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SPACES ON WHICH EVERY POINTWISE CONVERGENT SERIES OF CONTINUOUS FUNCTIONS CONVERGES PSEUDO-NORMALLY

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ABSTRACT. A topological space X is a $\Sigma\Sigma^*$-space provided that, for every sequence $\langle f_n\rangle_{n=0}^{\infty}$ of continuous functions from X to $\mathbb{R}$, if the series $\sum_{n=0}^{\infty}|f_n|$ converges pointwise, then it converges pseudo-normally. We show that every regular Lindelöf $\Sigma\Sigma^*$-space has the Rothberger property. We also construct, under the continuum hypothesis, a $\Sigma\Sigma^*$-subset of $\mathbb{R}$ of cardinality continuum.

1. INTRODUCTION

We will use standard set-theoretical notation as in [4]. In particular, the ordinal numbers will be identified with the sets of their predecessors and cardinals with the initial ordinals. Thus, $2 = \{0,1\}$ and the first infinite ordinal number ω is equal to the set of all natural numbers $\{0,1,2,\dots\}$. The family of all functions from a set X into Y is denoted by Y^X. Thus, for $n<\omega$ symbol 2^n will stand for the set of all sequences $s\colon\{0,1,2,\dots,n-1\}\to\{0,1\}$, while $2^{<\omega}=\bigcup_{n<\omega}2^n$ is the set of all finite sequences into 2. For a set X and a cardinal number κ we define $[X]^\kappa$ as the family of all subsets of X of cardinality κ. Families $[X]^{<\kappa}$ and $[X]^{\le\kappa}$ are defined similarly. We let $\mathfrak{C}\subset[0,1]$ be the classical Cantor middle-thirds set. For a topological space X we write $\mathcal{C}(X)$ for the family of continuous functions from X into the set $\mathbb{R}$ of real numbers. For a sequence $\langle f_n\rangle_{n<\omega}$ of continuous functions from X into $\mathbb{R}$ we let

$$S(\langle f_n\rangle)=\left\{x\in X\colon \sum_{n<\omega}|f_n(x)|<\infty\right\}.$$

A series $\sum_{n<\omega}f_n$ of functions from a set X into $\mathbb{R}$ *converges normally* (respectively, *pseudo-normally*) on a set $A\subset X$ if there exists a sequence $\langle\varepsilon_n : n<\omega\rangle$ of positive reals such that $\sum_{n<\omega}\varepsilon_n<\infty$ and for every $x\in A$ we have $|f_n(x)|\le\varepsilon_n$ for all (respectively, all but finitely many) $n<\omega$. Thus, a topological space X belongs to $\Sigma\Sigma^*$ (see [3]) provided that, for any sequence $\langle f_n\in\mathcal{C}(X)\colon n<\omega\rangle$ with $S(\langle f_n\rangle)=X$, the series $\sum_{n<\omega}f_n$ converges pseudo-normally on X.

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The class $\Sigma\Sigma^*$ has been studied, in a context of several similar classes of sets, by Bukovský, Recław, and Repický in [3]. In particular, it is known (see [3, Diagram 2]) that every $\Sigma\Sigma^*$-space X is a *σ-space*, that is, every G_δ subset of X is also an F_σ subset of X. One of the main results of this paper says that Lindelöf $\Sigma\Sigma^*$-spaces are small also in a different sense.

Theorem 1. *Every regular and Lindelöf $\Sigma\Sigma^*$-space has the Rothberger property.*

Recall here that a topological space X has the *Rothberger property* (called sometimes *property C''*) provided that, for every sequence $\langle \mathcal{G}_n : n < \omega \rangle$ of open covers of X, we can find $U_n \in \mathcal{G}_n$ for every $n < \omega$ such that $\{U_n : n < \omega\}$ covers X.

