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# UNIFORMLY ANTISYMMETRIC FUNCTIONS WITH BOUNDED RANGE

**Abstract**

The goal of this note is to construct a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with a bounded countable range. This answers Problem 1(b) of Ciesielski and Larson [6]. (See also the list of problems in Thomson [9] and Problem 2(b) from Ciesielski's survey [5].) A problem of existence of uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with finite range remains open.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *uniformly antisymmetric* [6] (or *nowhere weakly symmetrically continuous* [9]) provided for every $x \in \mathbb{R}$ the limit $\lim_{n\to\infty}(f(x+s_n) - f(x-s_n))$ equals 0 for no sequence $\{s_n\}_{n<\omega}$ converging to 0. Uniformly antisymmetric functions have been studied by Kostyrko [7], Ciesielski and Larson [6], Komjáth and Shelah [8], and Ciesielski [1, 2]. (A connection of some of these results to the paradoxical decompositions of the Euclidean space $\mathbb{R}^n$ is described in Ciesielski [3].) In particular in [6] the authors constructed a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{N}$ and noticed that the existence of a uniformly antisymmetric function cannot be proved without an essential use of the axiom of choice.

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The terminology and notation used in this note is standard and follows [4]. In particular for a set $X$ we will write $|X|$ for its cardinality and $\mathcal{P}(X)$ for its power set. Also $2^\omega$ will stand for the set of all functions from $\omega = \{0, 1, 2, \dots\}$ into $2 = \{0, 1\}$. We consider $2^\omega$ as ordered lexicographically.

**Theorem 1.** *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ with countable bounded range such that for every $x \in \mathbb{R}$ there exists an $\varepsilon_x > 0$ with the property that the set*

$$S_x = \{s \in \mathbb{R}\colon |f(x-s) - f(x+s)| < \varepsilon_x\}$$

*is finite. In particular $f$ is uniformly antisymmetric.*

Proof. First notice that it is enough to find a compact zerodimensional metric space $\langle T, d\rangle$ and a function $g$ from $\mathbb{R}$ into a countable subset $T_0$ of $T$ such that for every $x \in \mathbb{R}$ there is a $\delta_x > 0$ for which the set

$$\hat{S}_x = \{s \in \mathbb{R}\colon d(g(x-s), g(x+s)) < \delta_x\}$$

is finite.

To see this assume that such a function $g\colon \mathbb{R} \to T$ exists and take a homeomorphic embedding $h$ of $T$ into $\mathbb{R}$. We claim that $f = h \circ g\colon \mathbb{R} \to \mathbb{R}$ is as desired. Indeed, $f[\mathbb{R}] = h[g[\mathbb{R}]]$ is countable, as it is a subset of a countable set $h[T_0]$, and it is bounded, since it is a subset of a compact set $h[T]$. So take $x \in \mathbb{R}$ and $\delta_x > 0$ for which $\hat{S}_x$ is finite. Since $h^{-1}\colon h[T] \to T$ is uniformly continuous, we can find an $\varepsilon_x > 0$ such that

$$|y_1 - y_2| < \varepsilon_x \ \text{ implies } \ d(h^{-1}(y_1), h^{-1}(y_1)) < \delta_x$$

for every $y_1, y_2 \in h[T]$. But for such a choice of $\varepsilon_x$ we have

$$S_x = \{s \in \mathbb{R}\colon |h(g(x-s)) - h(g(x+s))| < \varepsilon_x\} \subset \hat{S}_x$$

proving that $S_x$ is finite.

Thus, we proceed to construct a function $g$ described above. The value of $g(x)$ will be defined with help of a representation of $x$ in a Hamel basis; i.e., a linear basis of $\mathbb{R}$ over $\mathbb{Q}$. For this we will use the following notation. Let $\{y_\eta\colon \eta \in 2^\omega\}$ be a one-to-one enumeration of a Hamel basis $\mathcal{H}$. For every $x \in \mathbb{R}$ let $\sum_{\eta\in 2^\omega} q_{x,\eta} y_\eta$, with $q_{x,\eta} \in \mathbb{Q}$ for $\eta \in 2^\omega$, be the unique representation of $x$ in basis $\mathcal{H}$ and let $w_x = \{\eta \in 2^\omega\colon q_{x,\eta} \neq 0\}$. Thus $w_x$ is finite and

