FUNDAMENTA
MATHEMATICAE
176 (2003)

Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$ and its consequences

by

Krzysztof Ciesielski (Morgantown, WV) and
Janusz Pawlikowski (Wrocław)

Abstract. We formulate a Covering Property Axiom $\mathrm{CPA}_{\mathrm{cube}}$, which holds in the iterated perfect set model, and show that it implies easily the following facts.

(a) For every $S \subset \mathbb{R}$ of cardinality continuum there exists a uniformly continuous function $g: \mathbb{R} \to \mathbb{R}$ with $g[S] = [0, 1]$.

(b) If $S \subset \mathbb{R}$ is either perfectly meager or universally null then S has cardinality less than $\mathfrak{c}$.

(c) $\mathrm{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$, i.e., the cofinality of the measure ideal $\mathcal{N}$ is ω_1.

(d) For every uniformly bounded sequence $\langle f_n \in \mathbb{R}^{\mathbb{R}} \rangle_{n<\omega}$ of Borel functions there are sequences: $\langle P_\xi \subset \mathbb{R}: \xi < \omega_1 \rangle$ of compact sets and $\langle W_\xi \in [\omega]^\omega: \xi < \omega_1 \rangle$ such that $\mathbb{R} = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$, $\langle f_n \restriction P_\xi \rangle_{n \in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.

(e) Total failure of Martin's Axiom: $\mathfrak{c} > \omega_1$ and for every non-trivial ccc forcing $\mathbb{P}$ there exist ω_1 dense sets in $\mathbb{P}$ such that no filter intersects all of them.

1. Axiom $\mathrm{CPA}_{\mathrm{cube}}$ and other preliminaries. Our set-theoretic terminology is standard and follows that of [4]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space `without isolated points`.

For a Polish space X, the symbol $\mathrm{Perf}(X)$ will stand for the collection of all subsets of X homeomorphic to $\mathfrak{C}$. We will consider $\mathrm{Perf}(X)$ to be ordered by inclusion. A family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *dense in* $\mathrm{Perf}(X)$ provided for every $P \in \mathrm{Perf}(X)$ there exists a $Q \in \mathcal{E}$ such that $Q \subset P$.

2000 *Mathematics Subject Classification*: Primary 03E35; Secondary 03E17, 26A03.

Key words and phrases: continuous images, perfectly meager, universally null, cofinality of null ideal, uniform convergence, Martin's Axiom.

The work of the first author was partially supported by NATO Grant PST.CLG. 977652.

The second author wishes to thank West Virginia University for its hospitality during 1998-2001, where the results presented here were obtained.

Axiom $\mathrm{CPA}_{\mathrm{cube}}$ is of the form

$\mathfrak{c} = \omega_2$ and if $\mathcal{E} \subset \mathrm{Perf}(X)$ is *appropriately* dense in $\mathrm{Perf}(X)$ then $|X \setminus \bigcup \mathcal{E}_0| < \mathfrak{c}$ for some $\mathcal{E}_0 \in [\mathcal{E}]^{\leq \omega_1}$.

If the word "appropriately" in the above is ignored, then it implies the following statement.

NAÏVE-CPA: If $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ then $|X \setminus \bigcup \mathcal{E}| < \mathfrak{c}$.

It is a very good candidate for our axiom in the sense that it implies all the properties we are interested in. It has, however, one major flaw—*it is false!* This is the case since $S \subset X \setminus \bigcup \mathcal{E}$ for some dense set $\mathcal{E}$ in $\mathrm{Perf}(X)$ provided

for each $P \in \mathrm{Perf}(X)$ there is a $Q \in \mathrm{Perf}(X)$ such that $Q \subset P \setminus S$.

This means that the family $\mathcal{G}$ of all sets of the form $X \setminus \bigcup \mathcal{E}$, where $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, coincides with the σ-ideal s_0 of Marczewski sets, since $\mathcal{G}$ is clearly hereditary. Thus we have

$$s_0 = \Big\{X \setminus \bigcup \mathcal{E} \colon \mathcal{E} \text{ is dense in } \mathrm{Perf}(X)\Big\}. \tag{1}$$

However, it is well known (see e.g. [17, Thm. 5.10]) that there are s_0-sets of cardinality $\mathfrak{c}$. Thus, our Naïve-CPA "axiom" cannot be consistent with ZFC.

In order to formulate the real axiom $\mathrm{CPA}_{\mathrm{cube}}$ we need the following terminology and notation. A subset C of a product $\mathfrak{C}^\eta$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n\in\eta} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each n. For a fixed Polish space X let $\mathcal{F}_{\mathrm{cube}}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set P from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\mathrm{cube}}$ from C onto P as a coordinate system imposed on P. We say that $P \in \mathrm{Perf}(X)$ is a *cube* if it is determined by an (implicitly given) witness function $f \in \mathcal{F}_{\mathrm{cube}}$ onto P, and Q is a *subcube of a cube* $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\mathrm{cube}}$ is a witness function for P and $C \subset \mathrm{dom}(f) \subset \mathfrak{C}^\omega$ is a perfect cube. Here and in what follows, $\mathrm{dom}(f)$ stands for the domain of f.

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *cube dense* in $\mathrm{Perf}(X)$ provided every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\mathrm{cube}}\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2}$$

It is easy to see that the notion of cube density is a generalization of the notion of density as defined in the first paragraph of this section:

(3) if $\mathcal{E}$ is cube dense in $\mathrm{Perf}(X)$ then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$.

On the other hand, the converse implication is not true, as shown by the following simple example.

EXAMPLE 1.1. *Let* $X = \mathfrak{C} \times \mathfrak{C}$ *and let* $\mathcal{E}$ *be the family of all* $P \in \mathrm{Perf}(X)$ *such that either*

- *all vertical sections* $P_x = \{y \in \mathfrak{C}\colon \langle x, y\rangle \in P\}$ *of* P *are countable, or*
- *all horizontal sections* $P^y = \{x \in \mathfrak{C}\colon \langle x, y\rangle \in P\}$ *of* P *are countable.*

Then $\mathcal{E}$ *is dense in* $\mathrm{Perf}(X)$, *but it is not cube dense in* $\mathrm{Perf}(X)$.

