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SEPARATELY NOWHERE CONSTANT FUNCTIONS; n-CUBE AND α-PRISM DENSITIES

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Abstract. A function f from a countable product $X = \prod_i X_i$ of Polish spaces into a Polish space is separately nowhere constant provided it is nowhere constant on every section of X. We show that every continuous separately nowhere constant function is one-to-one on a product of perfect subsets of X_i's. This result is used to distinguish between n-cube density notions for different $n \leq \omega$, where ω-cube density is a basic notion behind the Covering Property Axiom CPA formulated by Ciesielski and Pawlikowski. We will also distinguish, for different values of $\alpha < \omega_1$, between the notions of α-prism densities — the more refined density notions used also in CPA.

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1. Introduction

The set theoretical notation used in the paper is standard and follows [1]. For a topological space X a function $f\colon X \to Y$ is *nowhere constant* if f is not constant on any non-empty open subset of X. For a subset G of a product space $X = \prod_{i\in I} X_i$ we say that a function $f\colon G \to Y$ is *separately nowhere constant* if for every $t \in G$ and $k \in I$ function f restricted to the section $G^t_k = \{x \in G\colon x \restriction I \setminus \{k\} = t \restriction I \setminus \{k\}\}$ is nowhere constant. This notion is the most natural when $G = X$. In this case it is related in a natural way to the notion of a *separately continuous* function $f\colon X \to Y$, that is, such that f restricted to every section X^t_k is continuous.

Note that every separately nowhere constant function is nowhere constant. However, the converse implication is false, as shown by the polynomial functions from $\mathbb{R}^2$ into $\mathbb{R}$ defined by $w_0(x, y) = xy$ and $w_1(x, y) = x$. This implications pattern stays in contrast with the implications for separate continuity: continuity implies separate continuity, but the converse implication is false.

We will consider the notion of being separately nowhere constant only for the product of *Polish spaces* which we define here as a complete separable metric spaces `without isolated points`. (No function is nowhere constant if it is defined on a space containing isolated points.) Let $\mathfrak{C}$ stand for the Cantor set 2^ω and for a Polish space X let $\mathrm{Perf}(X)$ be the family of all subsets of X homeomorphic to $\mathfrak{C}$. Our main theorem on separately nowhere constant functions is the following result, where a subset P of $\prod_{i\in I} X_i$ is a *perfect cube* provided it is of the form $P = \prod_{i\in I} P_i$ with $P_i \in \mathrm{Perf}(X_i)$ for all $i \in I$.

Theorem 1. *Let G be a dense G_δ subset of a product $\prod_{i\in I} X_i$ of Polish spaces and let f be a continuous function from G into a Polish space Y. If f is separately nowhere constant then there is a perfect cube P in $\prod_{i\in I} X_i$ such that $P \subset G$ and f restricted to P is one-to-one.*

It is not difficult to see that the conclusion of the theorem remains true for the function $w_0(x, y) = xy$, despite the fact that w_0 is not separately nowhere constant. On the other hand, the theorem's conclusion is false for the nowhere constant function $w_1(x, y) = x$.

The theorem will be used to provide the examples distinguishing between the notions of n-cube densities defined as follows. For $0 < n \le \omega$ we say that a family $\mathcal{F} \subset \mathrm{Perf}(X)$ is *n-cube dense* provided that for every continuous injection $f\colon \mathfrak{C}^n \to X$ there is a perfect cube $C \subset \mathfrak{C}^n$ such that $f[C] \in \mathcal{F}$.

To put these notions in a better perspective notice that 1-cube density is just the standard perfect set density, that is, $\mathcal{F} \subset \mathrm{Perf}(X)$ is 1-cube dense provided every perfect subset of X contains a set from $\mathcal{F}$. To see that in general n-cube density is a stronger notion recall the following example, which is extracted from Miller's construction [6, Theorem 5.10] of a Marczewski s_0-set of cardinality continuum.

Example 1 ([2, 4])**.** Let $X = \mathfrak{C} \times \mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either all vertical sections of P are countable, or all horizontal sections of P are countable. Then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ but it is not 2-cube dense.

The interest in cube densities comes from the work of Ciesielski and Pawlikowski [4], in which authors formulate the Covering Property Axiom CPA, notice that it holds in the iterated perfect set model, and discuss in depth its consequences. The simplest form of the axiom is that of $\mathrm{CPA}_{\mathrm{cube}}$:

$\mathrm{CPA}_{\mathrm{cube}}$**:** $\mathfrak{c} = \omega_2$ and for every Polish space X and every ω-cube dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there exists an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| < \mathfrak{c}$ and $|X \setminus \bigcup \mathcal{E}_0| < \mathfrak{c}$.

Another version of CPA axiom, stronger than $\mathrm{CPA}_{\mathrm{cube}}$, is axiom $\mathrm{CPA}_{\mathrm{prism}}$ which is obtained from $\mathrm{CPA}_{\mathrm{cube}}$ by replacing the notion of ω-cube density with the weaker notion of density known as *prism density*: a family $\mathcal{F} \subset \mathrm{Perf}(X)$ is prism dense if it is α-prism dense for every $0 < \alpha < \omega_1$. Now, α-prism density is defined in a way similar to n-cube density. First one needs to identify the family $\mathbb{P}_\alpha$ of subsets of $\mathfrak{C}^\alpha$, termed *iterated perfect sets*. This family, which includes all perfect cubes in $\mathfrak{C}^\alpha$, will be defined in Section 3. Then, we say that a family $\mathcal{F} \subset \mathrm{Perf}(X)$ is *α-prism dense* provided that for every continuous injection $f\colon \mathfrak{C}^\alpha \to X$ there is a $P \in \mathbb{P}_\alpha$ such that $f[P] \in \mathcal{F}$.

The next two theorems discuss these notions of density. In particular, the first of them shows that essentially all these notions are different. The second theorem shows that any strengthening of the axiom $\mathrm{CPA}_{\mathrm{prism}}$ obtained by replacing the prism density with a proper subclass of the densities we consider leads to the statement which is false in ZFC.