$$x = \sum_{\eta \in w_x} q_{x,\eta} y_\eta.$$

The definition of the space $T$ is considerably more technical since it reflects several different cases of the proof that the sets $\hat{S}_x$ are indeed finite. To this end let $\{q_j : j < \omega\}$ be a one-to-one enumeration of $\mathbb{Q}$ with $q_0 = 0$. For $i < \omega$ let $\mathcal{P}_i = \mathcal{P}(\{q_j : j < i\})$ and put $P_i = \mathcal{P}(2^i \times \{0,1\} \times \mathcal{P}_i \times \mathcal{P}_i)$. Note that each $P_i$ is finite, so $T = \prod_{i<\omega} P_i$, considered as the standard product of discrete spaces, is compact zerodimensional. We equip $T$ with a distance function $d$ defined between different $s, t \in T$ by $d(s,t) = 2^{-\min\{i<\omega\colon s(i)\neq t(i)\}}$ and let

$$T_0 = \{t \in T\colon (\exists n < \omega)(\forall i \geq n)\ t(i) = \emptyset\}\,.$$

Clearly $T_0$ is countable.

Now we are ready to define $g\colon \mathbb{R} \to T_0 \subset T$. For this, however, we will need few more definitions. For $x \in \mathbb{R}$, $q \in \mathbb{Q}$, $i < \omega$, and $\zeta \in 2^i$ such that $\zeta \in \{(\eta \restriction i)\colon \eta \in w_x\}$ we define:

- $p(i) \in \{0,1\}$ as the parity of $i$, i.e., $p(i) = i \bmod 2$;
- $k_i(q) = \{q_j \in \mathbb{Q}\colon q_j < q \;\&\; j < i\} \in \mathcal{P}_i$;
- $\eta(x,\zeta)$ to be the minimum of $\{\eta \in w_x \colon \zeta \subset \eta\}$ (in the lexicographical order);
- $\xi(x,\zeta)$ to be the minimum of $\{\eta \in w_x\colon \zeta \subset \eta\} \setminus \{\eta(x,\zeta)\}$ provided $|\{\eta \in w_x\colon \zeta \subset \eta\}| \neq 1$; otherwise we put $\xi(x,\zeta) = \eta(x,\zeta)$;
- $n_x < \omega$ to be the smallest number $n > 0$ such that
  (i) $\eta \restriction n \neq \xi \restriction n$ for any different $\eta, \xi \in w_x$, and
  (ii) $q_{x,\eta} \in \{q_j : j < n\}$ for every $\eta \in w_x$.

Consider the function $g\colon \mathbb{R} \to T_0$ defined as follows. For every $x \in \mathbb{R}$ and $i < \omega$ we define $g(x)(i) \in P_i$ as

$$\big\{\langle \zeta, p(|\{\eta \in w_x \colon \zeta \subset \eta\}|), k_i(q_{x,\eta(x,\zeta)}), k_i(q_{x,\xi(x,\zeta)})\rangle \colon \zeta \in \{(\eta \restriction i)\colon \eta \in w_x\}\big\}$$

provided $i \leq n_x$ and we put $g(x)(i) = \emptyset$ for $n_x < i < \omega$. In the argument below the key role will be played by the function $k_i$ in general, and the coordinate $k_i(q_{x,\eta(x,\zeta)})$ in particular.

The key step in the proof that $g$ has the desired property is that for every $x \in \mathbb{R}$ and $s \neq 0$

$$\text{if}\quad n_x \leq \max\{n_{x-s}, n_{x+s}\} \quad\text{then}\quad g(x-s)(n_x) \neq g(x+s)(n_x). \tag{1}$$

To see (1) assume that $n_x \leq n_{x+s}$. If $n_{x-s} < n_x$ then $g(x-s)(n_x) = \emptyset \neq g(x+s)(n_x)$, where $g(x+s)(n_x) \neq \emptyset$ since $w_{x+s} \neq \emptyset$ as $n_{x-s} < n_x \leq n_{x+s}$ implies $x + s \neq 0$. Thus, we can assume that $n_x \leq \min\{n_{x-s}, n_{x+s}\}$. Take an

$\hat\eta \in w_{x-s} \cup w_{x+s}$ such that $q_{x-s,\hat\eta} \neq q_{x+s,\hat\eta}$ and let $\zeta = \hat\eta \restriction n_x$. Note that, by the definition of $n_x$, the set $S = \{\eta \in w_x \colon \zeta \subset \eta\}$ has at most one element.