Proof. To see that $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ let $R \in \mathrm{Perf}(X)$. We need to find a $P \subset R$ with $P \in \mathcal{E}$. Clearly at least one of the projections $\pi_0(R)$ or $\pi_1(R)$ is uncountable. Assume that $\pi_0(R)$ is uncountable and let $p\colon \pi_0(R) \to \mathfrak{C}$ be a Borel function. (For example, if p is defined by $p(x) = \min R_x$ then $p \subset R$ is Baire class 1.) So, there is a $Q \in \mathrm{Perf}(\mathfrak{C})$ such that $p{\restriction}Q$ is continuous. In particular, $p{\restriction}Q$ (identified with its graph) is a closed subset of $X = \mathfrak{C} \times \mathfrak{C}$. So $P = p{\restriction}Q \in \mathcal{E}$ is a subset of R.

To see that $\mathcal{E}$ is not $\mathcal{F}_{\mathrm{cube}}$-dense in $\mathrm{Perf}(X)$ it is enough to notice that $P = X = \mathfrak{C} \times \mathfrak{C}$ considered as a cube, where the second coordinate is identified with $\mathfrak{C}^{\omega\setminus\{0\}}$, has no subcube in $\mathcal{E}$. More formally, let h be a homeomorphism from $\mathfrak{C}$ onto $\mathfrak{C}^{\omega\setminus\{0\}}$, let $g\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}^\omega = \mathfrak{C} \times \mathfrak{C}^{\omega\setminus\{0\}}$ be given by $g(x, y) = \langle x, h(y)\rangle$, and let $f = g^{-1}\colon \mathfrak{C}^\omega \to \mathfrak{C} \times \mathfrak{C}$ be the coordinate function making $\mathfrak{C} \times \mathfrak{C} = \mathrm{range}(f)$ a cube. Then $\mathrm{range}(f)$ does not contain a subcube from $\mathcal{E}$. ∎

With these notions in hand we are ready to formulate our axiom $\mathrm{CPA}_{\mathrm{cube}}$.

$\mathrm{CPA}_{\mathrm{cube}}$: $\mathfrak{c} = \omega_2$ and for every Polish space X and every cube dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

The proof that $\mathrm{CPA}_{\mathrm{cube}}$ holds in the iterated perfect set model can be found in [6] and [7].

It is also worth noticing that in order to check that $\mathcal{E}$ is cube dense it is enough to consider in condition (2) only functions f defined on the entire space $\mathfrak{C}^\omega$, that is:

FACT 1.2. $\mathcal{E} \subset \mathrm{Perf}(X)$ *is cube dense if and only if*

$$\forall f \in \mathcal{F}_{\mathrm{cube}},\ \mathrm{dom}(f) = \mathfrak{C}^\omega,\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{4}$$

Proof. To see this, let Φ be the family of all bijections $h = \langle h_n\rangle_{n<\omega}$ between perfect subcubes $\prod_{n\in\omega} D_n$ and $\prod_{n\in\omega} C_n$ of $\mathfrak{C}^\omega$ such that each h_n is a homeomorphism between D_n and C_n. Then

$$f \circ h \in \mathcal{F}_{\mathrm{cube}} \quad \text{for every } f \in \mathcal{F}_{\mathrm{cube}} \text{ and } h \in \Phi \text{ with } \mathrm{range}(h) \subset \mathrm{dom}(f). \tag{5}$$

Now take an arbitrary $f\colon C \to X$ from $\mathcal{F}_{\mathrm{cube}}$ and choose an $h \in \Phi$ mapping $\mathfrak{C}^\omega$ onto C. Then $\widehat{f} = f \circ h \in \mathcal{F}_{\mathrm{cube}}$ maps $\mathfrak{C}^\omega$ into X and, using (4), we can find $\widehat{g} \in \mathcal{F}_{\mathrm{cube}}$ such that $\widehat{g} \subset \widehat{f}$ and $\mathrm{range}(\widehat{g}) \in \mathcal{E}$. Then $g = f{\restriction}h[\mathrm{dom}(\widehat{g})]$ satisfies condition (2). ∎

Next, let us consider

$$(6)\qquad s_0^{\mathrm{cube}} = \Big\{X \setminus \bigcup \mathcal{E}\colon \mathcal{E} \text{ is cube dense in } \mathrm{Perf}(X)\Big\}$$
$$= \{S \subset X\colon \forall \text{ cube } P \in \mathrm{Perf}(X)\ \exists \text{ subcube } Q \subset P \setminus S\}.$$

Notice that

FACT 1.3. $[X]^{<\mathfrak{c}} \subset s_0^{\mathrm{cube}} \subset s_0$ *for every Polish space* X.

An easy proof of this fact can be found in [7]. It can also be shown in ZFC that s_0^{cube} forms a σ-ideal. However, neither of these facts will be used in what follows. On the other hand we will be interested in the following proposition.

PROPOSITION 1.4. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds then* $s_0^{\mathrm{cube}} = [X]^{\le\omega_1}$.

Proof. It is obvious that $\mathrm{CPA}_{\mathrm{cube}}$ implies $s_0^{\mathrm{cube}} \subset [X]^{<\mathfrak{c}}$. We will not be interested in the other inclusion, though it follows immediately from Fact 1.3. ∎

REMARK 1.5. $s_0^{\mathrm{cube}} \neq [X]^{\le\omega_1}$ in a model obtained by adding Sacks numbers side-by-side. In particular $\mathrm{CPA}_{\mathrm{cube}}$ is false in this model.

Proof. This follows from the fact that $s_0^{\mathrm{cube}} = [X]^{\le\omega_1}$ implies property (a) (see Corollary 2.2) while it is false in the model mentioned above, as noticed by Miller in [16, p. 581]. (In this model the set X of all Sacks generic numbers cannot be mapped continuously onto $[0,1]$.) ∎

2. Continuous images of sets of cardinality $\mathfrak{c}$. An important quality of the ideal s_0^{cube}, and so the power of the assumption $s_0^{\mathrm{cube}} = [X]^{<\mathfrak{c}}$, is well depicted by the following fact.