Theorem 2. *For a Polish space X, a family $\mathcal{F} \subset \mathrm{Perf}(X)$, and $1 < \alpha < \omega_1$ consider the following sentences:*

C_α**:** *family $\mathcal{F}$ is β-cube dense for every $0 < \beta < \alpha$;*
P_α**:** *family $\mathcal{F}$ is β-prism dense for every $0 < \beta < \alpha$.*

Then, for $2 < m < n < \omega$ and $\omega + 1 < \alpha < \gamma < \omega_1$, they are related by the following implications.

$$
\begin{tikzcd}
\mathrm{C}_2 \arrow[d, leftrightarrow] & \mathrm{C}_m \arrow[l] \arrow[d] & \mathrm{C}_n \arrow[l] \arrow[d] & \mathrm{C}_\omega \arrow[l] \arrow[d] & \mathrm{C}_{\omega+1} \arrow[l] \arrow[r, leftrightarrow] \arrow[d] & \mathrm{C}_\alpha \arrow[r, leftrightarrow] \arrow[d] & \mathrm{C}_\gamma \arrow[d] \\
\mathrm{P}_2 & \mathrm{P}_m \arrow[l] & \mathrm{P}_n \arrow[l] & \mathrm{P}_\omega \arrow[l] & \mathrm{P}_{\omega+1} \arrow[l] & \mathrm{P}_\alpha \arrow[l] & \mathrm{P}_\gamma \arrow[l]
\end{tikzcd}
$$

Moreover, none of these implications can be reversed.

For $\alpha < \omega_1$ we say that a family $\mathcal{F} \subset \mathrm{Perf}(X)$ is α-prism* dense provided $\mathcal{F}$ is β-prism dense for every $0 < \beta < \alpha$ and it is n-cube dense for every $0 < n < \omega$.

Theorem 3. *For every $\alpha < \omega_1$ and for every Polish space X there is an α-prism* dense family $\mathcal{F} \subset \mathrm{Perf}(X)$ for which $|X \setminus \bigcup \mathcal{F}| = \mathfrak{c}$.*

These theorems will be proved in Section 3.

2. Separately nowhere constant functions

As usual $\mathbb{R}$ and $\mathbb{Q}$ represent the sets of real and rational numbers, respectively. The space $\mathbb{R}^\omega$ is the product space of countably many copies of $\mathbb{R}$ with its usual topology. This is a Polish space and a vector space over $\mathbb{R}$ with the operations defined pointwise from the usual operations in $\mathbb{R}$. In this context we consider for every $k < \omega$ the canonical unit vectors $\vec{e}_k \in \mathbb{R}^\omega$: $\vec{e}_k(k) = 1$ and $\vec{e}_k(i) = 0$ for all other $i < \omega$. If $S \subset \mathbb{R}^\omega$ and $\delta \in \mathbb{R}$ then $\delta \cdot \vec{e}_k + S = \{\delta \cdot \vec{e}_k + s : s \in S\}$. If $\varepsilon > 0$ and $x \in \mathbb{R}^\omega$ then $B(x, \varepsilon)$ denotes the open ball with center x and radius ε and $\overline{B}(x, \varepsilon)$ is the corresponding closed ball. If X is a Polish space and $A \subset X$, the closure of A is denoted by $\overline{A}$. If $m < \omega$ then we identify $\mathbb{R}^\omega$ with $\mathbb{R} \times \mathbb{R}^{\omega \setminus \{m\}}$; if $y \in \mathbb{R}^{\omega \setminus \{m\}}$ and $G \subset \mathbb{R}^\omega$ then the section of G along y is the set $(G)^y = \{x \in \mathbb{R} \colon \langle x, y \rangle \in G\}$.

The following variant of Kuratowski-Ulam theorem will be useful in what follows. Although it looks like it should be well known, we could not locate it in the literature. Thus, we present it here with a proof.

Lemma 2. *Let X be a Polish space and consider X^T with the product topology, where $T \neq \emptyset$ is an arbitrary set. Fix at most countable family $\mathcal{K}$ of sets $K \subsetneq T$. Then for every comeager set $H \subset X^T$ there exists a comeager set $G \subset H$ such that for every $x \in G$ and $K \in \mathcal{K}$ the set*

$$G_{x \restriction K} = \left\{ y \in X^{T \setminus K} \colon (x \restriction K) \cup y \in G \right\}$$

is comeager in $X^{T \setminus K}$.

Proof. Let $\{K_i : i < \omega\}$ be an enumeration of $\mathcal{K}$ with infinite repetitions. We construct, by induction on $i < \omega$, a decreasing sequence $\langle G_i : i < \omega \rangle$ of comeager subsets of H such that for every $i < \omega$

(i) the set $(G_i)_{x \restriction K_i}$ is comeager in $X^{T \setminus K_i}$ for every $x \in G_i$.

Put $G_{-1} = H$ and assume that for some $i < \omega$ the comeager set G_{i-1} is already constructed. To define G_i identify X^T with $X^{K_i} \times X^{T \setminus K_i}$. Then, by Kuratowski-Ulam theorem, the set

$$A = \left\{ y \in X^{K_i} : (G_{i-1})_y \text{ is comeager in } X^{T \setminus K_i} \right\}$$

is comeager in X^{K_i}. Put $G_i = G_{i-1} \cap (A \times X^{T \setminus K_i})$.

Clearly $G_i \subset G_{i-1}$ is a comeager subset of X^T. If $x \in G_i$ then $x \restriction K_i \in A$ so $(G_i)_{x \restriction K_i} = (G_{i-1})_{x \restriction K_i}$ is comeager in $X^{T \setminus K_i}$. So, (i) holds. This completes the definition of the sequence $\langle G_i : i < \omega \rangle$.

Let $G = \bigcap_{i<\omega} G_i$. Clearly $G \subset H$ is comeager in X^T. To see the additional part, take a $K \in \mathcal{K}$. Since $G = \bigcap \{G_i : i < \omega \ \& \ K_i = K\}$, for every $x \in G$ the set

$$G_{x \restriction K} = \bigcap \{(G_i)_{x \restriction K_i} : i < \omega \ \& \ K_i = K\}$$

is comeager in $X^{T \setminus K}$. □

The next lemma is an immediate consequence of Lemma 2 applied to $X = \mathbb{R}$, $T = \omega$, and $\mathcal{K} = \{\omega \setminus \{n\} : n < \omega\}$.

Lemma 3. *For every comeager set $G \subset \mathbb{R}^\omega$ there exists a comeager set $H \subset G$ such that for every $x \in H$ and $n < \omega$ the set $H \cap (x + \mathbb{R} \cdot \vec{e}_n)$ is comeager in $x + \mathbb{R} \cdot \vec{e}_n$.*

The following lemma will facilitate the inductive step in the next theorem.