Certainly the most interesting case of Theorem 1 is when the space is separable and metric. It should be stressed that although many classes from [3, Diagram 2] are contained in the class of σ-spaces, within the class of separable metric spaces only $\Sigma\Sigma^*$-spaces have the Rothberger property. Recall also that every metric space X with the Rothberger property is of strong measure zero. (See e.g. [1, thm. 8.1.11].) Since there are models of ZFC in which every strong measure zero subset of $\mathbb{R}$ is countable [8, thm. 3.2], an uncountable $\Sigma\Sigma^*$-subset of $\mathbb{R}$ cannot be constructed in ZFC. Moreover, so far there has been no consistent example of a $\Sigma\Sigma^*$-subset of $\mathbb{R}$ of cardinality continuum. This state changes with the following theorem.

Theorem 2. *If the continuum hypothesis holds then there exists a set $X \subset [0,1]$ of cardinality continuum that belongs to $\Sigma\Sigma^*$.*

Note that the existence of such a set does not seem to follow from any other known constructions, since $\Sigma\Sigma^*$ does not contain any other known (to us) class for which the analogous result is known. Indeed, one of the smallest classes of sets that admits the consistent examples of cardinality continuum subsets of $\mathbb{R}$ is the class of strong γ-sets.[1] (See [5].) However, there are consistent examples of strong γ-sets that are not in $\Sigma\Sigma^*$. Actually, slightly modifying the proof of [5, thm. 8] it is easy to construct, under Martin's axiom, a strong γ-subset of $\mathbb{R}$ of cardinality continuum that is continuum-concentrated on a countable subset. By theorems 3.12 and 4.1 from [3], such a set is not in $\Sigma\Sigma^*$.

2. Every regular Lindelöf $\Sigma\Sigma^*$-space has the Rothberger property

We start with the following variation of a well-known characterization of the Rothberger property.

Lemma 3. *Let $\langle m_n \rangle_{n<\omega}$ be a sequence of positive integers. Then a topological space X has the Rothberger property if and only if*

(∗) *for every sequence $\langle \mathcal{U}_n \rangle_{n<\omega}$ of open covers of X there exist an increasing sequence $\langle n_i < \omega : i < \omega \rangle$ and sets $\mathcal{V}_{n_i} \subset \mathcal{U}_{n_i}$ with $|\mathcal{V}_{n_i}| \le m_{n_i}$ such that $X = \bigcup_{i<\omega} \bigcup \mathcal{V}_{n_i}$.*

Proof. Clearly every space with the Rothberger property satisfies (∗).

To see the other implication let $\langle \mathcal{G}_n : n < \omega \rangle$ be a sequence of open covers of X. For $n < \omega$ put $p_n = \sum_{k=0}^{n} m_k$ and let $\mathcal{U}_n$ be an open cover of X refining covers $\mathcal{G}_k$ for $k < p_n$. Let also $p_{-1} = 0$. Applying (∗) to $\langle \mathcal{U}_n \rangle_{n<\omega}$ we can find appropriate sequences $\langle n_i \rangle_{i<\omega}$ and $\langle \mathcal{V}_{n_i} \rangle_{i<\omega}$. For $n < \omega$ let $\mathcal{W}_n = \{W_k \in \mathcal{U}_n : p_{n-1} \le k < p_n\}$

[1] Every strong γ-set has the γ-property, which is the strongest property in Scheepers' diagram [7] admitting consistent examples of cardinality continuum.

be such that $\mathcal{V}_{n_i} \subset \mathcal{W}_{n_i}$ for every $i < \omega$. (Thus, for n not equal to any n_i the sets in $\mathcal{W}_n$ are arbitrary.) Then we have $\bigcup_{k<\omega} W_k \supset \bigcup_{i<\omega} \bigcup \mathcal{V}_{n_i} = X$. But for $p_{n-1} \leq k < p_n$ the set W_k belongs to $\mathcal{U}_n$, which is a refinement of $\mathcal{G}_k$. So, we can find $U_k \in \mathcal{G}_k$ for which $W_k \subset U_k$. Therefore, $X = \bigcup_{k<\omega} U_k$. □

Proof of Theorem 1. Since X is regular and Lindelöf, it is completely regular. Thus, being a $\Sigma\Sigma^*$-space, X must be zero-dimensional — see [2, cor. 4.5]. Let $\langle \mathcal{U}_n \rangle_{n<\omega}$ be a sequence of open covers of X. We will show that the condition $(*)$ from Lemma 3 is fulfilled.