PROPOSITION 2.1. *If* X *is a Polish space and* $S \subset X$ *does not belong to* s_0^{cube} *then there exist a* $T \in [S]^{\mathfrak{c}}$ *and a uniformly continuous function* h *from* T *onto* $\mathfrak{C}$.

Proof. Take an S as above and let $f\colon \mathfrak{C}^\omega \to X$ be a continuous injection such that $f[C] \cap S \neq \emptyset$ for every perfect cube C. Let $g\colon \mathfrak{C} \to \mathfrak{C}$ be a continuous function such that $g^{-1}(y)$ is perfect for every $y \in \mathfrak{C}$. Then $h_0 = g \circ \pi_0 \circ f^{-1}\colon f[\mathfrak{C}^\omega] \to \mathfrak{C}$ is uniformly continuous. Moreover, if $T = S \cap f[\mathfrak{C}^\omega]$ then $h_0[T] = \mathfrak{C}$ since

$$T \cap h_0^{-1}(y) = T \cap f[\pi_0^{-1}(g^{-1}(y))] = S \cap f[g^{-1}(y) \times \mathfrak{C} \times \mathfrak{C} \times \ldots] \neq \emptyset$$

for every $y \in \mathfrak{C}$. ∎

COROLLARY 2.2. *Assume* $s_0^{\mathrm{cube}} = [X]^{<\mathfrak{c}}$ *for a Polish space* X. *If* $S \subset X$ *has cardinality* $\mathfrak{c}$ *then there exists a uniformly continuous function* $f\colon X \to [0,1]$ *such that* $f[S] = [0,1]$. *In particular*, $\mathrm{CPA}_{\mathrm{cube}}$ *implies property* (a).

Proof. If S is as above then, by $\mathrm{CPA}_{\mathrm{cube}}$, $S \notin s_0^{\mathrm{cube}}$. Thus, by Proposition 2.1 there exists a uniformly continuous function h from a subset of S onto $\mathfrak{C}$. Consider $\mathfrak{C}$ as a subset of $[0,1]$ and let $\widehat{h}\colon X \to [0,1]$ be a uniformly continuous extension of h. If $g\colon [0,1] \to [0,1]$ is continuous and such that $g[\mathfrak{C}] = [0,1]$ then $f = g \circ \widehat{h}$ is as desired. ∎

The fact that (a) holds in the iterated perfect set model was first proved by A. Miller in [16].

It is worth noting here that the function f in Corollary 2.2 cannot be required to be either monotone or in the class "D^1" of all functions having finite or infinite derivative at every point. This follows immediately from the following proposition, since each function which is either monotone or "D^1" belongs to the Banach class

$$(T_2) = \{f \in \mathcal{C}(\mathbb{R})\colon \{y \in \mathbb{R}\colon |f^{-1}(y)| > \omega\} \in \mathcal{N}\}.$$

(See [10] or [19, p. 278].)

Proposition 2.3. *There exists, in ZFC, an $S \in [\mathbb{R}]^{\mathfrak{c}}$ such that $[0,1] \not\subset f[S]$ for every $f \in (T_2)$.*

Proof. Let $\{f_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of all functions from (T_2) whose range contains $[0,1]$. Construct by induction a sequence $\langle\langle s_\xi, y_\xi\rangle\colon \xi<\mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$,

(i) $y_\xi \in [0,1] \setminus f_\xi[\{s_\zeta\colon \zeta < \xi\}]$ and $|f_\xi^{-1}(y_\xi)| \le \omega$.

(ii) $s_\xi \in \mathbb{R} \setminus (\{s_\zeta\colon \zeta < \xi\} \cup \bigcup_{\zeta\le\xi} f_\zeta^{-1}(y_\zeta))$.

Then the set $S = \{s_\xi\colon \xi < \mathfrak{c}\}$ is as required since $y_\xi \in [0,1] \setminus f_\xi[S]$ for every $\xi < \mathfrak{c}$. ∎

3. Perfectly meager and universally null sets. The fact that (b) holds in the iterated perfect set model was first proved by A. Miller in [16].

Theorem 3.1. *If $S \subset \mathbb{R}$ is either perfectly meager or universally null then $S \in s_0^{\mathrm{cube}}$. In particular,* $\mathrm{CPA}_{\mathrm{cube}}$ *implies property* (b).

Proof. Take an $S \subset \mathbb{R}$ which is either perfectly meager or universally null and let $f\colon \mathfrak{C}^\omega \to \mathbb{R}$ be a continuous injection. Then $S \cap f[\mathfrak{C}^\omega]$ is either meager or null in $f[\mathfrak{C}^\omega]$. Thus $G = \mathfrak{C}^\omega \setminus f^{-1}(S)$ is either comeager or of full measure in $\mathfrak{C}^\omega$. Hence the theorem follows immediately from the following claim. ∎

Claim 3.2. *Consider $\mathfrak{C}^\omega$ with standard topology and standard product measure. If G is a Borel subset of $\mathfrak{C}^\omega$ which is either of second category or of positive measure then G contains a perfect cube $\prod_{i<\omega} P_i$.*

The measure version of the claim is a variant the following theorem:

(m) for every full measure subset H of $[0,1]\times[0,1]$ there are a perfect set $P\subset[0,1]$ and a positive inner measure subset $\widehat{H}$ of $[0,1]$ such that $P\times\widehat{H}\subset H$,

which was proved by Eggleston [9] and, independently, by Brodskiĭ [3]. The category version of the claim is a consequence of the category version of (m):

(c) for every Polish space X and every comeager subset G of $X\times X$ there are a perfect set $P\subset X$ and a comeager subset $\widehat{G}$ of X such that $P\times\widehat{G}\subset G$.

This well known result can be found in [12, Exercise 19.3]. (Its version for $\mathbb{R}^2$ is also proved, for example, in [8, condition ($\star$), p. 416].) For completeness, we will show here in detail how to deduce the claim from (m) and (c).

