

Differentiability versus continuity: monstrous examples and extension theorems

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Based on papers/notes with links 121, 129, S21, S22
see also <http://www.math.wvu.edu/~kcies/publications.html>.

Summer Symposium in Real Analysis XLI
The College of Wooster, Wooster, OH, June 18–24, 2017.

Preamble: new, 2017, results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f: \mathbb{R} \rightarrow \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism f of a compact perfect $X \subset \mathbb{R}$ with $f' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$:

- there is perfect $P \subset \mathbb{R}$ s.t. $f \upharpoonright P$ is Lipschitz;
- there is C^1 map $g: \mathbb{R} \rightarrow \mathbb{R}$ with $f \cap g$ uncountable.

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, then any differentiable $f: Q \rightarrow \mathbb{R}$ has differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$. This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

No familiarity
with Lebesgue measure
is needed to follow the proofs
sketched in this talk

Outline

- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: Differentiable Monster
- 3 Differentiability from continuity: differentiable restrictions
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$
- 5 Differentiable extensions: Jarník and Whitney theorems

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Positive, well known results

Clearly any differentiable function $F: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

True question: *To what extent $f = F'$ must be continuous?*

Theorem (Classic)

The set C_f of points of continuity of $f = F'$ is a dense G_δ -set.

Pr. F' is Baire 1: $F'(x) = \lim_n F_n(x)$, $F_n(x) = \frac{f(x+1/n) - f(x)}{1/n}$ cont.

Density of C_f : C_f contains a residual set

$\bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \text{int}(\{x \in \mathbb{R} : |F_k(x) - F_m(x)| \leq 1/n \text{ for all } m, k \geq N\})$.

C_f is a G_δ -set:

$C_f = \bigcap_{n=1}^{\infty} \bigcup_{\delta > 0} \{x : |f(s) - f(t)| < 1/n \text{ for all } s, t \in (x - \delta, x + \delta)\}$.

This is it:

Theorem (Classic)

For any dense G_δ -set G there is a derivative f with $C_f = G$.

Pr. Enough to prove for maps on $(0, 1)$.

Let $(0, 1) \setminus G = \bigcup_{n=1}^{\infty} P_n$, each P_n closed nowhere dense.

Define $f_n: (0, 1) \rightarrow \mathbb{R}$ as 0 on P_n and as

$$f_n(x) = \frac{(x-a)^2(x-b)^2}{(b-a)^2} \left[\sin\left(\frac{1}{x-a}\right) + \sin\left(\frac{1}{x-b}\right) \right]$$

on any component (a, b) of $(0, 1) \setminus P_n$.

Then $f = \sum_{n=1}^{\infty} 3^{-n} f_n$ is as needed. □

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Differentiable monster

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; and many others)

There is differentiable $f: \mathbb{R} \rightarrow \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x: f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

A variant of Pompeiu function, of 1907

Fix $r \in (0, 1)$ and $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ such that $|q_i| \leq i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

- (i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is continuous, "differentiable," strictly increasing, onto \mathbb{R} , with $g'(q) = \infty$ for all $q \in \mathbb{Q}$.
- (ii) $h = g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is everywhere differentiable with $h' \geq 0$ and $Z = \{x \in \mathbb{R} : h'(x) = 0\}$ being a dense G_δ -set.
- (iii) $Z^c = \mathbb{R} \setminus Z$ is also dense in \mathbb{R} .

Pr. (i) Continuity follows from $|g(x)| \leq \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3(x - q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y) - \psi_i(x)}{y - x} \leq 6\psi_i'(x)$.

(ii) and (iii) easily follow from (i).



New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h: \mathbb{R} \rightarrow \mathbb{R}$ with $Z = \{x \in \mathbb{R}: h'(x) = 0\}$ being a dense G_δ -set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. $h' > 0$ on D .

Any t in residual $G = \bigcap_{d \in D} ((-d + Z) \cap (d - Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D : $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

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How much differentiability continuous map must have

None?

Example (Weierstrass Monster)

There exists continuous $F: \mathbb{R} \rightarrow \mathbb{R}$ differentiable at no point.

Proof.

Put $F(x) = \sum_{n=0}^{\infty} 4^n \min\{|x - \frac{k}{8^n}| : k \in \mathbb{Z}\}$.

Continuous at each $x \in \mathbb{R}$, since for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$
 $|f(y) - f(x)| \leq |\sum_{i=0}^n 4^i f_i(y) - \sum_{i=0}^n 4^i f_i(x)| + \frac{1}{2^n}$.

Not differentiable at $x \in \mathbb{R}$, since for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with
 $x \in [\frac{k}{8^n}, \frac{k+1}{8^n}]$ there exists a $y_n \in \{\frac{k}{8^n}, \frac{k+1}{8^n}\} \setminus \{x\}$ such that

$$\left| \frac{f(x) - f(y_n)}{x - y_n} \right| \geq \left| \frac{f(\frac{k+1}{8^n}) - f(\frac{k}{8^n})}{\frac{k+1}{8^n} - \frac{k}{8^n}} \right| = \left| \sum_{i=0}^{n-1} \frac{f_i(\frac{k+1}{8^n}) - f_i(\frac{k}{8^n})}{\frac{k+1}{8^n} - \frac{k}{8^n}} 4^i \right| =$$

$$\left| \sum_{i=0}^{n-1} \pm 4^i \right| \geq \frac{2}{3} 4^{n-1}. \quad \square$$

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \upharpoonright Q$ is differentiable.