Since X is Lindelöf and zero-dimensional we can assume, replacing with a refinement, if necessary, that each cover $\mathcal{U}_n$ is countable and consists of pairwise disjoint clopen sets. For every $n < \omega$ let $\{U_k : k \in P_n\}$ be an enumeration of $\mathcal{U}_n$, where $\{P_n : n < \omega\}$ is an appropriate partition of ω.

Let $\langle m_n \rangle_{n<\omega}$ be a sequence of positive integers such that $r = \sum_{n=0}^{\infty} 1/m_n < \infty$. For every $n < \omega$ and $k \in P_n$ let $f_k = \frac{1}{m_n}\chi_{U_k}$, where $\chi_U : X \to \{0,1\}$ is the characteristic function of U. Then for every $x \in X$

$$\sum_{k<\omega} f_k(x) = \sum_{n<\omega} \sum_{k\in P_n} f_k(x) = \sum_{n<\omega} \frac{1}{m_n} = r.$$

Since X is a $\Sigma\Sigma^*$-space, there exists a sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive reals such that $\sum_{n<\omega} \varepsilon_n < \infty$, and for every $x \in X$ we have $f_n(x) \leq \varepsilon_n$ for all but finitely many $n < \omega$. For $i, n < \omega$ let $X_i = \{x \in X : f_k(x) \leq \varepsilon_k \text{ for all } i \leq k < \omega\}$ and put $T_i^n = \{k \in P_n : X_i \cap U_k \neq \emptyset\}$. We claim that for any $i < \omega$

(1) there exist infinitely many $n \in \omega$ such that $|T_i^n| \leq m_n$.

To see this, by way of contradiction assume that there is an $i < \omega$ for which (1) is false. So, there exists an n_0 such that $|T_i^n| > m_n$ for each $n \geq n_0$. Moreover, increasing n_0, if necessary, we can assume that $k \geq i$ for any $k \in T_i^n$ and $n \geq n_0$. Hence for every $k \in T_i^n$ and $n \geq n_0$ there is an $x \in X_i \cap U_k$, and so $\frac{1}{m_n} = f_k(x) \leq \varepsilon_k$. Consequently $\sum_{k=0}^{\infty} \varepsilon_k \geq \sum_{n\geq n_0} \sum_{k\in T_i^n} \varepsilon_k \geq \sum_{n \geq n_0} |T_i^n| \cdot 1/m_n = \infty$, which is a contradiction.

Now let $\langle n_i < \omega : i < \omega \rangle$ be an increasing sequence such that $|T_i^{n_i}| \leq m_{n_i}$. Setting $\mathcal{V}_{n_i} = \{U_j^{n_i} : j \in T_i^{n_i}\} \subset \mathcal{U}_{n_i}$ we obtain a cover satisfying $(*)$ from Lemma 3. □

3. $\Sigma\Sigma^*$-subset of $\mathbb{R}$ of cardinality continuum

We start with recalling Egoroff's theorem (see e.g. [10, p. 73]):

> *Let μ be a finite countably additive measure on a set X and let $\langle g_k \rangle_{k<\omega}$ be a pointwise convergent sequence of measurable functions from X into $\mathbb{R}$. Then for every $\varepsilon > 0$ there exists a measurable set $E \subset X$ such that $\mu(X \setminus E) < \varepsilon$ and $\langle g_k \rangle_{k<\omega}$ converges uniformly on E.*

In the proof of the theorem we will use the following result, which is of interest on its own.

