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# MEASURE ZERO SETS WHOSE ALGEBRAIC SUM IS NON-MEASURABLE

**Abstract**

In this note we will show that for every natural number $n > 0$ there exists an $S \subset [0,1]$ such that its $n$-th algebraic sum $nS = S + \cdots + S$ is a nowhere dense measure zero set, but its $n+1$-st algebraic sum $nS + S$ is neither measurable nor it has the Baire property. In addition, the set $S$ will be also a Hamel base, that is, a linear base of $\mathbb{R}$ over $\mathbb{Q}$.

We use the standard notation as in [2]. Thus symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathfrak{c}$ stand for the set of real numbers, the set of rational numbers, the set of integers, and the cardinality of $\mathbb{R}$, respectively. The set of natural numbers $\{0,1,2,\ldots\}$ will be denoted by either $\mathbb{N}$ or $\omega$, and $|X|$ will stand for the cardinality of a set $X$. For $A, B \subseteq \mathbb{R}$ we put $A + B = \{a + b \colon a \in A \ \&\ b \in B\}$ and $\mathrm{LIN}_{\mathbb{Q}}(A)$ will stand for the linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by $A$. In addition for $0 < n < \omega$ symbol $[X]^n$ will stand for the family of all $n$-element subsets of $X$ and $nA$ for the $n$-th algebraic sum of $A$, that is,

$$nA = \left\{ \sum_{i=1}^{n} a_i \colon a_i \in A \ \text{ for all } i = 1, 2, \ldots, n \right\}.$$

Thus $2S = S + S$. For a Polish space $X$ we say that $B \subset X$ is a *Bernstein set (in $X$)* provided $B$ and $X \setminus B$ intersect every perfect set $P \subset X$. Clearly every Bernstein subset of an interval in $\mathbb{R}$ is neither measurable nor it has the Baire property.

For $0 < p < q < \omega$ let $Z_q^p$ stand for the set of all numbers $x \in [0,1]$ which in base $q$ have representations formed with digits $0, \ldots, p$, that is,

$$Z_q^p = \left\{ \sum_{i=1}^{\infty} \frac{a(i)}{q^i} \colon a(i) \in \{0, \ldots, p\} \text{ for every } i = 1, 2, \ldots \right\}.$$

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Clearly if $p < q-1$ then the set $Z_q^p$ is closed, nowhere dense, and has Lebesgue measure zero. Also, for every $k \in \mathbb{N}$

$$\text{if } 0 < kp < q \text{ then } kZ_q^p = Z_q^{kp}. \tag{1}$$

In what follows we will also need the following lemma.

**Lemma 1.** *Let $n > 0$ be a natural number, $q = 2n+2$, and $A \subset \mathbb{R}$ with $|A| < \mathfrak{c}$. Then for every $x \in [0,1] \setminus \mathrm{LIN}_{\mathbb{Q}}(A)$ there exists $\{x_0, \ldots, x_n\} \in [Z_q^2]^{n+1} \setminus A$ such that $x = x_0 + \cdots + x_n$ and $\{x_0, \ldots, x_n\} \cup A$ is linearly independent over $\mathbb{Q}$.*

PROOF. In the proof any sequence $\langle z(i) \in \{0, \ldots, q-1\} \colon i = 1, 2, \ldots \rangle$ will be treated as a base $q$ representation of a number $z \in [0,1]$, that is, $z = \sum_{i=1}^{\infty} \frac{z(i)}{q^i}$.

Let $x \in [0,1] \setminus \mathrm{LIN}_{\mathbb{Q}}(A)$. Then $x > 0$ and it can be represented as sequence $\langle x(i) \in \{0, \ldots, q-1\} \colon i = 1, 2, \ldots \rangle$. We can also assume that the set $T = \{i \colon x(i) > 0\}$ is infinite, since any number with almost all $x(i)$'s being zero has also another representation with almost all $x(i)$'s equal to $q-1$.

Let $\{T_0, \ldots, T_{n-1}\}$ be a partition of $T$ onto infinite sets. For each $i \notin T$ and $j \le n$ define $x_j^*(i) = 0$. For $i \in T_k$ we choose $x_j^*(i) \in \{0, 1, 2\}$ such that

$$x_k^*(i) \in \{0,1\},\ x_n^*(i) \in \{1,2\}, \text{ and } x_0^*(i) + \cdots + x_n^*(i) = x(i). \tag{2}$$

Such a choice can be made since $0 < x(i) \le 2n+1$ for any such $i$. Next, by induction on $k < n$, we will choose the sequences $\langle s_k(i) \in \{0,1\} \colon i < \omega \rangle$ such that

$$s_k(i) = 0 \text{ for all } i \notin T_k. \tag{3}$$

We aim for $x_n = x_n^* - \sum_{j=0}^{n-1} s_j$ and $x_k = x_k^* + s_k$ for every $k < n$. Notice that, by (2) and (3), such a definition ensures that $x_j$'s belong to $Z_q^2$ and that their sum is equal to $x$. The freedom of choice of $s_i$'s will allow us to enforce required linear independence.