If $S = \emptyset$ then $\{\eta \in w_{x-s} \colon \zeta \subset \eta\} = \{\eta \in w_{x+s} \colon \zeta \subset \eta\} \neq \emptyset$ and so $\eta(x-s,\zeta) = \eta(x+s,\zeta) \notin w_x$ while $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$. Thus $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$ implying that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$. Therefore $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, assume that $S \neq \emptyset$ and let $\eta'$ be the only element of $S$. Then $\eta' \in w_{x-s} \cup w_{x+s}$. If $\eta'$ belongs to precisely one of the sets $w_{x+s}$ and $w_{x-s}$, say $w_{x+s}$, then $\{\eta \in w_{x+s} \colon \zeta \subset \eta\} = \{\eta \in w_{x-s} \colon \zeta \subset \eta\} \cup \{\eta'\}$. In particular, $p(|\{\eta \in w_{x+s} \colon \zeta \subset \eta\}|) \neq p(|\{\eta \in w_{x-s} \colon \zeta \subset \eta\}|)$ implying that $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, we can assume that $\eta' \in w_{x-s} \cap w_{x+s}$. Then $\{\eta \in w_{x-s} \colon \zeta \subset \eta\} = \{\eta \in w_{x+s} \colon \zeta \subset \eta\}$ and $\eta(x-s,\zeta) = \eta(x+s,\zeta)$. We will consider three cases.

CASE 1: $\eta' \neq \eta(x-s,\zeta) = \eta(x+s,\zeta)$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$, so $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 2: $\eta' = \eta(x-s,\zeta) = \eta(x+s,\zeta)$ and $q_{x-s,\eta(x-s,\zeta)} \neq q_{x+s,\eta(x+s,\zeta)}$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 2q_{x,\eta'}$ and, by the definition of $n_x$, $q_{x,\eta'} \in \{q_j \colon j < n_x\}$. Since $q_{x,\eta'}$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$ we conclude that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 3: $\eta' = \eta(x-s,\zeta) = \eta(x+s,\zeta)$ and $q_{x-s,\eta(x-s,\zeta)} = q_{x+s,\eta(x+s,\zeta)}$. Then $Z = \{\eta \in w_{x-s} \colon \zeta \subset \eta\} \setminus \{\eta(x-s,\zeta)\} = \{\eta \in w_{x+s} \colon \zeta \subset \eta\} \setminus \{\eta(x+s,\zeta)\}$ is non-empty, since it contains $\hat\eta$, and so $\xi(x-s,\zeta) = \xi(x+s,\zeta) \notin w_x$. Therefore, as in Case 1, $q_{x-s,\xi(x-s,\zeta)} + q_{x+s,\xi(x+s,\zeta)} = 0$, so $q_0 = 0$ separates $q_{x-s,\xi(x-s,\zeta)}$ and $q_{x+s,\xi(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\xi(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\xi(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

This finishes the proof of (1).

Next, for every $x \in \mathbb{R}$ put $\delta_x = 2^{-n_x}$. To finish the proof of the theorem it is enough to show that every $\hat S_x$ defined for such a choice of $\delta_x$ is a subset of a finite set

$$Z_x = \{s \in \mathbb{R} \colon w_{x+s} \subset w_x \ \&\ n_{x+s} < n_x\} = \left\{ \sum_{\eta \in w_x} p_\eta y_\eta \colon p_\eta \in \{q_j \colon j < n_x\} \right\}.$$

Indeed, take an $s \in \hat S_x$. Then, by (1) and the definition of the distance function $d$, we have $\max\{n_{x-s}, n_{x+s}\} < n_x$. Notice also that if $n_{x-s} \neq n_{x+s}$, say $n_{x-s} < n_{x+s}$, then $g(x-s)(n_{x+s}) = \emptyset \neq g(x+s)(n_{x+s})$ implying that $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$, which contradicts $s \in \hat S_x$. So, we have $n_{x-s} = n_{x+s}$. To prove that $s \in Z_x$ it is enough to show that

$w_{x+s} \subset w_x$. But if it is not the case then there exists an $\eta \in w_{x+s} \setminus w_x$. Moreover, $q_{x+s,\eta} = -q_{x-s,\eta} \neq 0$ and $\eta = \eta(x+s,\zeta) = \eta(x-s,\zeta)$, where $\zeta = \eta \restriction n_{x+s}$. In particular, $q_0 = 0$ separates $q_{x+s,\eta(x+s,\zeta)}$ and $q_{x-s,\eta(x-s,\zeta)}$. Therefore $k_{n_{x+s}}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_{x+s}}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_{x+s}) \neq g(x+s)(n_{x+s})$. So $d(g(x+s),g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$ again contradicting $s \in \hat{S}_x$. Thus, $w_{x+s} \subset w_x$ and $s \in Z_x$. $\square$

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[1]Preprints marked by $^\star$ are available in electronic form from *Set Theoretic Analysis Web Page:* http://www.math.wvu.edu/homepages/kcies/STA/STA.html