Lemma 4. *Let G be a comeager subset of $\mathbb{R}^\omega$ such that*

(•) *$G \cap (x + \mathbb{R} \cdot \vec{e}_k)$ is comeager in $x + \mathbb{R} \cdot \vec{e}_k$ for every $x \in G$ and $k < \omega$.*

Let f be a continuous separately nowhere constant function from G into a Polish space Y. If $S \in [G]^{<\omega}$ is such that f is one-to-one on S, then for every $k < \omega$ and $\varepsilon > 0$ there exists a $\delta \in (0, \varepsilon)$ such that $(S + \delta \cdot \vec{e}_k) \subset G$ and f is one-to-one on $S \cup (S + \delta \cdot \vec{e}_k)$.

Proof. Let $S = \{x_i : i < n\} \subset G$ be such that $f \restriction S$ is one-to-one. Since f is continuous, decreasing ε if necessary, we can assume that

(∗) if $S^* = \{x_i^* : i < n\}$ is such that $x_i^* \in G \cap B(x_i, \varepsilon)$ for every $i < n$, then f is also one-to-one on S^*.

For each $x \in S$ consider the sets $M_x = \{\delta \in \mathbb{R}\colon x + \delta \cdot \vec{e}_k \in G\}$, which by ($\bullet$) are comeager, and $N_x = \{\delta \in M_x\colon f(x + \delta \cdot \vec{e}_k) \in f[S]\}$. Since $f \restriction G \cap (x + \mathbb{R} \cdot \vec{e}_k)$ is nowhere constant, the set N_x is meager in $\mathbb{R}$. So, $B = \bigcap_{x \in S} M_x \setminus \bigcup_{x \in S} N_x$ is comeager in $\mathbb{R}$.

Pick a $\delta \in (0, \varepsilon) \cap B$. Then $S + \delta \cdot \vec{e}_k \subset G$ as $\delta \in \bigcap_{x \in S} M_x$. To see that f is one-to-one on $S \cup (S + \delta \cdot \vec{e}_k)$ take $x \neq y$ in this set. We need to show that $f(x) \neq f(y)$. This follows from the assumption when $x, y \in S$, from ($*$) when $x, y \in S + \delta \cdot \vec{e}_k$, and from $\delta \notin \bigcup_{x \in S} N_x$ otherwise. □

Theorem 4. *Let G be a comeager subset of $\mathbb{R}^\omega$ and let f be a continuous separately nowhere constant function from G into a Polish space Y. Then there is a perfect cube P in $\mathbb{R}^\omega$ such that $P \subset G$ and f is one-to-one on P.*

Proof. Let $\{m_k\colon k < \omega\}$ be an enumeration of ω where every natural number appears infinitely often. By Lemma 3, shrinking G if necessary, we can assume that G satisfies the condition ($\bullet$) from Lemma 4. Since G is a dense G_δ subset of $\mathbb{R}^\omega$ we have $G = \bigcap_{n<\omega} G_n$, where each G_n is open and dense subset of $\mathbb{R}^\omega$.

We will construct by induction on $k < \omega$ the sequences $\langle S_k \in [G]^{2^k} : k < \omega\rangle$, $\langle \varepsilon_k\colon k < \omega\rangle$, and $\langle \delta_k\colon k < \omega\rangle$ such that for every $k < \omega$:

(1) $0 < \delta_k < \varepsilon_k \leq 2^{-k}$,
(2) $S_{k+1} = S_k \cup (\delta_k \cdot \vec{e}_{m_k} + S_k) \subset \bigcup\{B(x, \varepsilon_k)\colon x \in S_k\}$,
(3) $\overline{B}(x, \varepsilon_k) \subset G_k$ for every $x \in S_k$,
(4) $f[\overline{B}(x, \varepsilon_k)] \cap f[\overline{B}(x^*, \varepsilon_k)] = \emptyset$ for every distinct $x, x^* \in S_k$.

We start the construction with an arbitrary $S_0 = \{s\} \subset G$, and $\varepsilon_0 \leq 1$ ensuring (3). If for some $k < \omega$ the set S_k and ε_k are already constructed we choose δ_k using Lemma 4 with $k = m_k$ and $\varepsilon \leq \varepsilon_k$ small enough that it insures (2) and $|S_{k+1}| = 2^{k+1}$. Then f is one-to-one on $S_{k+1} \subset G$ and, using continuity of f, we can choose ε_{k+1} satisfying (1), (3), and (4). This finishes the construction.

If for $n, k < \omega$ we put $A_{k,n} = \{x(n)\colon x \in S_k\}$ then it is easy to see that:

(a) $S_k = \prod_{n<\omega} A_{k,n}$,
(b) $A_{k+1,n} = A_{k,n}$ for every $n \neq m_{k+1}$,
(c) $A_{k+1,m_{k+1}} = A_{k,m_{k+1}} \cup (\delta_k + A_{k,m_{k+1}})$.

We define $P_n = \overline{\bigcup_{k<\omega} A_{k,n}}$ and put $P = \prod_{n<\omega} P_n$. We will show that each P_n is a perfect subset of $\mathbb{R}$, $P \subset G$, and f is one-to-one on P. Notice that this will finish the proof, because as a final adjustment (necessary, when P_n has a non-empty interior in $\mathbb{R}$) we can shrink each P_n to a subset from $\operatorname{Perf}(\mathbb{R})$.

Clearly each P_n is closed and, by (1) and (2), it has no isolated points. We need to show that

$$\overline{\bigcup_{k<\omega} S_k} = \prod_{n<\omega} P_n.$$

The inclusion $\overline{\bigcup_{k<\omega} S_k} \subset \prod_{n<\omega} P_n$ follows from (a). In order to prove the other inclusion pick an $x \in \prod_{n<\omega} P_n$. Then for every $n < \omega$ there exists a sequence $\{a_i^n : i < \omega\} \subset \bigcup_{k<\omega} A_{k,n}$ with distinct terms such that $\lim_{i\to\infty} a_i^n = x(n)$. For every $m < \omega$ let $x_m \in \mathbb{R}^\omega$ be defined as $x_m(i) = a_i^m$ if $i \le m$ and $x_m(i) = s(i)$ if $i > m$. Then $\{x_m : m < \omega\} \subset \bigcup_{k<\omega} S_k$ and $\lim_{m\to\infty} x_m = x$. This proves that $\prod_{n<\omega} P_n \subset \overline{\bigcup_{k<\omega} S_k}$.