In our inductive construction we will use the following notation for $Z \subset \mathbb{R}$ and $T \subset \{1, 2, 3, \ldots\}$:

$$Z \restriction T = \left\{ z \restriction T \colon\ z \text{ maps } \{1,2,3,\ldots\} \text{ into } \{0, \ldots, q-1\} \ \ \&\ \ \sum_{i=1}^{\infty} \frac{z(i)}{q^i} \in Z \right\}.$$

Now, since $A_0 = (\mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\})) \restriction T_0$ has cardinality less than $\mathfrak{c}$ there is an $s_0$ as in (3) for which $\langle x_0^*(i) + s_0(i) \colon i \in T_0 \rangle \notin A_0$. This clearly implies that $x_0 = x_0^* + s_0 \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\})$. In general, if for some $0 < k < n$ the sequences $s_i$'s (so also $x_i$'s) are already defined for all $i < k$ we put

$A_k = (\mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\} \cup \{x_i : i < k\})) \restriction T_k$ and choose $s_k$ as in (3) for which $\langle x_k^*(i) + s_k(i) : i \in T_k \rangle \notin A_k$. This ensures that

$$x_k = x_k^* + s_k \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x\} \cup \{x_i : i < k\}). \tag{4}$$

This finishes the inductive construction.

Finally, by (4), $x_n = x - (x_0 + \cdots + x_{n-1}) \notin \mathrm{LIN}_{\mathbb{Q}}(A \cup \{x_i : i < n\})$ since $x \notin \mathrm{LIN}_{\mathbb{Q}}(A)$. □

**Theorem 2.** *For every natural number $n > 0$ there exists a Hamel basis $H \subset Z^2_{2n+2}$ such that for every natural number $m > n$ the set $[0,1] \cap mH$ is Bernstein in $[0,1]$.*

Before we prove the theorem, we like to list some of its corollaries. First note that if $H$ is as in the theorem then, by (1), for every natural number $k \leq n$ we have $kH \subset Z^{2k}_{2n+2}$. So $kH$ is nowhere dense and it has measure 0. In particular,

**Corollary 3.** *For every natural number $n > 0$ there exists an $S \subset [0,1]$ such that for every $k \in \mathbb{N}$, $kS$ is Lebesgue measurable if and only if $k \leq n$.*

Corollary 3 used with $n = 1$ implies that there exists a measure 0 subset $S$ of $[0,1]$ such that $S + S$ is non-measurable. This fact has been known for quite a while and was used by several authors. (See e.g. [5, 3, 1].) However, we have not been able to locate its proof in the literature. This prompted the author to write this note. At the same time we should note here that it is very easy to find a meager measure zero set $E \subset \mathbb{R}$ such that for some natural number $n$ the $n$-th algebraic sum $nE$ of $E$ is neither measurable nor it has the Baire property. For this take a meager measure zero Hamel basis $H \subset \mathbb{R}$ and note that $E = \{qh : q \in \mathbb{Q} \ \&\ h \in H\}$ has these properties, since $\mathbb{R} = \bigcup_{n<\omega} nE$. Sets with the properties similar to these of our set $E$ has been also investigated in [4].

Notice also, that if we put $S = \mathbb{Z} + H$ then we get the following

**Corollary 4.** *For every natural number $n > 0$ there exists an $S \subset \mathbb{R}$ such that for every $k \in \mathbb{N}$*

- *if $k \leq n$ then $kS$ is meager and it has Lebesgue measure 0;*
- *if $k > n$ then $kS$ is a Bernstein set in $\mathbb{R}$.*

It is also not difficult to complicate a bit shifts in the definition of $S$ from Corollary 4 to ensure that $S$ is still a Hamel basis.

**Corollary 5.** *There exists a Bernstein set $B \subset \mathbb{R}$ such that $kB$ is also Bernstein for every natural $k > 0$.*

PROOF. Just put $B = S + S$ where $S$ is from Corollary 4 used with $n = 1$. $\square$

PROOF OF THEOREM 2. Fix an $n > 0$ and let $\{P_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all perfect subsets of $[0,1]$. By transfinite induction on $\xi < \mathfrak{c}$ construct a sequence $\langle H_\xi \in [Z^2_{2n+2}]^{n+1} \colon \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$

- $\sum_{y \in H_\xi} y \in P_\xi$ and $H_\xi$ is linearly independent over $\mathrm{LIN}_{\mathbb{Q}}\left(\bigcup_{\zeta<\xi} H_\zeta\right)$.

To make such a choice, just put $A = \bigcup_{\zeta<\xi} H_\zeta$, pick an $x \in P_\xi \setminus \mathrm{LIN}_{\mathbb{Q}}(A)$, and apply Lemma 1 to find $H_\xi$.

The set $\bar{H} = \bigcup_{\xi<\mathfrak{c}} H_\xi \subset Z^2_{2n+2}$ is linearly independent over $\mathbb{Q}$. Since $(n+1)Z^2_{2n+2} \supset [0,1]$ we can find an $H \subset Z^2_{2n+2}$ containing $\bar{H}$ which is a Hamel basis. We will show that $H$ is as desired.

For this notice that

(∗) for every $m > n$ the set $mH$ intersects every $P \subset [0,1]$.

This is clear for $m = n+1$, since for $\xi < \mathfrak{c}$ such that $P = P_\xi$ the number $\sum_{y \in H_\xi} y \in P_\xi$ belongs to $P$ and $(n+1)H_\xi \subset (n+1)H$.

So, take an $m > n+1$ and let $k = m - (n+1)$. Taking a subset of $P$, if necessary, we can assume that there exists an $\varepsilon > 0$ such that $P \subset (\varepsilon, 1]$. By the part with $m = n+1$ the set $H$ contains arbitrary small elements. So, there exists a $z \in (0,\varepsilon) \cap kH$. Then $P - z \subset [0,1]$ and, again by case $m = n+1$, there exists an $x \in (P - z) \cap (n+1)H$. But then $x + z \in P \cap mH$. So (∗) is proved.

To finish the theorem it is enough to note that for every $m > n$ and perfect set $P \subset [0,1]$ by (∗) we have $P \cap (m+1)H \neq \emptyset$, while $(m+1)H \subset \mathbb{R} \setminus mH$ (since $H$ is a Hamel basis). $\square$

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