Proposition 4. *Let P be a perfect subset of $\mathbb{R}$ and $\langle f_n \in \mathcal{C}(P) : n < \omega \rangle$. If $S(\langle f_n \rangle) = P$, then there exists a sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive numbers such that $\sum_{n<\omega} \varepsilon_n < \infty$ and the closed set $K = \bigcap_{n<\omega} \{x \in P : |f_n(x)| \leq \varepsilon_n\}$ has cardinality continuum.*

Proof. For $k<\omega$ and $x\in P$ define $g_k(x)=\sum_{n=k}^{\infty}|f_n(x)|$. Then functions $g_k\colon P\to\mathbb{R}$ are measurable (in fact, Baire class one) and they converge pointwise to 0. By Egoroff's theorem (used with a countably additive Borel probability measure on P that vanishes on points) there exists a perfect subset Q of P on which the sequence $\langle g_k\rangle$ converges uniformly. By induction on $k<\omega$ we will construct the sequences $\langle \varepsilon_n^k>0\colon k,n<\omega\rangle$ and $\langle x_s\in Q\colon s\in 2^k\ \&\ k<\omega\rangle$ such that for every $0<k<\omega$, $s\in 2^{k-1}$, $t\in 2^k$, and $n<\omega$ the following holds true:

(i) $x_\emptyset\in Q$ is arbitrary, $\varepsilon_n^0=|f_n(x_\emptyset)|+2^{-n}$; thus $\sum_{m=0}^{\infty}\varepsilon_m^0<\infty$;

(ii) $x_{s\frown 0}=x_s$ and $0<|x_{s\frown 1}-x_s|<4^{-k}\delta_{k-1}$, where the number δ_{k-1} is defined as $\min\left(\{|x_{s_0}-x_{s_1}|>0\colon s_0,s_1\in 2^{k-1}\}\cup\{1\}\right)$;

(iii) $|f_n(x_t)|<\varepsilon_n^k$ and $\varepsilon_n^{k-1}\le\varepsilon_n^k$;

(iv) $\sum_{m=0}^{\infty}\varepsilon_m^k\le 2^{-k}+\sum_{m=0}^{\infty}\varepsilon_m^{k-1}$ and $\varepsilon_i^k=\varepsilon_i^i$ for $i<k$.

First notice that if such sequences can be constructed and we put $\varepsilon_n=\varepsilon_n^n$, then the sequence $\langle\varepsilon_n\rangle_{n<\omega}$ is as desired. Indeed, the series converges, since by (iv) for every $k<\omega$ we have

$$\sum_{m=0}^{k}\varepsilon_m=\sum_{m=0}^{k}\varepsilon_m^m=\sum_{m=0}^{k}\varepsilon_m^k\le\sum_{m=0}^{\infty}\varepsilon_m^k\le\sum_{m=1}^{k}2^{-m}+\sum_{m=0}^{\infty}\varepsilon_m^0<1+\sum_{m=0}^{\infty}\varepsilon_m^0,$$

so $\sum_{m=0}^{\infty}\varepsilon_m\le 1+\sum_{m=0}^{\infty}\varepsilon_m^0<\infty$.

To see that the set $K=\bigcap_{n<\omega}\{x\in P\colon |f_n(x)|\le\varepsilon_n^n\}$ has cardinality continuum, notice that K is closed and that, by conditions (iii) and (iv), we have $x_t\in K$ for every $t\in 2^{<\omega}$. So K contains the closure of the set $\{x_t\colon t\in 2^{<\omega}\}$. But, by (ii), the mapping associating $\lim_{k\to\infty}x_{\varphi\restriction k}\in K$ to each $\varphi\in 2^\omega$ is one-to-one.