Next notice that if for $k < \omega$ we put $T_k = \bigcup\{\overline{B}(x, \varepsilon_k) : x \in S_k\}$ then condition (2) gives us $\overline{\bigcup_{k<\omega} S_k} \subset \bigcap_{k<\omega} T_k$ while the other inclusion is obvious. In particular we have

$$\prod_{n<\omega} P_n = \overline{\bigcup_{k<\omega} S_k} = \bigcap_{k<\omega} T_k.$$

In order to prove that f is one-to-one on $\prod_{n<\omega} P_n$ pick distinct x and y from $\prod_{n<\omega} P_n$. Then there are sequences $\{x_m\}$ and $\{y_m\}$ such that for every $m < \omega$ we have $x_m, y_m \in S_m$, $x \in \overline{B}(x_m, \varepsilon_{m+1})$, and $y \in \overline{B}(y_m, \varepsilon_{m+1})$. Since $x \neq y$, there is an $m < \omega$ such that $x_m \neq y_m$. So, by (5), we have $f[\overline{B}(x_m, \varepsilon_{m+1})] \cap f[\overline{B}(y_m, \varepsilon_{m+1})] = \emptyset$. Hence $f(x) \neq f(y)$. This shows that f is one-to-one on $\prod_{n<\omega} P_n$.

Finally note that by (3) we have $\prod_{n<\omega} P_n = \bigcap_{k<\omega} T_k \subset G$. □

Corollary 5. *Let $\{X_n : n < \omega\}$ be a family of Polish spaces, G be a dense G_δ subset of $\prod_{n<\omega} X_n$, and let f be a continuous separately nowhere constant function from G into a Polish space Y. Then there exist perfect sets $P_n \in \mathrm{Perf}(X_n)$, $n < \omega$, such that f is one-to-one on $\prod_{n<\omega} P_n$.*

Proof. For every $n < \omega$ let G_n be a dense G_δ subset of X_n homeomorphic to the Baire space ω^ω.[1] Since ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, there is a homeomorphism $h_n : G_n \to \mathbb{R} \setminus \mathbb{Q}$. Then, $h : \prod_{n<\omega} G_n \to (\mathbb{R} \setminus \mathbb{Q})^\omega$ defined by $h = \langle h_n : n < \omega \rangle$ is a cube-preserving homeomorphism. We can apply Theorem 4 to the function $f \circ h^{-1}$ on a dense G_δ subset $h\left[G \cap \prod_{n<\omega} G_n\right]$ of

[1] Every Polish space X (without isolated points) has a dense subspace G homeomorphic to ω^ω constructed as follows. Let $\{B_n : n < \omega\}$ be a basis for X. Then $Y = X \setminus \bigcup_{n<\omega} \mathrm{bd}(B_n)$ is a zero-dimensional dense G_δ subspace of X, where $\mathrm{bd}(B_n)$ denotes the boundary of B_n. Take a countable dense subset D of Y and put $G = Y \setminus D$. Then G is a dense G_δ subspace of X. Also, G is Polish, zero-dimensional, and every compact subset of G has an empty interior. So, by Alexandrov-Urysohn theorem [5, Theorem 7.7], it is homeomorphic to ω^ω.

$\mathbb{R}^\omega$ to obtain a perfect cube $\prod_{n<\omega} Q_n$ on which $f\circ h^{-1}$ is one-to-one. Then, $h^{-1}\left[\prod_{n<\omega} Q_n\right]$ is a perfect cube in $\prod_{n<\omega} X_n$ on which f is one-to-one. $\square$

Proof of Theorem 1. We can assume that the index set I is a cardinal number κ. Let $X=\prod_{i\in\kappa} X_i$.

The case $\kappa=\omega$ is true by Corollary 5.

If $\kappa=n<\omega$ and $f\colon G\to Y$ is continuous and separately nowhere constant consider $F\colon G\times\mathfrak{C}^{\omega\setminus n}\to Y\times\mathfrak{C}^{\omega\setminus n}$ defined by $F(x)=(f(x\restriction n),x\restriction\omega\setminus n)$. Then, F is continuous and separately nowhere constant function defined on a dense G_δ subset of $\prod_{i\in\omega} X_i$ where $X_i=\mathfrak{C}$ for every $i\in\omega\setminus n$. Thus, by case $\kappa=\omega$, there are $\{P_i\in \mathrm{Perf}(X_i)\colon i<\omega\}$ such that F is one-to-one on $\prod_{i<\omega} P_i\subset G\times\mathfrak{C}^{\omega\setminus n}$. This implies that f is one-to-one on $\prod_{i<n} P_i\subset G$.

If $\kappa>\omega$ then the result is trivial because in this case f cannot be simultaneously continuous and separately nowhere constant on a dense G_δ subset G of X. To see this first notice that G contains a subset of the form $H\times\prod_{i\in\kappa\setminus A} X_i$, where A is a countable subset of κ and H is a dense G_δ subset of $\prod_{i\in A} X_i$. This is the case, since every dense open subset U of X contains a dense open subset in similar form: a union of a maximal pairwise disjoint family of basic open subsets of U. So, we can assume that G is in this form. Pick an $x_0\in G$. By the continuity of f at x_0, for every $n<\omega$ there exists an open subset U_n in X containing x_0 such that the diameter of $f\left[G\cap U_n\right]$ is less than 2^{-n}. By the definition of the product topology each U_n contains a set of the form $\prod_{i\in F_n}\{x_0(i)\}\times\prod_{i\in\kappa\setminus F_n} X_i$, where each $F_n\subset\kappa$ is finite. Put $F=A\cup\bigcup_{n<\omega} F_n$ and notice that $Z=\prod_{i\in F}\{x_0(i)\}\times\prod_{i\in\kappa\setminus F} X_i\subset G\cap\bigcap_{n<\omega} U_n$. So, $f[Z]$ has the diameter equal to 0, that is, f is constant on Z. But this contradicts the fact that f is separately nowhere constant on G, since for $\xi\in\kappa\setminus F$ set Z contains the section $\{x\in X\colon x\restriction\kappa\setminus\{\xi\}=x_0\restriction\kappa\setminus\{\xi\}\}$. $\square$