We will start the argument with a simple fact, in which we will use the following notations. If X is a Polish space endowed with a Borel measure then $\psi_0(X)$ will stand for the sentence

$\psi_0(X)$: For every full measure subset H of $X\times X$ there are a perfect set $P\subset X$ and a positive inner measure subset $\widehat{H}$ of X such that $P\times\widehat{H}\subset H$.

Thus $\psi_0([0,1])$ is a restatement of (m). We will also use the following seemingly stronger variants of $\psi_0(X)$.

$\psi_1(X)$: For every full measure subset H of $X\times X$ there are a perfect set $P\subset X$ and a subset $\widehat{H}$ of X of full measure such that $P\times\widehat{H}\subset H$.

$\psi_2(X)$: For a subset H of $X\times X$ of positive inner measure there are a perfect set $P\subset X$ and a positive inner measure subset $\widehat{H}$ of X such that $P\times\widehat{H}\subset H$.

Fact 3.3. *Let $n=1,2,3,\ldots$*

(i) *If $E\subset\mathbb{R}^n$ has a positive Lebesgue measure then the set $\mathbb{Q}^n+E=\bigcup_{q\in\mathbb{Q}^n}(q+E)$ has a full measure.*

(ii) *$\psi_k(X)$ holds for all $k<3$ and $X\in\{[0,1],(0,1),\mathbb{R},\mathfrak{C}\}$.*

Proof. (i) Let λ be the Lebesgue measure on $\mathbb{R}^n$ and for $\varepsilon>0$ and $x\in\mathbb{R}^n$ let $B(x,\varepsilon)$ be the open ball in $\mathbb{R}^n$ of radius ε centered at x. By way of contradiction assume that there exists a positive measure set $A\subset\mathbb{R}^n$ disjoint from $\mathbb{Q}^n+E$. Let $a\in A$ and $x\in E$ be Lebesgue density points of A and X, respectively. Take an $\varepsilon>0$ such that $\lambda(A\cap B(a,\varepsilon))>(1-4^{-n})\lambda(B(a,\varepsilon))$ and $\lambda(E\cap B(x,\varepsilon))>(1-4^{-n})\lambda(B(x,\varepsilon))$. Now, if $q\in\mathbb{Q}^n$ is such that $q+x\in B(a,\varepsilon/2)$ then $A\cap(q+E)\cap B(a,\varepsilon/2)\neq\emptyset$ since $B(a,\varepsilon/2)\subset B(a,\varepsilon)\cap B(q+x,\varepsilon)$ and thus

$$\lambda(A\cap(q+E)\cap B(a,\varepsilon/2))>\lambda(B(a,\varepsilon/2))-2\cdot4^{-n}\lambda(B(a,\varepsilon))\geq 0.$$

Hence $A\cap(\mathbb{Q}^n+E)\neq\emptyset$, contradicting the choice of A.

(ii) First note that $\psi_k(\mathbb{R}) \Leftrightarrow \psi_k((0,1)) \Leftrightarrow \psi_k([0,1]) \Leftrightarrow \psi_k(\mathfrak{C})$ for every $k < 3$. This is justified by the fact that for the mappings $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = \cot(x\pi)$, the identity mapping id: $(0,1) \to [0,1]$, and $d\colon \mathfrak{C} \to [0,1]$ given by $d(x) = \sum_{i<\omega} x(i)/2^{i+1}$, the image and the preimage of any measure zero set (resp. full measure set) is of measure zero (resp. full measure).

Since, by (m), $\psi_0([0,1])$ is true, we deduce that $\psi_0(X)$ also holds for $X \in \{(0,1), \mathbb{R}, \mathfrak{C}\}$. To finish the proof it is enough to show that $\psi_0(\mathbb{R})$ implies $\psi_1(\mathbb{R})$ and $\psi_2(\mathbb{R})$.

To prove $\psi_1(\mathbb{R})$ let H be a full measure subset of $\mathbb{R} \times \mathbb{R}$ and define $H_0 = \bigcap_{q\in\mathbb{Q}}(\langle 0, q\rangle + H)$. Then H_0 is still of full measure so, by $\psi_0(\mathbb{R})$, there are a perfect set $P \subset \mathbb{R}$ and a positive inner measure subset $\widehat{H}_0$ of $\mathbb{R}$ such that $P \times \widehat{H}_0 \subset H_0$. Thus, for every $q \in \mathbb{Q}$ we also have $P \times (q + \widehat{H}_0) \subset \langle 0, q\rangle + H_0 = H_0$. Let $\widehat{H} = \bigcup_{q\in\mathbb{Q}}(q + \widehat{H}_0)$. Then $P \times \widehat{H} \subset H_0 \subset H$ and, by (i), $\widehat{H}$ has full measure. So, $\psi_1(\mathbb{R})$ is proved.

To prove $\psi_2(\mathbb{R})$ let $H \subset \mathbb{R} \times \mathbb{R}$ be of positive inner measure. Decreasing H if necessary, we can assume that H is compact. Let $H_0 = \mathbb{Q}^2 + H$. Then, by (i), H_0 is of full measure so, by $\psi_0(\mathbb{R})$, there are a perfect set $P_0 \subset \mathbb{R}$ and a positive inner measure subset $\widehat{H}_0$ of $\mathbb{R}$ such that $P_0 \times \widehat{H}_0 \subset H_0$. Once again, decreasing P_0 and $\widehat{H}_0$ if necessary, we can assume that they are homeomorphic to $\mathfrak{C}$ and that no relatively open subset of $\widehat{H}_0$ has measure zero. Since $P_0 \times \widehat{H}_0 \subset \bigcup_{q\in\mathbb{Q}^2}(q + H)$ is covered by countably many compact sets $(P_0 \times \widehat{H}_0) \cap (q + H)$ with $q \in \mathbb{Q}^2$, there is a $q = \langle q_0, q_1\rangle \in \mathbb{Q}^2$ such that $(P_0 \times \widehat{H}_0) \cap (q + H)$ has a non-empty interior in $P_0 \times \widehat{H}_0$. Let U and V be non-empty clopen subsets of P_0 and $\widehat{H}_0$, respectively, such that $U \times V \subset (P_0 \times \widehat{H}_0) \cap (q + H) \subset \langle q_0, q_1\rangle + H$. Then U and V are perfect and V has positive measure. Let $P = -q_0 + U$ and $\widehat{H} = -q_1 + V$. Then $P \times \widehat{H} = (-q_0 + U) \times (-q_1 + V) = -\langle q_0, q_1\rangle + (U \times V) \subset H$, so $\psi_2(\mathbb{R})$ holds. ∎