To make an inductive step in the construction take a $k>0$ and assume that the $(k-1)$-th step of the construction is already done. Find a $j<\omega$ such that $j>k$ and $g_j(x)<4^{-k}$ for all $x\in Q$. Let $0<\delta\le 4^{-k}\delta_{k-1}$ be such that for every $s\in 2^{k-1}$, $n\le j$, and $x\in Q$, if $|x-x_s|<\delta$, then $|f_n(x)|<\varepsilon_n^{k-1}$. Existence of such a δ follows from the inductive assumption (iii) and the continuity of functions f_n. For $s\in 2^{k-1}$ pick $x_{s\frown 1}\in Q$ such that $0<|x_{s\frown 1}-x_s|<\delta$. This insures (ii). For $n<j$ put $\varepsilon_n^k=\varepsilon_n^{k-1}$ and for $j\le n<\omega$ define $\varepsilon_n^k=\varepsilon_n^{k-1}+\sum_{t\in 2^k}|f_n(x_t)|$. This clearly guarantees (iii) and the second part of (iv). To see the first part of (iv) notice that

$$\begin{aligned}
\sum_{m=0}^{\infty}\varepsilon_m^k &= \sum_{m<j}\varepsilon_m^k+\sum_{m=j}^{\infty}\varepsilon_m^k\\
&= \sum_{m<j}\varepsilon_m^{k-1}+\sum_{m=j}^{\infty}\left(\varepsilon_m^{k-1}+\sum_{t\in 2^k}|f_m(x_t)|\right)\\
&= \sum_{t\in 2^k}\sum_{m=j}^{\infty}|f_m(x_t)|+\sum_{m=0}^{\infty}\varepsilon_m^{k-1}\\
&= \sum_{t\in 2^k}g_j(x_t)+\sum_{m=0}^{\infty}\varepsilon_m^{k-1}\\
&\le 2^k\,4^{-k}+\sum_{m=0}^{\infty}\varepsilon_m^{k-1}=2^{-k}+\sum_{m=0}^{\infty}\varepsilon_m^{k-1}.
\end{aligned}$$

This finishes the proof of Proposition 4. $\square$

In Proposition 4, if we require only that the sequence $\langle \varepsilon_n : n < \omega \rangle$ converges to zero then, by Egoroff's theorem, we can additionally assume that the set K has a positive measure. However, the set K from Proposition 4 need not have positive measure.

In the proof that follows we will use the following simple fact.

Fact 5. *For every $X \subset \mathbb{R}$ and $\langle f_n \in \mathcal{C}(X) \colon n < \omega \rangle$ such that $S(\langle f_n \rangle) = X$ there exists a Borel set $B \subset \mathbb{R}$ containing X and extensions $f_n^* \in \mathcal{C}(B)$ of the functions f_n such that $S(\langle f_n^* \rangle) = B$.*

Proof. For every $n < \omega$ there exists a G_δ set $G_n \subset \mathbb{R}$ containing X and an extension $\hat{f}_n \in \mathcal{C}(G_n)$ of f_n. Let $G = \bigcap_{n<\omega} G_n \supset X$, and let us define B as $\bigcup_{m<\omega} \bigcap_{k<\omega} \left\{ x \in G \colon \sum_{n \leq k} |\hat{f}_n(x)| \leq m \right\}$. Then B is a Borel set containing X and $B = S(\langle \hat{f}_n \restriction G \rangle)$. Thus functions $f_n^* = \hat{f}_n \restriction B$ are as required. □