3. Cube and prism densities

We start with recalling the definition of the family $\mathbb{P}_\alpha$ of iterated perfect sets in $\mathfrak{C}^\alpha$, where $0<\alpha<\omega_1$. So, let Φ_α be the family of all continuous injections $f\colon\mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ such that for every $\beta<\alpha$

$$f\restriction\restriction\beta\stackrel{\text{def}}{=}\{\langle x\restriction\beta,y\restriction\beta\rangle\colon\langle x,y\rangle\in f\}$$

is a one-to-one function from $\mathfrak{C}^\beta$ into $\mathfrak{C}^\beta$. For example, if $\alpha=3=\{0,1,2\}$ then $f\in\Phi_\alpha$ provided there exist continuous functions $f_0\colon\mathfrak{C}\to\mathfrak{C}$, $f_1\colon\mathfrak{C}^2\to\mathfrak{C}$, and $f_2\colon\mathfrak{C}^3\to\mathfrak{C}$ such that $f(x_0,x_1,x_2)=\langle f_0(x_0),f_1(x_0,x_1),f_2(x_0,x_1,x_2)\rangle$ for all $x_0,x_1,x_2\in\mathfrak{C}$ and maps f_0, $\langle f_0,f_1\rangle$, and f are one-to-one. Note that Φ_α is closed under compositions and that for every $0 < \beta < \alpha$ if $f \in \Phi_\alpha$ then $f \restriction\restriction \beta \in \Phi_\beta$. We define

$$\mathbb{P}_\alpha = \{\text{range}(f) \colon f \in \Phi_\alpha\}.$$

The following properties can be easily deduced from these definitions. (For (D) see [4, (3.13)].) Here π_β is the projection from $\mathfrak{C}^\alpha$, for some $\alpha \geq \beta$, onto the first β coordinates, that is, $\pi_\beta(x) = x \restriction \beta$.

(A) Every perfect cube in $\mathfrak{C}^\alpha$ belongs to $\mathbb{P}_\alpha$.
(B) If $P \in \mathbb{P}_{\alpha+1}$ and $x \in \pi_\alpha[P]$ then $|(\{x\} \times \mathfrak{C}) \cap P| = \mathfrak{c}$.
(C) If $0 < \beta < \alpha$ and $P \in \mathbb{P}_\alpha$ then $\pi_\beta[P] \in \mathbb{P}_\beta$.
(D) If $0 < \beta < \alpha$ then $Q = \{x \in P \colon \pi_\beta(x) \in R\} \in \mathbb{P}_\alpha$ for every $P \in \mathbb{P}_\alpha$ and $R \in \mathbb{P}_\beta$ with $R \subset \pi_\beta[P]$.

Lemma 6. *For $0 < n < \omega$ and any continuous $f \colon \mathfrak{C}^n \to Y$ there exist a basic clopen subset $U = \prod_{i<n} U_i$ of $\mathfrak{C}^n$, an $A \subset n$, and, if $A \neq n$, a dense G_δ subset G of $W = \prod_{i \in n \setminus A} U_i$ such that*

- *$f \restriction U$ does not depend on the variables x_j for $j \in A$;*
- *if $A \neq n$ then $f \restriction U$, considered as a function of the variables x_i with $i \in n \setminus A$, is separately nowhere constant on G.*

Proof. We proceed by induction on n. If $n = 1$ the lemma is true by the definition of nowhere constant function. Suppose the lemma is true for n and let $f \colon \mathfrak{C}^{n+1} \to Y$ be continuous. Denote by $\mathcal{B}_0$ a countable basis for the topology on $\mathfrak{C}$ consisting of non-empty clopen sets. For each $i \leq n$ and $V \in \mathcal{B}_0$ consider the closed set $S_i(V) = \left\{\vec{x} \in \mathfrak{C}^{n+1\setminus\{i\}} \colon f \restriction \{\vec{x}\} \times V \text{ is constant}\right\}$.

First assume that for every $i \leq n$ and $V \in \mathcal{B}_0$ the set $S_i(V)$ has empty interior. Then each set $H_i = \mathfrak{C}^{\{i\}} \times \left(\mathfrak{C}^{n+1\setminus\{i\}} \setminus \bigcup\{S_i(V) \colon B \in \mathcal{B}_0\}\right)$ is comeager in $\mathfrak{C}^{n+1}$. So, we can apply Lemma 2 to $X = \mathfrak{C}$, $T = n+1$, $\mathcal{K} = \{n+1 \setminus \{i\} \colon i \leq n\}$, and $H = \bigcap_{i \leq n} H_i$ to find a comeager set $G \subset H$ such that for every $x \in G$ and $i \leq n$ the set $G_{x \restriction n+1\setminus\{i\}} = \left\{y \in \mathfrak{C}^{\{i\}} \colon (x \restriction n+1 \setminus \{i\}) \cup y \in G\right\}$ is comeager in $\mathfrak{C}^{\{i\}}$. Note also that this last property implies that $f \restriction G$ is separately nowhere-constant, since for every $x \in G$ its restriction $x \restriction n+1 \setminus \{i\}$ does not belong to $\bigcup\{S_i(V) \colon B \in \mathcal{B}_0\}$. Thus, in this case the lemma is satisfied with $U = \mathfrak{C}^{n+1}$, $A = \emptyset$, and the above chosen G.

So, assume that there exist $i \leq n$ and $V_i \in \mathcal{B}_0$ such that the set $S_i(V_i)$ has non-empty interior. Let $V^* \subset S_i(V_i)$ be a non-empty basic clopen subset of $\mathfrak{C}^{n+1\setminus\{i\}}$. Then $V^* = \prod_{j \neq i} V_j$, where $V_j \subset \mathfrak{C}$ is a basic clopen set for every $j \neq i$. If $V = \prod_{j \leq n} V_j$ then V is homeomorphic to $\mathfrak{C}^{n+1}$, $f \restriction V$ does not depend on the variable x_i, and we can consider $f \restriction V$ as a function g from V^* to Y. By applying our inductive hypothesis to g we can find a basic clopen subset $U^* = \prod_{j\neq i} U_j$ of V^*, a set $A^* \subset n+1\setminus\{i\}$, and, if $A^* \neq n+1\setminus\{i\}$, a dense G_δ subset G of $W = \prod_{n+1\setminus A} U_j$, where $A = A^*\cup\{i\}$, satisfying the lemma for g. But then $U = \prod_{j\leq n} U_j$, where $U_i = V_i$, and the sets A and G are as desired. □

Here is the main example of the paper.

