\{a_\alpha\} \cup \bigcup_{\xi<\alpha}\{a_\xi, b_\xi\} \cup \bigcup_{\xi<\alpha} g_\xi^{-1}(d_\alpha)\right)$.

(iv) $c_\alpha \in Y \setminus \left(\{d_\alpha\} \cup \bigcup_{\xi<\alpha}\{c_\xi, d_\xi\} \cup \{g_\xi(a_\alpha)\colon \xi < \alpha\}\right)$.

The choice as in (iii) can be made since the set $\bigcup_{\xi<\alpha} g_\xi^{-1}(d_\alpha)$ is a union of less than continuum many nowhere dense sets, so it cannot cover G. It is easy to see that g constructed that way is a well defined one-to-one function from G onto Y. It is SZ by Lemma 1. Indeed, let $g_\alpha \in \mathcal{C}^\star_{G_\delta}$ and $x \in \mathbb{R}$ be such that $g(x) = g_\alpha(x)$. Then there exist $\xi < \mathbf{c}$ such that $x \in \{a_\xi, b_\xi\}$. To finish the argument it is enough to notice that $\xi \le \alpha$. But if $\xi > \alpha$ then, by (iv), $g(a_\xi) = c_\xi \neq g_\alpha(a_\xi)$ and, by (iii), $g(b_\xi) = d_\xi \neq g_\alpha(b_\xi)$ implying that $g(x) \neq g_\alpha(x)$. □

Corollary 6 *If* $\mathbb{R}$ *cannot be covered by less than* $\mathbf{c}$ *many meager sets then there exists an* SZ*-bijection* $h\colon \mathbb{R} \to \mathbb{R}$ *such that* h^{-1} *is not an* SZ*-function.*

Proof: Let f be as in Theorem 2 and let $C \subset [0,1]$ be a Cantor set. Apply Lemma 5 for $G = \mathbb{R} \setminus C$ and $X = \mathbb{R} \setminus f[C]$ to find an SZ-bijection $g: G \to X$. Then $h = g \cup f|C$ is an SZ bijection and $h^{-1}|f[C]$ is continuous. Thus h^{-1} is not an SZ-function. $\Box$

The next theorem shows in particular that under appropriate set theoretical assumptions there exist also SZ-bijections $f: \mathbb{R} \to \mathbb{R}$ such that f^{-1} is also an SZ-function.

Theorem 7 *Assume that* $\mathbb{R}$ *cannot be covered by less than* $\mathbf{c}$ *many meager sets. Then there exists an* SZ *bijection* $f: \mathbb{R} \to \mathbb{R}$ *such that* $f^{-1} = f$.

Proof: Let $\{g_\alpha: \alpha < \mathbf{c}\}$ and $\{r_\alpha: \alpha < \mathbf{c}\}$ be the enumerations of $\mathcal{C}^\star_{G_\delta}$ and $\mathbb{R}$, respectively. We will construct by induction on $\alpha < \mathbf{c}$ a family of pairwise disjoint sets $\{\{a_\alpha, b_\alpha\} \in [\mathbb{R}]^2: \alpha < \mathbf{c}\}$ such that the following conditions hold for every $\alpha < \mathbf{c}$:

(i) $r_\alpha \in \bigcup_{\xi \le \alpha} \{a_\xi, b_\xi\}$; and,

(ii) $\langle a_\alpha, b_\alpha \rangle \notin \bigcup_{\xi \le \alpha} (g_\xi \cup g_\xi^{-1})$, where $g_\xi^{-1} = \{\langle y, x \rangle: y = g_\xi(x)\}$.

The construction is aimed to define f by putting $f(a_\alpha) = b_\alpha$ and $f(b_\alpha) = a_\alpha$ for every $\alpha < \mathbf{c}$. This clearly will imply that $f^{-1} = f$, while (i) will guarantee that f is defined for all real numbers.