Proof of Theorem 2. Let $\mathcal{B}$ be a fixed countable clopen basis for $\mathfrak{C}$. For $G \subset \mathbb{R}$ let $\mathcal{K}(G)$ be the family of all sequences $\langle f_n \in \mathcal{C}(G) \colon n < \omega \rangle$ with $S(\langle f_n \rangle) = G$ and let $\langle \langle f_n^\xi \rangle_{n<\omega} \colon \xi < \omega_1 \rangle$ be an enumeration of the family

$$\mathcal{K} = \bigcup \{ \mathcal{K}(G) \colon G \text{ is a Borel subset of } \mathfrak{C} \}$$

such that every f_n^0 is the constant zero function on $\mathfrak{C}$. By induction on $\xi < \omega_1$ we will construct a sequence $\langle \langle x_\xi, \langle \varepsilon_n^\xi \rangle_{n<\omega}, \langle P_n^\xi \rangle_{n<\omega}, P_\xi, \mathcal{G}_\xi \rangle \colon \xi < \omega_1 \rangle$ such that $x_0 = 0$, $\varepsilon_n^0 = 2^{-n}$, $P_0 = P_n^0 = \mathfrak{C}$, $\mathcal{G}_0 = \{\mathfrak{C}\}$, and the following conditions are satisfied for every $0 < \xi < \omega_1$ and $k < \omega$:

(a) $P_\xi = \bigcup_{n<\omega} P_n^\xi$, $0 < \varepsilon_k^\xi < \infty$, and $\sum_{n<\omega} \varepsilon_n^\xi < \infty$.
(b) If $\bigcap_{\eta<\xi} P_\eta \not\subset S(\langle f_n^\xi \rangle_{n<\omega})$, then
  – $x_\xi \in \bigcap_{\eta<\xi} P_\eta \setminus S(\langle f_n^\xi \rangle)$, $\varepsilon_k^\xi = 2^{-k}$, and $P_k^\xi = \mathfrak{C}$;
 otherwise,
  – $P_k^\xi = \bigcap_{k<n<\omega} \{ x \in G_\xi \colon |f_n^\xi(x)| \leq \varepsilon_n^\xi \}$, where G_ξ is the domain of the functions f_n^ξ, $n < \omega$;
  $x_\xi \in \bigcap_{\eta\leq\xi} P_\eta \setminus \{ x_\eta \colon \eta < \xi \}$.
(c) $\mathcal{G}_\xi$ is a non-empty countable family of perfect subsets of $\bigcap_{\eta\leq\xi} P_\eta$ such that
  – if $T \in \mathcal{G}_\xi$, $U \in \mathcal{B}$, and $T \cap U \neq \emptyset$, then $T \cap U \in \mathcal{G}_\xi$;
  – every $H \in \bigcup_{\eta<\xi} \mathcal{G}_\eta$ contains a subset T from the family $\mathcal{G}_\xi$.

It should be clear that if we put $X = \{ x_\xi \colon \xi < \omega_1 \}$, then X is uncountable and belongs to $\Sigma\Sigma^*$.

Indeed, X is uncountable, since $S(\langle f_n^\xi \rangle_{n<\omega}) = \mathfrak{C} \supset \bigcap_{\eta<\xi} P_\eta$ for uncountably many $\xi < \omega_1$. To see that $X \in \Sigma\Sigma^*$ take a sequence $\langle f_n \in \mathcal{C}(X) \colon n < \omega \rangle$ for which $S(\langle f_n \rangle) = X$. By Fact 5 we can find a Borel set $B \subset \mathbb{R}$ containing X and extensions $f_n^* \in \mathcal{C}(B)$ of the functions f_n such that $S(\langle f_n^* \rangle) = B$. Then, there exists a $\xi < \omega_1$ such that $\langle f_n^\xi \rangle_{n<\omega} = \langle f_n^* \rangle_{n<\omega}$. Notice that $\bigcap_{\eta<\xi} P_\eta \subset S(\langle f_n^\xi \rangle_{n<\omega})$, since otherwise $x_\xi \in X \setminus S(\langle f_n^\xi \rangle_{n<\omega})$ contradicting $X = S(\langle f_n \rangle) \subset S(\langle f_n^\xi \rangle_{n<\omega})$. Therefore, $\{ x_\zeta \colon \xi \leq \zeta < \omega_1 \} \subset P_\xi = \bigcup_{k<\omega} \bigcap_{k<n<\omega} \{ x \in G_\xi \colon |f_n^\xi(x)| \leq \varepsilon_n^\xi \}$. Since the sequence $\langle \varepsilon_n^\xi \rangle_{n<\omega}$ works for all but countably many points of X, the existence of an appropriate sequence $\langle \varepsilon_n \rangle_{n<\omega}$ follows from an obvious fact that the class $\Sigma\Sigma^*$ is countably additive.