Proof of Claim 3.2. Since the natural homeomorphism between $\mathfrak{C}$ and $\mathfrak{C}^{\omega\setminus\{0\}}$ preserves product measure, we can identify $\mathfrak{C}^\omega = \mathfrak{C} \times \mathfrak{C}^{\omega\setminus\{0\}}$ with $\mathfrak{C} \times \mathfrak{C}$ considered with its usual topology and its usual product measure. With this identification the result follows easily, by induction on coordinates, from the following fact.

(•) For every Borel subset H of $\mathfrak{C} \times \mathfrak{C}$ which is of second category (resp. of positive measure) there are a perfect set $P \subset \mathfrak{C}$ and a second category (resp. positive measure) subset $\widehat{H}$ of $\mathfrak{C}$ such that $P \times \widehat{H} \subset H$.

The measure version of (•) is a restatement of $\psi_2(\mathfrak{C})$, which was proved in Fact 3.3(ii). To see the category version of (•) let H be a Borel subset of $\mathfrak{C} \times \mathfrak{C}$ of second category. Then there are clopen subsets U and V of $\mathfrak{C}$ such that $H_0 = H \cap (U \times V)$ is comeager in $U \times V$. Since U and V are homeomorphic to $\mathfrak{C}$, we can apply (c) to H_0 and $U \times V$ to find a perfect set $P \subset U$ and a comeager Borel subset $\widehat{H}$ of V such that $P \times \widehat{H} \subset H_0 \subset H$, finishing the proof. ∎

4. $\mathrm{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c}$**.** Next we show that $\mathrm{CPA}_{\mathrm{cube}}$ implies that $\mathrm{cof}(\mathcal{N}) = \omega_1$. So, under $\mathrm{CPA}_{\mathrm{cube}}$, all cardinals from Cichoń's diagram (see e.g. [1]) are equal to ω_1. The fact that this holds in the iterated perfect set model has been well known.

Let $\mathcal{C}_H$ be the family of all subsets $\prod_{n<\omega} T_n$ of ω^ω such that $T_n \in [\omega]^{\le n+1}$ for all $n < \omega$. We will use the following characterization.

PROPOSITION 4.1 (Bartoszyński [1, Thm. 2.3.9]).

$$\mathrm{cof}(\mathcal{N}) = \min\Big\{|\mathcal{F}|\colon \mathcal{F} \subset \mathcal{C}_H \ \&\ \bigcup \mathcal{F} = \omega^\omega\Big\}.$$

LEMMA 4.2. *The family* $\mathcal{C}_H^* = \{X \subset \omega^\omega\colon X \subset T$ *for some* $T \in \mathcal{C}_H\}$ *is* $\mathcal{F}_{\mathrm{cube}}$*-dense in* $\mathrm{Perf}(\omega^\omega)$.

Proof. Let $f\colon \mathfrak{C}^\omega \to \omega^\omega$ be a continuous function. By (4) it is enough to find a perfect cube C in $\mathfrak{C}^\omega$ such that $f[C] \in \mathcal{C}_H^*$.

Construct, by induction on $n < \omega$, the families $\{E^i_s\colon s \in 2^n \ \&\ i < \omega\}$ of non-empty clopen subsets of $\mathfrak{C}$ such that for every $n < \omega$ and $s, t \in 2^n$:

(i) $E^i_s = E^i_t$ for every $n \le i < \omega$;
(ii) $E^i_{s^\frown 0}$ and $E^i_{s^\frown 1}$ are disjoint subsets of E^i_s for every $i < n+1$;
(iii) for every $\langle s_i \in 2^n\colon i < \omega\rangle$,

$$f(x_0)\restriction 2^{(n+1)^2} = f(x_1)\restriction 2^{(n+1)^2} \quad \text{for every } x_0, x_1 \in \prod_{i<\omega} E_{s_i}.$$

For each $i < \omega$ the fusion of $\{E^i_s\colon s \in 2^{<\omega}\}$ will give us the ith coordinate set of the desired perfect cube C.

Condition (iii) can be ensured by uniform continuity of f. Indeed, let $\delta > 0$ be such that $f(x_0)\restriction 2^{(n+1)^2} = f(x_1)\restriction 2^{(n+1)^2}$ for every $x_0, x_1 \in \mathfrak{C}^\omega$ at distance less than δ. Then it is enough to choose $\{E^i_s\colon s \in 2^n \ \&\ i < \omega\}$ such that (i) and (ii) are satisfied and every set $\prod_{i<\omega} E_{s_i}$ from (iii) has diameter less than δ. This finishes the construction.

Next for every $i, n < \omega$ let $E^i_n = \bigcup\{E^i_s\colon s \in 2^n\}$ and $E_n = \prod_{i<\omega} E^i_n$. Then $C = \bigcap_{n<\omega} E_n = \prod_{i<\omega}\big(\bigcap_{n<\omega} E^i_n\big)$ is a perfect cube, since $\bigcap_{n<\omega} E^i_n \in \mathrm{Perf}(\mathfrak{C})$ for every $i < \omega$. Thus, to finish the proof it is enough to show that $f[C] \in \mathcal{C}_H^*$.