To see that the choice of such a sequence is possible, assume that for some $\alpha < \mathbf{c}$ the sequence $\{\{a_\xi, b_\xi\} \in [\mathbb{R}]^2: \xi < \alpha\}$ is already defined. Then we define a_α as r_α if $r_\alpha \notin \bigcup_{\xi < \alpha} \{a_\xi, b_\xi\}$ and as an arbitrary element of the set $\mathbb{R} \setminus \bigcup_{\xi < \alpha} \{a_\xi, b_\xi\}$ otherwise. Now, to have (ii) it is enough to choose b_α such that

$$b_\alpha \in \mathbb{R} \setminus \left(\{a_\alpha\} \cup \bigcup_{\xi < \alpha} \{a_\xi, b_\xi\} \cup \bigcup_{\xi \le \alpha} (\{g_\xi(a_\alpha)\} \cup g_\xi^{-1}(a_\alpha)) \right), \tag{1}$$

which is possible since according to our assumption less than continuum many nowhere dense sets $g_\xi^{-1}(a_\alpha)$ does not cover $\mathbb{R}$.

To verify that $f \in \mathrm{SZ}$ by Lemma 1 it is enough to show that $|f \cap g| < \mathbf{c}$ for any $g \in \mathcal{C}^\star_{G_\delta}$. So, fix a $g \in \mathcal{C}^\star_{G_\delta}$ and let $x \in \mathbb{R}$ be such that $f(x) = g(x)$. Then there exist $\alpha, \xi < \mathbf{c}$ such that $g = g_\alpha$ and $x \in \{a_\xi, b_\xi\}$. Observe that $\xi \leq \alpha$. Indeed, if $\xi > \alpha$ then, by (1), $g_\alpha(a_\xi) \neq b_\xi = f(a_\xi)$ and $g_\alpha(b_\xi) \neq a_\xi = f(b_\xi)$, as $b_\xi \notin g_\alpha^{-1}(a_\xi)$, a contradiction. Therefore $|f \cap g| \leq |\alpha| < \mathbf{c}$. $\square$

Theorem 7 and Corollary 6 show that under appropriate set theoretic assumptions there are SZ bijections f from $\mathbb{R}$ onto $\mathbb{R}$ such that f^{-1} is SZ and such that f^{-1} is not SZ. We also know that these results cannot be obtained in ZFC. (See [1].) On the other hand, by Corollary 3, there is a ZFC example of an SZ injection $g\colon \mathbb{R} \to \mathbb{R}$ with $g^{-1} \notin \mathrm{SZ}$. Thus, to complete the picture, it is reasonable to ask the question whether there is a ZFC example of an SZ injection $g\colon \mathbb{R} \to \mathbb{R}$ for which $g^{-1} \in \mathrm{SZ}$. The negative answer for this question will be deduced from the following characterization.

Theorem 8 *The following conditions are equivalent:*

(i) *for each bijection f from a set $X \in [\mathbb{R}]^{\mathbf{c}}$ onto a set $Y \in [\mathbb{R}]^{\mathbf{c}}$ either $f \notin \mathrm{SZ}$ or $f^{-1} \notin \mathrm{SZ}$;*

(ii) *there exists a family of functions $\mathcal{H} \subset \mathcal{C}_{G_\delta}$ of cardinality less than $\mathbf{c}$ such that $\mathbb{R}^2 = \bigcup \mathcal{H} \cup \bigcup\{h^{-1}\colon h \in \mathcal{H}\}$, where $h^{-1} = \{\langle y, x\rangle\colon y = h(x)\}$.*

Proof: (ii)$\Rightarrow$(i) Assume that there exists $\mathcal{H} \subset \mathcal{C}_{G_\delta}$ with $|\mathcal{H}| = \kappa < \mathbf{c}$ satisfying (ii). First note that this implies that

$$\mathbf{c} = \kappa^+. \tag{2}$$

(This is an easy generalization of property P_1 from [6]. See also [3, Thm 6.1.8].)

Indeed, by way of contradiction assume that $\mathbf{c} > \kappa^+$ and take $X \subset \mathbb{R}$ with $|X| = \kappa^+$. Then $Y = \bigcup\{h[X]: h \in \mathcal{H}\}$ has cardinality $\leq \kappa^+ < \mathbf{c}$. Take $y \in \mathbb{R} \setminus Y$ and $x \in X \setminus \{h(y): h \in \mathcal{H}\}$. (This can be done, since $|\{h(y): h \in \mathcal{H}\}| \leq |\mathcal{H}| = \kappa < |X|$.) Then $\langle x, y\rangle \notin \bigcup \mathcal{H}$ by the choice of y and $\langle x, y\rangle \notin \bigcup\{h^{-1}: h \in \mathcal{H}\}$ by the choice of x, a contradiction.