Example 7. For every $0<\alpha<\omega_1$ there is a family $\mathcal{G}_\alpha \subset \mathrm{Perf}(\mathfrak{C}^\alpha)$ such that

(a) $\mathcal{G}_\alpha$ does not contain any iterated perfect set, that is, $\mathcal{G}_\alpha\cap\mathbb{P}_\alpha=\emptyset$;
(b) $\mathcal{G}_\alpha$ is γ-prism dense for every $0<\gamma<\alpha$;
(c) $\mathcal{G}_\alpha$ is n-cube dense for every $0<n<\min\{\alpha,\omega\}$;
(d) if $\mathcal{G}^*\in[\mathcal{G}_\alpha]^{<\mathfrak{c}}$ then $|\mathfrak{C}^\alpha\setminus\bigcup\mathcal{G}^*|=\mathfrak{c}$.

Proof. For $\xi<\alpha$ let $\mathcal{K}_\xi = \{P\in \mathrm{Perf}(\mathfrak{C}^\alpha)\colon \pi_\xi\restriction\pi_{\xi+1}[P] \text{ is one-to-one}\}$, where in case of $\xi=0$ we understand the definition of $\mathcal{K}_0$ as $\{P\colon \pi_1[P] \text{ is a singleton}\}$. It is worth to note that $\{P\in\mathrm{Perf}(\mathfrak{C}^\alpha)\colon \pi_\xi\restriction P$ is one-to-one$\}\subset\mathcal{K}_\xi$. We define $\mathcal{G}_\alpha=\bigcup_{\xi<\alpha}\mathcal{K}_\xi$.

To see (a) take $P\in\mathbb{P}_\alpha$ and $\xi<\alpha$. We need to show that $P\notin\mathcal{K}_\xi$. But by (C) we have $\pi_{\xi+1}[P]\in\mathbb{P}_{\xi+1}$ and then (B) shows that $P\notin\mathcal{K}_\xi$.

We prove (b) by induction on α. Clearly it holds for $\alpha=1$. So, assume that for some $1<\alpha<\omega_1$ condition (b) holds for every non-zero $\alpha'<\alpha$. To see that (b) holds for α fix $0<\gamma<\alpha$ and a continuous injection $f\colon\mathfrak{C}^\gamma\to\mathfrak{C}^\alpha$. We need to find a $Q\in\mathbb{P}_\gamma$ for which $f[Q]\in\mathcal{G}_\alpha$.

Let $g=\pi_\gamma\circ f$. By [4, Lemma 3.2.2] there exist $P\in\mathbb{P}_\gamma$ and $0<\beta\leq\gamma$ such that $h=g\circ\pi_\beta^{-1}$ is a function on $\pi_\beta[P]$ (i.e., $g\restriction P$ does not depend on coordinates $\delta\geq\beta$) and this function is either one-to-one or constant. If h is constant then $\pi_1[f[P]]$ is a singleton and $f[P]\in\mathcal{K}_0\subset\mathcal{G}_\alpha$. So, assume that h is one-to-one.

If $\beta=\gamma$ then $g=\pi_\gamma\circ f$ is one-to-one on P and so π_γ is one-to-one on $f[P]$. Then $f[P]\in\mathcal{K}_\gamma\subset\mathcal{G}_\alpha$. So, assume that $\beta<\gamma$. Then h is an injection from $\pi_\beta[P]\in\mathbb{P}_\beta$ into $\mathfrak{C}^\gamma$. Let $\varphi\in\Phi_\beta$ witness $\pi_\beta[P]\in\mathbb{P}_\beta$. Then φ maps $\mathfrak{C}^\beta$ onto $\pi_\beta[P]$. Since $h\circ\varphi\colon\mathfrak{C}^\beta\to\mathfrak{C}^\gamma$ is a continuous injection, by the inductive hypothesis used for $\alpha'=\gamma$ there exists an $E\in\mathbb{P}_\beta$ such that $Z=h\circ\varphi[E]\in\mathcal{G}_\gamma$, that is, there exists a $\xi<\gamma$ for which $\pi_\xi\restriction\pi_{\xi+1}[Z]$ is one-to-one.

Next notice that $R=\varphi[E]\in\mathbb{P}_\beta$, since Φ_β is closed under the composition, and $R\subset\pi_\beta[P]$. So, by (D), $Q=\{x\in P\colon \pi_\beta(x)\in R\}\in\mathbb{P}_\alpha$. Moreover,

$$Z=h\circ\varphi[E]=h[R]=(g\circ\pi_\beta^{-1})[\pi_\beta[Q]]=g[P]=\pi_\gamma[f[Q]]$$

and so $\pi_{\xi+1}[Z]=\pi_{\xi+1}[\pi_\gamma[f[Q]]]=\pi_{\xi+1}[f[Q]]$. Thus, $\pi_\xi\restriction\pi_{\xi+1}[f[Q]]$ is one-to-one and so $f[Q]\in\mathcal{K}_\xi\subset\mathcal{G}_\alpha$.

To show (c) we will prove by induction on $0 < n < \omega$ the statement

for every $0 < \alpha < \omega_1$ if $n < \alpha$ then $\mathcal{G}_\alpha$ is n-cube dense.

So, take $0 < n < \omega$ and assume that the statement holds for all non-zero $k < n$. Take an $\alpha > n$. To prove that $\mathcal{G}_\alpha$ is n-cube dense fix a continuous injection $f\colon \mathfrak{C}^n \to \mathfrak{C}^\alpha$. Then $\pi_n \circ f\colon \mathfrak{C}^n \to \mathfrak{C}^n$ is continuous. Apply Lemma 6 to $\pi_n \circ f$ to find $U = \prod_{i<n} U_i \subset \mathfrak{C}^n$, $A \subset n$, and G satisfying the lemma.

If $A = n$ then $\pi_n[f[U]] = (\pi_n \circ f)[U]$ is a singleton and $f[U] \in \mathcal{K}_0 \subset \mathcal{G}_\alpha$.

If $A = \emptyset$ then $\pi_n \circ f \restriction G$ is continuous separately nowhere constant. So, by Theorem 1, there exist perfect sets $\{P_i \subset U_i\colon i < n\}$ such that $(\pi_n \circ f) \restriction \prod_{i<n} P_i$ is one-to-one. Then, $\pi_n \restriction f\left[\prod_{i<n} P_i\right]$ is one-to-one and so $f\left[\prod_{i<n} P_i\right] \in \mathcal{K}_n \subset \mathcal{G}_\alpha$.