So, for every $k < \omega$ let $n < \omega$ be such that $2^{n^2} \le k+1 < 2^{(n+1)^2}$, put

$$\begin{aligned} T_k &= \{f(x)(k)\colon x \in E_n\} \\ &= \Big\{f(x)(k)\colon x \in \prod_{i<\omega} E_{s_i} \text{ for some } \langle s_i \in 2^n\colon i < \omega\rangle\Big\}, \end{aligned}$$

and notice that T_k has at most $2^{n^2} \le k+1$ elements. Indeed, by (iii), the set $\{f(x)(k)\colon x \in \prod_{i<\omega} E_{s_i}\}$ has precisely one element for every $\langle s_i \in 2^n\colon i < \omega\rangle$ while (i) implies that $\{\prod_{i<\omega} E_{s_i}\colon \langle s_i \in 2^n\colon i<\omega\rangle\}$ has 2^{n^2} elements. Therefore $\prod_{k<\omega} T_k \in \mathcal{C}_H$.

To finish the proof it is enough to notice that $f[C] \subset \prod_{k<\omega} T_k$. ∎

COROLLARY 4.3. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds then* $\mathrm{cof}(\mathcal{N}) = \omega_1$.

Proof. By $\mathrm{CPA}_{\mathrm{cube}}$ and Lemma 4.2 there exists an $\mathcal{F} \in [\mathcal{C}_H]^{\le\omega_1}$ such that $|\omega^\omega \setminus \bigcup \mathcal{F}| \le \omega_1$. This and Proposition 4.1 imply $\mathrm{cof}(\mathcal{N}) = \omega_1$. ∎

5. Pointwise convergence of subsequences of real-valued functions. A sequence $\langle f_n\rangle_{n<\omega}$ of real-valued functions is *uniformly bounded* provided there exists an $r \in \mathbb{R}$ such that $\mathrm{range}(f_n) \subset [-r, r]$ for every n. In 1932 Mazurkiewicz [15] proved the following variant of Egorov's theorem.

> *For every uniformly bounded sequence $\langle f_n\rangle_{n<\omega}$ of real-valued continuous functions defined on a Polish space X there exists a subsequence which is uniformly convergent on some perfect set P.*

The main result of this section is the following theorem.

THEOREM 5.1. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds then*

(∗) *for every Polish space X and every uniformly bounded sequence $\langle f_n\colon X \to \mathbb{R}\rangle_{n<\omega}$ of Borel measurable functions there are sequences: $\langle P_\xi\colon \xi < \omega_1\rangle$ of compact subsets of X and $\langle W_\xi \in [\omega]^\omega\colon \xi < \omega_1\rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$, $\langle f_n \restriction P_\xi\rangle_{n\in W_\xi}$ is a monotone uniformly convergent sequence of uniformly continuous functions.*

Theorem 5.1 is a variant of [5, Thm. 2] and its corollary, according to which condition (∗) for continuous functions f_n can be deduced from the assumptions that $\mathrm{cof}(\mathcal{N}) = \omega_1$ and there exists a selective ω_1-generated ultrafilter on ω.

Proof of Theorem 5.1. We first note that the family $\mathcal{E}$ of all $P \in \mathrm{Perf}(X)$ for which there exists a $W \in [\omega]^\omega$ such that

$$\text{the sequence } \langle f_n \restriction P\rangle_{n\in W} \text{ is monotone and uniformly convergent}$$

is $\mathcal{F}_{\mathrm{cube}}$-dense in $\mathrm{Perf}(X)$.

Indeed, let $g \in \mathcal{F}_{\mathrm{cube}}$, $g\colon \mathfrak{C}^\omega \to X$, and consider the functions $h_n = f_n \circ g$. Since $h = \langle h_n\colon n<\omega\rangle\colon \mathfrak{C}^\omega \to \mathbb{R}^\omega$ is Borel measurable, there is a dense G_δ subset G of $\mathfrak{C}^\omega$ such that $h\restriction G$ is continuous. So, we can find a perfect cube $C \subset G \subset \mathfrak{C}^\omega$, and for this C the function $h\restriction C$ is continuous. Thus, identifying the coordinate spaces of C with $\mathfrak{C}$, without loss of generality we can assume that $C = \mathfrak{C}^\omega$, that is, each function $h_n\colon \mathfrak{C}^\omega \to \mathbb{R}$ is continuous. Now, by [21, Thm. 6.9], there is a perfect cube C in $\mathfrak{C}^\omega$ and a $W \in [\omega]^\omega$ such that the sequence $\langle h_n \restriction C\rangle_{n\in W}$ is monotone and uniformly convergent ([1]). So $P = g[C]$ is in $\mathcal{E}$.

Now, by $\mathrm{CPA}_{\mathrm{cube}}$, there exists an $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ such that $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$. Then $\{P_\xi : \xi < \omega_1\} = \mathcal{E}_0 \cup \{\{x\} : x \in X \setminus \bigcup \mathcal{E}_0\}$ is as desired: if $P_\xi \in \mathcal{E}_0$ then the existence of an appropriate W_ξ follows from the definition of $\mathcal{E}$. If P_ξ is a singleton, then the existence of W_ξ follows from the fact that every sequence of reals has a monotone subsequence. ■

6. Total failure of Martin's Axiom. In this section we prove that $\mathrm{CPA}_{\mathrm{cube}}$ implies the total failure of Martin's Axiom, that is, the property that

> for every non-trivial ccc forcing $\mathbb{P}$ there exist ω_1 dense sets in $\mathbb{P}$ such that no filter intersects all of them.

The consistency of this fact with $\mathfrak{c} > \omega_1$ was first proved by Baumgartner [2] in a model obtained by adding Sacks reals side-by-side. The topological and boolean-algebraic formulations of the theorem follow immediately from the following proposition.

Proposition 6.1. *The following conditions are equivalent.*

(a) *For every non-trivial ccc forcing $\mathbb{P}$ there exist ω_1 dense sets in $\mathbb{P}$ such that no filter intersects all of them.*

(b) *Every compact ccc topological space without isolated points is a union of ω_1 nowhere dense sets.*

(c) *For every atomless ccc complete Boolean algebra B there exist ω_1 dense sets in B such that no filter intersects all of them.*

(d) *For every atomless ccc complete Boolean algebra B there exist ω_1 maximal antichains in B such that no filter intersects all of them.*

(e) *For every countably generated atomless ccc complete Boolean algebra B there exist ω_1 maximal antichains in B such that no filter intersects all of them.*

Proof. The equivalence of the conditions (a), (b), (c), and (d) is well known. In particular, the equivalence of (a)–(c) is explicitly given in [2, Thm. 0.1]. Clearly (d) implies (e). The remaining implication, (e)⇒(d), is a version of the theorem from [14, p. 158]. However, it is expressed there in a bit different language, so we include here its proof.