Remark

There are continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2): [0, 1] \rightarrow [0, 1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and \mathcal{I} -approximately differentiable. So it is as in the remark. □

New proof of differentiable restriction theorem

Goal: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont, then $f \upharpoonright Q$ is diff. for some perfect Q .

Theorem (With new simple proof, by KC)

For every continuous increasing $f: [a, b] \rightarrow \mathbb{R}$ there is perfect P such that $f \upharpoonright P$ is Lipschitz.

Proof based on the following results, due to Riesz:

Lemma (Rising sun lemma, proof is an easy exercise)

If $g: [a, b] \rightarrow \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every component (c, d) of $\{x \in [a, b]: g(x) < g(y) \text{ for some } y \in (x, b]\}$.

Fact (Proved by induction)

Let $a < b$ and \mathcal{J} be a family of open intervals with $\bigcup \mathcal{J} \subset (a, b)$.

- (i) If $[\alpha, \beta] \subset \bigcup \mathcal{J}$, then $\sum_{I \in \mathcal{J}} \ell(I) > \beta - \alpha$.
- (ii) If $I \in \mathcal{J}$ are pairwise disjoint, then $\sum_{I \in \mathcal{J}} \ell(I) \leq b - a$.

Proof of Lipschitz part

Goal: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont \nearrow , then $f \upharpoonright P$ is Lipschitz for a perfect P .

Have: If $g: [a, b] \rightarrow \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every comp. (c, d) of $\{x \in [a, b]: g(x) < g(y) \text{ for some } y \in (x, b)\}$.

Sketch of proof. Fix $L > \frac{f(b)-f(a)}{b-a}$, put $g(t) = f(t) - Lt$, and

$$U = \{x \in [a, b]: g(y) > g(x) \text{ for some } y \in (x, b)\}.$$

f is Lipschitz on $P = [\bar{a}, b] \setminus U$ with constant L , where

$\bar{a} = \sup\{x: [a, x) \subset U\}$. Fix $X = \{x_n: n \in \mathbb{N}\}$. Need $P \setminus X \neq \emptyset$.

If $\mathcal{J} =$ open components of U , then $\ell(f[\mathcal{J}]) \geq L\ell(\mathcal{J})$ for $\mathcal{J} \in \mathcal{J}$.

By Fact (ii), $\sum_{\mathcal{J} \in \mathcal{J}} \ell(f[\mathcal{J}]) \leq f(b) - f(\bar{a})$. So,

$$\sum_{\mathcal{J} \in \mathcal{J}} \ell(\mathcal{J}) \leq \frac{1}{L} \sum_{\mathcal{J} \in \mathcal{J}} \ell(f[\mathcal{J}]) \leq \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}, \text{ and by Fact (i),}$$

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly \mathcal{J} .

End of proof of differentiable restriction theorem

Goal: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont, then $f \upharpoonright Q$ is diff. for some perfect Q .

Have: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont \nearrow , then $f \upharpoonright P$ is Lipschitz for a perfect P .

Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P : proved above for somewhere monotone f ; otherwise f is constant on some perfect set.

For function $f \upharpoonright P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \upharpoonright Q$ is uniformly continuous. Then Q is as needed. □

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Are differentiable $f: P \rightarrow \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good?

Not at all!

Example (Ciesielski & Jasinski 2016 example simplified)

There exists differentiable auto-homeomorphism f of a compact perfect subset \mathfrak{X} of the Cantor ternary set \mathfrak{C} such that $f' \equiv 0$.

Counterintuitive, as f is shrinking at every $x \in \mathfrak{X}$

($|f(x) - f(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact \mathfrak{X} **onto** itself.

$f = h \circ \sigma \circ h^{-1}$, where $h: 2^\omega \rightarrow \mathbb{R}$ is embedding and $\sigma: 2^\omega \rightarrow 2^\omega$ is the “add one and carry” adding machine:

$$\sigma(\mathbf{s}) = \begin{cases} \langle 0, 0, 0, \dots \rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots \rangle & \text{if } s_k = 0 \text{ and } s_i = 1 \text{ for } i < k. \end{cases}$$

Definition of $h: 2^\omega \rightarrow \mathbb{R}$ with $f' \equiv 0$ for $f = h \circ \sigma \circ h^{-1}$

$$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s \upharpoonright n)}, \text{ where } N(s \upharpoonright 0) = 1 \text{ and}$$

$$N(s \upharpoonright n) = \sum_{i < n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \text{ for } n > 0.$$