So, assume that $\emptyset \neq A \neq n$ and let $k = |n \setminus A|$. Then $0 < k < n$. Since $(\pi_n \circ f) \restriction U$ does not depend on the variables x_j for $j \in A$, it can be considered as a function g on $W = \prod_{i \in n\setminus A} U_i$. Moreover, $g \restriction G$ is separately nowhere constant. Thus, by Theorem 1, we can find a perfect cube $P = \prod_{i\in n\setminus A} P_i \subset G \subset \prod_{i\in n\setminus A} U_i$ on which g is one-to-one. Thus, g is a continuous injection from P, which can be identified with $\mathfrak{C}^k$, into $\mathfrak{C}^n$. Since, by the inductive assumption, $\mathcal{G}_n$ is k-cube dense, there exists a perfect cube $C = \prod_{i\in n\setminus A} C_i \subset \prod_{i\in n\setminus A} P_i$ such that $g[C] \in \mathcal{G}_n$. Let $C_i = U_i$ for $i \in A$. Then $Q = \prod_{i<n} C_i \subset \prod_{i<n} U_i$ is a perfect cube and $\pi_n[f[Q]] = (\pi_n \circ f)[Q] = g[C] \in \mathcal{G}_n$. So, there exists a $\xi < n$ such that π_ξ is one-to-one on $\pi_{\xi+1}[\pi_n[f[Q]]] = \pi_{\xi+1}[f[Q]]$, So, $f[Q] \in \mathcal{K}_\xi \subset \mathcal{G}_\alpha$.

Now, to argue for (d) fix a $\mathcal{G}^* \in [\mathcal{G}_\alpha]^{<\mathfrak{c}}$. We need to show that $|\mathfrak{C}^\alpha \setminus \bigcup \mathcal{G}^*| = \mathfrak{c}$. For $\xi < \alpha$ let $\mathcal{G}^*_\xi = \mathcal{G}^* \cap \mathcal{K}_\xi$. By induction on $\xi < \alpha$ choose

$$x(\xi) \in \mathfrak{C} \setminus \{z(\xi)\colon z \in \mathcal{G}^*_\xi \;\&\; z(\eta) = x(\eta) \text{ for every } \eta < \xi\}.$$

Note that at each step we have less than continuum many restricted points since for every $z \in \mathcal{K}_\xi$ the set $\{z(\xi)\colon z(\eta) = x(\eta) \text{ for every } \eta < \xi\}$ may have at most one element. It is easy to see that $x = \langle x(\xi)\colon \xi < \alpha\rangle \in \mathfrak{C}^\alpha \setminus \bigcup_{\xi<\alpha} \mathcal{G}^*_\xi = \mathfrak{C}^\alpha \setminus \mathcal{G}^*$. To finish the proof it is enough to notice that the value of $x(0)$ can be chosen in continuum many ways, so indeed $|\mathfrak{C}^\alpha \setminus \bigcup \mathcal{G}^*| = \mathfrak{c}$. $\square$

To transport the above example into an arbitrary Polish space we will use the following simple fact.

Fact 8. *Let h be a homeomorphic embedding of a Polish space Y into a Polish space X, let $\mathcal{F} \subset \mathrm{Perf}(Y)$, and put $\mathcal{F}^* = \{h[F]\colon F \in \mathcal{F}\} \cup$ Perf$(X \setminus h[Y[)$. Then for every $1 \le \alpha \le \omega_1$ the following conditions are equivalent.*

(a) *$\mathcal{F}$ is α-cube (α-prism) dense in Y.*
(b) *$\mathcal{F}^*$ is α-cube (α-prism) dense in X.*

Proof. "(a)$\Longrightarrow$(b)" Let $f\colon \mathfrak{C}^\alpha \to X$ be injective and continuous. Since $h[Y]$ is a G_δ-set in X we can apply [2, Claim 3.2] (see also [4, Claim 1.21.5]) to find a perfect cube $C \subset \mathfrak{C}^\alpha$ such that either $f[C] \subset h[Y]$ or $f[C] \cap h[Y] = \emptyset$.

If $f[C] \cap h[Y] = \emptyset$ then $f[C] \in \mathcal{F}^*$ and we are done. If $f[C] \subset h[Y]$ then $h^{-1} \circ f\colon C \to Y$ is a continuous injection. Identifying C with $\mathfrak{C}^\alpha$ and using to $h^{-1} \circ f$ the α-cube (α-prism) density of $\mathcal{F}$ in Y we can find a $C' \subset C$ such that C' is a perfect cube (belongs to $\mathbb{P}_\alpha$) and $F = (h^{-1} \circ f)[C'] \in \mathcal{F}$. So $f[C'] = h[F] \in \mathcal{F}^*$. The family $\mathcal{F}^*$ is as desired.

The other implication is easy. □

Corollary 9. *For every $1 < \alpha < \omega_1$ and every Polish space X there exists a family $\mathcal{F}_\alpha \subset \mathrm{Perf}(X)$ such that: $\mathcal{F}_\alpha$ is not α-prism dense; $\mathcal{F}_\alpha$ is β-prism dense for every $0 < \beta < \alpha$; $\mathcal{F}_\alpha$ is n-cube dense for every $0 < n < \min\{\alpha, \omega\}$; $|X \setminus \bigcup \mathcal{F}_\alpha| = \mathfrak{c}$.*

Proof. First note that it is enough to find such an $\mathcal{F}_\alpha$ for $X = \mathfrak{C}^\alpha$. Indeed, if $\mathcal{F}$ is such a family and h, X is an arbitrary Polish space, and h is an embedding from $\mathfrak{C}^\alpha$ into X, then the family $\mathcal{F}^*$ from Fact 8 is as desired.

Thus, it is enough to notice that the family $\mathcal{G}_\alpha$ from Example 7 is not α-prism dense. But this is the case since for the identity function f on $\mathfrak{C}^\alpha$ there is no $P \in \mathbb{P}$ for which $f[P] = P \in \mathcal{G}_\alpha$. □

Proof of Theorem 3. Use Corollary 9 with $\alpha \geq \omega$. □

Proof of Theorem 2. The vertical implications, that α-cube density implies α-prism density, follows from the fact (A), that every perfect cube in $\mathfrak{C}^\alpha$ is also in $\mathbb{P}_\alpha$. For $0 < \beta < \alpha < \omega_1$ the implications "$\mathrm{C}_\alpha \Longrightarrow \mathrm{C}_\beta$" and "$\mathrm{P}_\alpha \Longrightarrow \mathrm{P}_\beta$" are obvious. P_2 implies C_2 since 1-prism density is just perfect set density ($\mathbb{P}_1 = \mathrm{Perf}(\mathfrak{C}^1)$, as Φ_1 consists just of autohomeomorphisms of $\mathfrak{C}^1$) and so it implies 1-cube density.