Thus, it is enough to show that the inductive construction is possible. The key point here is to prove that the intersection $\bigcap_{\eta<\xi} P_\eta$ is uncountable, so that we could have a chance to choose the next x_ξ. This will be done with the help of the families $\mathcal{G}_\eta$.

First notice that

(2) for every $H \in \bigcup_{\eta<\xi} \mathcal{G}_\eta$ there is a perfect set $H^* \subset H \cap \bigcap_{\eta<\xi} P_\eta$.

If ξ is a successor ordinal, say $\xi = \alpha+1$, this is obvious: by the inductive assumption there is an $H^* \in \mathcal{G}_\alpha$ that is a subset of $H \cap \bigcap_{\eta\le\alpha} P_\eta = H \cap \bigcap_{\eta<\xi} P_\eta$. So, assume that ξ is a limit ordinal, and let $H \in \mathcal{G}_\eta$ for some $\eta < \xi$. Let $\eta = \eta_0 < \eta_1 < \cdots$ be such that $\xi = \bigcup_{n<\omega} \eta_n$. Define a family $\{H_s \colon s \in 2^{<\omega}\}$ by induction on the length of s. We put $H_\emptyset = H$ and, if for some $s \in 2^n$ the set $H_s \in \mathcal{G}_{\eta_n}$ is already defined, choose disjoint perfect subsets $H_{s\hat{}0}$ and $H_{s\hat{}1}$ from $\mathcal{G}_{\eta_{n+1}}$. The choice can be made by the inductive assumption (c) applied to the family $\mathcal{G}_{\eta_{n+1}}$ and disjoint clopen portions of H_s. Then $H^* = \bigcap_{n<\omega} \bigcup_{s\in 2^n} H_s$ is a perfect subset of $\bigcap_{\eta<\xi} P_\eta$.

Now, let $\mathcal{H} = \{H^* \colon H \in \bigcup_{\eta<\xi} \mathcal{G}_\eta\} = \{H_k \colon k < \omega\}$. We need to find a sequence $\langle \varepsilon_n^\xi \rangle_{n<\omega}$ for which

(3) $H^* \cap P_\xi$ contains a perfect subset for every $H^* \in \mathcal{H}$.

If $\bigcap_{\eta<\xi} P_\eta \not\subset S(\langle f_n^\xi \rangle_{n<\omega})$, then $P_\xi = \mathfrak{C}$, and this is obvious. So, assume that $\bigcap_{\eta<\xi} P_\eta \subset S(\langle f_n^\xi \rangle_{n<\omega})$. Then, by Proposition 4, for every $k < \omega$ there exists a sequence $\langle \hat{\varepsilon}_n^k > 0 \colon n < \omega \rangle$ such that $\sum_{n<\omega} \hat{\varepsilon}_n^k < \infty$ and there is a perfect subset T_k of $\bigcap_{n<\omega} \{x \in H_k \colon |f_n(x)| \le \hat{\varepsilon}_n^k\}$. Let $\varepsilon_n^\xi = \max\{\hat{\varepsilon}_n^k \colon k \le k(n)\}$, where the sequence $k(0) \le k(1) \le \cdots$ converges to ∞ slowly enough that $\sum_{n<\omega} \varepsilon_n^\xi < \infty$. Then $T_n \subset P_\xi$ for every $n < \omega$ and the family

$$\mathcal{G}_\xi = \{T_n \cap B \neq \emptyset \colon n < \omega \ \& \ B \in \mathcal{B}\}$$

satisfies (c). $\square$

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