So, let $\langle B, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ be an atomless ccc complete Boolean algebra. For every $\sigma \in 2^{<\omega_1}$ define, by induction on the length $\mathrm{dom}(\sigma)$ of the sequence σ, a $b_\sigma \in B$ such that:

([1]) Actually [21, Thm. 6.9] is stated for functions defined on $[0,1]^\omega$. However, the proof presented there works also for functions defined on $\mathfrak{C}^\omega$.

- $b_\emptyset = \mathbf{1}$.
- b_σ is a disjoint union of $b_{\sigma\hat{}0}$ and $b_{\sigma\hat{}1}$.
- If $b_\sigma > \mathbf{0}$ then $b_{\sigma\hat{}0} > \mathbf{0}$ and $b_{\sigma\hat{}1} > \mathbf{0}$.
- If $\lambda = \mathrm{dom}(\sigma)$ is a limit ordinal then $b_\sigma = \bigwedge_{\xi<\lambda} b_{\sigma\restriction\xi}$.

Let $T = \{s \in 2^{<\omega_1}: b_s > \mathbf{0}\}$. Then T is a subtree of $2^{<\omega_1}$; its levels determine antichains in B, so they are countable.

First assume that T has a countable height. Then T itself is countable. Let B_0 be the smallest complete subalgebra of B containing $\{b_\sigma: \sigma \in T\}$ and notice that B_0 is atomless. Indeed, if there were an atom a in B_0 then $S = \{\sigma \in T: a \le b_\sigma\}$ would be a branch in T so that $\delta = \bigcup S$ would belong to $2^{<\omega_1}$. Since $b_\delta \ge a > \mathbf{0}$, we would also have $\delta \in T$. But then $a \le b_\delta = b_{\delta\hat{}0} \vee b_{\delta\hat{}1}$ so that either $\delta\hat{}0$ or $\delta\hat{}1$ belongs to S, which is impossible.

Thus, B_0 is a complete, countably generated, atomless subalgebra of B. So, by (e), there exists a family $\mathcal{A}$ of ω_1 maximal antichains in B_0 with no filter in B_0 intersecting all of them. But then each $A \in \mathcal{A}$ is also a maximal antichain in B and no filter in B would intersect all of them. So, we have (d).

Next, assume that T has height ω_1 and for every $\alpha < \omega_1$ let

$$T_\alpha = \{\sigma \in T: \mathrm{dom}(\sigma) = \alpha\}$$

be the αth level of T. Also let $b_\alpha = \bigvee_{\sigma \in T_\alpha} b_\sigma$. Notice that $b_\alpha = b_{\alpha+1}$ for every $\alpha < \omega_1$. On the other hand, it may happen that $b_\lambda > \bigwedge_{\alpha<\lambda} b_\alpha$ for some limit $\lambda < \omega_1$; however, this may happen only countably many times, since B is ccc. Thus, there is an $\alpha < \omega_1$ such that $b_\beta = b_\alpha$ for every $\alpha < \beta < \omega_1$.

Now, let B_0 be the smallest complete subalgebra of B below $\mathbf{1} \setminus b_\alpha$ containing $\{b_\sigma \setminus b_\alpha: \sigma \in T\}$. Then B_0 is countably generated and, as before, it can be shown that B_0 is atomless. Thus, there exists a family $\mathcal{A}_0$ of ω_1 maximal antichains in B_0 with no filter in B_0 intersecting all of them. Then no filter in B containing $\mathbf{1} \setminus b_\alpha$ intersects every $A \in \mathcal{A}_0$. But for every $\alpha < \beta < \omega_1$ the set $A^\beta = \{b_\sigma: \sigma \in T_\beta\}$ is a maximal antichain in B below b_α. Therefore, $\mathcal{A}_1 = \{A^\beta: \alpha < \beta < \omega_1\}$ is an uncountable family of maximal antichains in B below b_α with no filter in B containing b_α intersecting every $A \in \mathcal{A}_1$. Then it is easy to see that $\mathcal{A} = \{A_0 \cup A_1: a_0 \in \mathcal{A}_0 \ \&\ A_1 \in \mathcal{A}_1\}$ is a family of ω_1 maximal antichains in B with no filter in B intersecting all of them. This proves (d). ∎

Theorem 6.2. $\mathrm{CPA}_{\mathrm{cube}}$ *implies the total failure of Martin's Axiom.*

Proof. Let $\mathcal{A}$ be a countably generated atomless ccc complete Boolean algebra and let $\{A_n: n < \omega\}$ generate $\mathcal{A}$. By Proposition 6.1 it is enough to show that $\mathcal{A}$ contains ω_1 maximal antichains such that no filter in $\mathcal{A}$ intersects all of them.

Next let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathfrak{C} = 2^\omega$. Recall that it is a free countably generated σ-algebra, with free generators $B_i = \{s \in \mathfrak{C}:$

$s(i)=0\}$. Define h_0: $\{B_n\colon n<\omega\}\to\{A_n\colon n<\omega\}$ by $h_0(B_n)=A_n$ for all $n<\omega$. Then h_0 can be uniquely extended to a σ-homomorphism $h\colon \mathcal{B}\to\mathcal{A}$ between σ-algebras $\mathcal{B}$ and $\mathcal{A}$. (See e.g. [20, 34.1, p. 117].) Let $\mathcal{I}=\{B\in\mathcal{B}\colon h[B]=\mathbf{0}\}$. Then $\mathcal{I}$ is a σ-ideal in $\mathcal{B}$ and the quotient algebra $\mathcal{B}/\mathcal{I}$ is isomorphic to $\mathcal{A}$. (Compare also the Loomis–Sikorski theorem in [20, p. 117] or [13].) In particular, $\mathcal{I}$ contains all singletons and is ccc, since $\mathcal{A}$ is atomless and ccc.

