

The authors formulate and explore a new axiom of set theory, CPA, the Covering Property Axiom. CPA is consistent with the usual ZFC axioms; indeed, it is true in the iterated Sacks model and actually captures the combinatorial core of this model. A plethora of results known to be true in the Sacks model easily follow from CPA. Replacing iterated forcing arguments with deductions from CPA simplifies proofs, provides deeper insight, and leads to new results. One may say that CPA is similar in nature to Martin's axiom, as both capture the essence of the models of ZFC in which they hold.

The exposition is self-contained, and there are natural applications to real analysis and topology. Researchers who use set theory in their work will find much of interest in this book.

Krzysztof Ciesielski is Professor of Mathematics at West Virginia University.

Janusz Pawlikowski is Professor of Mathematics at Wrocław University.

Krzysztof Ciesielski Janusz Pawlikowski
West Virginia University *Wrocław University*

The Covering Property Axiom, CPA
A Combinatorial Core of the Iterated
Perfect Set Model



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