To see that for $\omega < \alpha < \omega_1$ we have "$\mathrm{C}_{\omega+1} \Longrightarrow \mathrm{C}_\alpha$" it is enough to notice that any ω-cube dense family is also β-cube dense for any $\omega \leq \beta < \omega_1$. This is the case since the coordinatewise homeomorphism between $\mathfrak{C}^\omega$ and $\mathfrak{C}^\beta$ preserves perfect cubes.

The fact that no other horizontal implication can be reversed is justified by the family $\mathcal{F}_\alpha$ from Corollary 9 for different values of α. Indeed, $\mathcal{F}_\alpha$ clearly justifies "$\mathrm{P}_\alpha \not\Longrightarrow \mathrm{P}_\gamma$" for any $1 < \alpha < \gamma$ since it satisfies P_α but not P_γ as it is not α-prism dense. If $1 < m < n \leq \omega$ then $\mathcal{F}_m$ also witness "$\mathrm{C}_m \not\Longrightarrow \mathrm{C}_n$" since it satisfies C_m but not C_n, since it cannot be m-cube dense without being m-prism dense.

The fact that none of the vertical implications "$\mathrm{C}_\alpha \Longrightarrow \mathrm{P}_\alpha$", for $2 < \alpha < \omega_1$, can be reversed is justified by any family which is α-prism dense for every α but is not 2-cube dense. There are many such families. For example, this is the case for the family $\mathcal{F}$ of all linearly independent (over $\mathbb{Q}$) subsets of $\mathbb{R}$. It is shown in [3] (see also [4, Corollary 5.1.2]) that this $\mathcal{F}$ is α-prism dense for every $0 < \alpha < \omega_1$. On the other hand it is not 2-cube dense, as shown by the following function f. (See [3, Remark 5.2] or [4, Remark 5.1.4].) Let P_1 and P_2 be disjoint perfect subsets of $\mathbb{R}$ such that $P_1 \cup P_1$ is linearly independent over $\mathbb{Q}$. Let $f\colon P_1 \times P_2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = x_1 + x_2$. Identifying P_1 and P_2 with $\mathfrak{C}$ we think about f as defined on $\mathfrak{C}^2$. It is easy to see that if each of the sets $Q_1 \subset P_1$ and $Q_2 \subset P_2$ has at least two elements then $f[Q_1 \times Q_2]$ is linearly dependent.

Another such example is a family $\mathcal{F}$ of all $P \in \mathrm{Perf}(\mathfrak{C}^2)$ such that the projection on one of the coordinates is one-to-one. It follows quite easily from [4, Lemma 3.2.2] that $\mathcal{F}$ is α-prism dense for every α. (See e.g. [4, Proposition 4.1.3].) It is not 2-cube dense since for the identity function $f\colon \mathfrak{C}^2 \to \mathfrak{C}^2$ there is no perfect cube C for which $f[P] \in \mathcal{F}$. □

4. Final remarks

It is also worth to notice that we have the following implications.

Proposition 10. *If $\beta + 1 \le \alpha < \omega_1$ then every α-prism (α-cube) dense family is also $(\beta+1)$-prism ($(\beta+1)$-cube) dense. In particular, if $0 < m < n \le \omega$ then every n-cube dense family is also m-cube dense.*

Proof. Let $g\colon \mathfrak{C}^{\alpha\setminus\beta} \to \mathfrak{C}$ be a homeomorphism, and let $h\colon \mathfrak{C}^{\alpha} \to \mathfrak{C}^{\beta+1}$ be defined by $h(x)(\xi) = x(\xi)$ for every $\xi < \beta$ and $h(x)(\beta) = g(x \restriction \alpha \setminus \beta)$. It is easy to see that h is a homeomorphism and that if $P \subset \mathfrak{C}^{\alpha}$ is a perfect cube (belongs to $\mathbb{P}_\alpha$) then $h[P]$ is a perfect cube (belongs to $\mathbb{P}_{\beta+1}$).

Now, let $\mathcal{F} \subset \mathrm{Perf}(X)$ be α-prism α-cube) dense in X. To see that $\mathcal{F}$ is $(\beta+1)$-prism ($(\beta+1)$-cube) dense take a continuous injection $f\colon \mathfrak{C}^{\beta+1} \to X$. Then $f \circ h\colon \mathfrak{C}^{\alpha} \to X$ is also a continuous injection. Since $\mathcal{F}$ is α-prism (α-cube) dense, there exists a $P \subset \mathfrak{C}^{\alpha}$ such that P belongs to $\mathbb{P}_\alpha$ (is a perfect cube) and $f[h[P]] = (f \circ h)[P] \in \mathcal{F}$. But $h[P]$ belongs to $\mathbb{P}_{\beta+1}$ (is a perfect cube), so $\mathcal{F}$ is $(\beta+1)$-prism ($(\beta+1)$-cube) dense. □

We do not know if, in general, for a limit ordinal $\lambda < \omega_1$ the $(\lambda+1)$-prism density implies λ-prism density.

The next example shows that Lemma 6 fails, in a strong way, for functions defined on infinite product.

Example 11. There exists a continuous function $f\colon \mathfrak{C}^\omega \to \mathfrak{C}^\omega$ such that for every perfect cube P there is an $n<\omega$ such that $f \restriction P$ is one-to-one on some section of n-th variable, and is constant on some other sections of the same variable.

Proof. For $n<\omega$ let $f_n\colon \mathfrak{C}^2 \to \mathfrak{C}$ be defined by $f_n(x,y)(i) = y(n)\cdot x(i)$. Clearly f_n is continuous. Moreover, if $y(n)=1$ then $f_n(\cdot,y)$ is the identity function, so it is one-to-one; if $y(n)=0$ then $f_n(\cdot,y)$ is constant equal to 0.

For $\langle x_n\colon n<\omega\rangle \in \mathfrak{C}^\omega$ define $f(\langle x_n\colon n<\omega\rangle) = \langle f_n(x_{n+1},x_0)\colon n<\omega\rangle$. Then f is clearly continuous. Consider f restricted to a perfect cube $P = \prod_{n<\omega} P_n$. Let $a,b\in P_0$ be distinct and let $n<\omega$ be such that $a(n)\neq b(n)$. Assume that $a(n)=0$ and let $z\in \mathfrak{C}^{\omega\setminus\{n+1\}}$. Look at $f\restriction P$ on a section given by z and note that: if $z(0)=a$ then $f\restriction P$ is constant on this section; if $z(0)=b$ then $f\restriction P$ is one-to-one on this section. □

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