To show (i) take a bijection $f: X \to Y$ for some $X, Y \in [\mathbb{R}]^{\mathbf{c}}$. Then, by the Pigeon Hole Principle and the regularity of κ^+, there exists $h \in \mathcal{H}$ such that either $|f \cap h| = \mathbf{c}$, or $|f^{-1} \cap h| = |f \cap h^{-1}| = \mathbf{c}$. Thus either $f \notin SZ$ or $f^{-1} \notin SZ$.

(i)$\Rightarrow$(ii) Let $\mathcal{C}_{G_\delta} = \{g_\alpha: \alpha < \mathbf{c}\}$ and for each $\alpha < \mathbf{c}$ let $\mathcal{H}_\alpha = \{g_\beta: \beta \leq \alpha\}$. Suppose $\neg$(ii). We choose inductively a sequence $\{\langle x_\alpha, y_\alpha\rangle: \alpha < \mathbf{c}\}$ of points such that

(1) $x_\alpha \neq x_\beta$ for $\beta < \alpha$;

(2) $y_\alpha \neq y_\beta$ for $\beta < \alpha$;

(3) $\langle x_\alpha, y_\alpha\rangle \in \mathbb{R}^2 \setminus (\bigcup \mathcal{H}_\alpha \cup \bigcup\{h^{-1}: h \in \mathcal{H}_\alpha\})$.

Such a choice is possible by the negation of (ii).

Put $X = \{x_\alpha: \alpha < \mathbf{c}\}$, $Y = \{y_\alpha: \alpha < \mathbf{c}\}$ and $f(x_\alpha) = y_\alpha$. Then $X, Y \in [\mathbb{R}]^{\mathbf{c}}$ and $f: X \to Y$ is a bijection. To prove that f is SZ function observe that for each $\alpha < \mathbf{c}$ we have $|f \cap g_\alpha| < \mathbf{c}$ since $\{x \in X: f(x) = g_\alpha(x)\} \subset \{x_\beta: \beta \leq \alpha\}$. Similarly, for every $\alpha < \mathbf{c}$ we have $|f^{-1} \cap g_\alpha| < \mathbf{c}$ since $\{\beta: \langle y_\beta, x_\beta\rangle \in g_\alpha\} \subset \alpha$. So f^{-1} is an SZ-function, contrary to (i). $\Box$

Corollary 9 *It is consistent with ZFC that there is no bijection f from a set $X \in [\mathbb{R}]^{\mathbf{c}}$ onto a set $Y \in [\mathbb{R}]^{\mathbf{c}}$ such that both $f \in \mathrm{SZ}$ and $f^{-1} \in \mathrm{SZ}$.*

Proof: It follows from Theorem 5.1 and Corollary 3.3 of a recent paper of Steprāns [8] (see also Definition 2.3 and Lemma 2.2) that the condition (ii) from Theorem 8 is consistent with ZFC axioms. $\Box$

References

[1] M. Balcerzak, K. Ciesielski, T. Natkaniec, *Sierpiński–Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic, **37(1)** (1997), 29–35. (Preprint⋆ available.[1])

[2] A. M. Bruckner, *Differentiation of real functions*, CMR Series vol. 5, Amer. Math. Soc., 1994.

[3] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts **39**, Cambridge Univ. Press, 1997.

[4] K. Ciesielski, *Set Theoretic Real Analysis*, J. Appl. Anal., **3(2)** (1997), 143–190.

[5] K. Ciesielski, T. Natkaniec, *Algebraic properties of the class of Sierpiński-Zygmund functions*, Topology Appl., **79** (1997), 75–99. (Preprint⋆ available.)

[6] W. Sierpiński, *Hypothèse du continu*, Monografie Matematyczne, Tom IV, Warsaw 1934.

[7] W. Sierpiński, A. Zygmund, *Sur une fonction qui est discontinue sur tout ensemble de puissance du continu*, Fund. Math., **4** (1923), 316–318.

[8] J. Steprāns, *Decomposing Euclidean Space with a Small Number of Smooth Sets*, Trans. Amer. Math. Soc., to appear. (Preprint⋆ available.)

West Virginia University
Morgantown, WV 26506-6310
e-mail address: kcies@wvnvms.wvnet.edu

[1]Preprints and papers marked by ⋆ are available in electronic form accessible from *Set Theoretic Analysis Web Page:*
http://www.math.wvu.edu/homepages/kcies/STA/STA.html

Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland
e-mail address: mattn@ksinet.univ.gda.pl