It follows that we only need to consider complete Boolean algebras of the form $\mathcal{B}/\mathcal{I}$, where $\mathcal{I}$ is some ccc σ-ideal of Borel sets containing all singletons. To prove that such an algebra has ω_1 maximal antichains as desired, it is enough to prove that

$(*)$ $\quad\mathfrak{C}$ is a union of ω_1 perfect sets $\{N_\xi\colon \xi<\omega_1\}$ which belong to $\mathcal{I}$.

Indeed, assume that $(*)$ holds and for every $\xi<\omega_1$ let $\mathcal{D}^*_\xi$ be a family of all $B\in\mathcal{B}\setminus\mathcal{I}$ with closures $\mathrm{cl}(B)$ disjoint from N_ξ. Then $\mathcal{D}_\xi=\{B/\mathcal{I}\colon B\in\mathcal{D}^*_\xi\}$ is dense in $\mathcal{B}/\mathcal{I}$, since $\mathfrak{C}\setminus N_\xi$ is σ-compact and $\mathcal{B}/\mathcal{I}$ is a σ-algebra. Let $\mathcal{A}^*_\xi\subset\mathcal{D}^*_\xi$ be such that $\mathcal{A}_\xi=\{B/\mathcal{I}\colon B\in\mathcal{A}^*_\xi\}$ is a maximal antichain in $\mathcal{B}/\mathcal{I}$. It is enough to show that no filter intersects all $\mathcal{A}_\xi$'s. But if there were a filter $\mathcal{F}$ in $\mathcal{B}/\mathcal{I}$ intersecting all $\mathcal{A}_\xi$'s then for every $\xi<\omega_1$ there would exist a $B_\xi\in\mathcal{A}^*_\xi$ with $B_\xi/\mathcal{I}\in\mathcal{F}\cap\mathcal{A}_\xi$. Thus, the set $\bigcap_{\xi<\omega_1}\mathrm{cl}(B_\xi)$ would be non-empty, despite the fact that it is disjoint from $\bigcup_{\xi<\omega_1}N_\xi=\mathfrak{C}$.

To finish the proof it is enough to show that $(*)$ follows from $\mathrm{CPA}_{\mathrm{cube}}$. But this follows immediately from the fact that any cube P in $\mathfrak{C}$ contains a subcube $Q\in\mathcal{I}$ as any cube P can be partitioned into $\mathfrak{c}$ disjoint subcubes and, by the ccc property of $\mathcal{I}$, only countably many of them can be outside $\mathcal{I}$. ∎

The authors like to thank Professor Anastasis Kamburelis for many helpful comments concerning the results presented here. We are especially grateful to him for pointing out a gap in an earlier version of the proof of Theorem 6.2.

Other consequences of $\mathrm{CPA}_{\mathrm{cube}}$ can be found in [6], [11], [18], and in the monograph in preparation [7].

References

[1] T. Bartoszyński and H. Judah, *Set Theory*, A K Peters, 1995.
[2] J. Baumgartner, *Sacks forcing and the total failure of Martin's axiom*, Topology Appl. 19 (1985), 211–225.
[3] M. L. Brodskiĭ, *On some properties of sets of positive measure*, Uspekhi Mat. Nauk 4 (1949), no. 3, 136–138.
[4] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Stud. Texts 39, Cambridge Univ. Press, 1997.

[5] K. Ciesielski and J. Pawlikowski, *Small combinatorial cardinal characteristics and theorems of Egorov and Blumberg*, Real Anal. Exchange 26 (2000–2001), 905–911. (Preprint* available.) ([1])

[6] —, —, *Crowded and selective ultrafilters under the Covering Property Axiom*, preprint*.

[7] —, —, *Covering Property Axiom CPA*, version of September 2002, work in progress*.

[8] K. Ciesielski and J. Wojciechowski, *Sums of connectivity functions on* $\mathbb{R}^n$, Proc. London Math. Soc. 76 (2) (1998), 406–426. (Preprint* available.)

[9] H. G. Eggleston, *Two measure properties of Cartesian product sets*, Quart. J. Math. Oxford (2) 5 (1954), 108–115.

[10] J. Foran, *Continuous functions: a survey*, Real Anal. Exchange 2 (1977), 85–103.

[11] F. Jordan, *Generalizing the Blumberg theorem*, Real Anal. Exchange 27 (2001–2002), 243–255. (Preprint* available.)

[12] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, Berlin, 1995.

[13] L. H. Loomis, *On the representation of the* σ*-complete Boolean algebra*, Bull. Amer. Math. Soc. 53 (1947), 757–760.

[14] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2 (1970), 143–178.

[15] S. Mazurkiewicz, *Sur les suites de fonctions continues*, Fund. Math. 18 (1932), 114–117.

[16] A. W. Miller, *Mapping a set of reals onto the reals*, J. Symbolic Logic 48 (1983), 575–584.

[17] —, *Special subsets of the real line*, in: Handbook of Set-theoretic Topology, K. Kunen and J. E. Vaughan (eds.), North-Holland, 1984, 201–233.

[18] A. Nowik, *Possibly there is no uniformly completely Ramsey null set of size* 2^ω, Colloq. Math. 93 (2002), 251–258. (Preprint* available.)

[19] S. Saks, *Theory of the Integral*, 2nd ed., Monograf. Mat., 7, PWN, Warszawa, 1937.

[20] R. Sikorski, *Boolean Algebras*, 3rd ed., Springer, New York, 1969.

[21] S. Todorčević and I. Farah, *Some Applications of the Method of Forcing*, Yenisei Series in Pure and Applied Mathematics, Yenisei, Moscow; Lycée, Troitsk, 1995.

Department of Mathematics
West Virginia University
Morgantown, WV 26506-6310, U.S.A.
E-mail: K_Cies@math.wvu.edu
Web page: http://www.math.wvu.edu/~kcies

Department of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: pawlikow@math.uni.wroc.pl

Received 19 December 2001

([1]) Preprints marked by * are available in electronic form from *Set Theoretic Analysis Web Page:* http://www.math.wvu.edu/~kcies/STA/STA.html