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega : s_i \neq t_i\}$, then

$$3^{-(n+1)N(s \upharpoonright n)} \stackrel{(i)}{\leq} |h(s) - h(t)| \stackrel{(ii)}{\leq} 3 \cdot 3^{-(n+1)N(s \upharpoonright n)}.$$

Proof. For $h_n(s) = \sum_{k < n} 2s_k 3^{-(k+1)N(s \upharpoonright k)}$ we have

$$h_n(s) + 2s_n 3^{-(n+1)N(s \upharpoonright n)} \leq h(s) \leq h_n(s) + (2s_n + 1)3^{-(n+1)N(s \upharpoonright n)},$$

as $h(s) = h_n(s) + 2s_n 3^{-(n+1)N(s \upharpoonright n)} + 2 \sum_{k > n} 3^{-(k+1)N(s \upharpoonright k)}$ and

$$0 \leq 2 \sum_{k > n} 3^{-(k+1)N(s \upharpoonright k)} \leq 2 \sum_{i=1}^{\infty} 3^{-[(n+1)N(s \upharpoonright n) + i]} = 3^{-(n+1)N(s \upharpoonright n)}$$

So, (i): $|h(s) - h(t)| \geq 3^{-(n+1)N(s \upharpoonright n)}$;

and (ii): $|h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s \upharpoonright n)}$.

Proof of $f' \equiv 0$ for $f = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)} N(s \upharpoonright n)$,

Have: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega : s_i \neq t_i\}$, then

$$3^{-(n+1)} N(s \upharpoonright n) \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)} N(s \upharpoonright n).$$

Also (a): $\forall s \in 2^\omega \exists k < \omega N(\sigma(s) \upharpoonright n) = N(s \upharpoonright n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \dots, s_{n-2}, s_{n-1}, \dots \rangle = \langle 1, \dots, 1, 0, \dots \rangle$.

Proof of $f' \equiv 0$.

To see $f'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t.

$0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega : s_i \neq t_i\} > k$. Then,

$$\frac{|f(h(s)) - f(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)} N(\sigma(s) \upharpoonright n)}{3^{-(n+1)} N(s \upharpoonright n)} = 3 \cdot 3^{-(n+1)}.$$

So $f'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ . \square

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Notation

For $J = (a, a + h)$ let $I_J = [a + h/3, a + 2h/3]$, middle third of J .

For closed $Q \subset \mathbb{R}$ and $f: Q \rightarrow \mathbb{R}$ let

$\hat{Q} = Q \cup \bigcup \{I_J : J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}$,

$\bar{f}: \mathbb{R} \rightarrow \mathbb{R}$ — “the” linear interpolation of f , $\hat{f} = \bar{f} \upharpoonright \hat{Q}$.

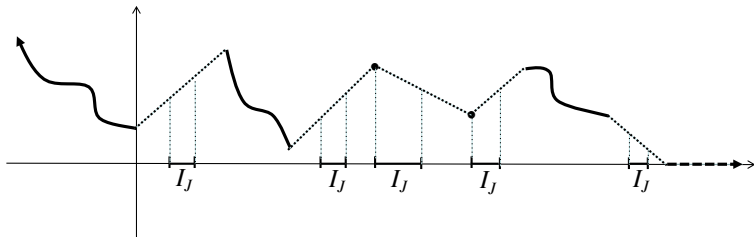


Figure: The linear interpolation \bar{f} of f , represented by thick curves.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, then any differentiable $f: Q \rightarrow \mathbb{R}$ has differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$. This F is C^1 iff such extension exists iff \hat{f} is continuously differentiable.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ there is C^1 map $g: \mathbb{R} \rightarrow \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \upharpoonright Q$ is uniformly continuous.

It is easy to see that \hat{h} is continuously differentiable for such h .

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f: Q \rightarrow \mathbb{R}$ has differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f: Q \rightarrow \mathbb{R}$ is differentiable, then the unilateral derivatives of \bar{f} exist and are finite everywhere.

So, \bar{f} is differentiable at any $x \in \mathbb{R}$ which is not in the set E of the end-points of connected components of $\mathbb{R} \setminus \tilde{Q}$.

Proof.

Clear, unless $x \in \tilde{Q} \setminus E$.

Estimate is clear, unless $y \in (a, b)$ for a component of \tilde{Q}^c , when

$\frac{|\bar{f}(y) - \bar{f}(x)|}{y-x}$ is between $\frac{|f(a) - f(x)|}{a-x}$ and $\frac{|f(b) - f(x)|}{b-x}$, so

$$\left| f'(x) - \frac{|\bar{f}(y) - \bar{f}(x)|}{y-x} \right| \leq \max \left\{ \left| f'(x) - \frac{|f(a) - f(x)|}{a-x} \right|, \left| f'(x) - \frac{|f(b) - f(x)|}{b-x} \right| \right\}$$

□



From linear interpolation to differentiable extension

Goal: Differentiable $f: Q \rightarrow \mathbb{R}$ has differ. extension $F: \mathbb{R} \rightarrow \mathbb{R}$.

Have: Linear interpolation almost works.

Solution: Small modification of linear interpolation, $F = \bar{f} + g$:

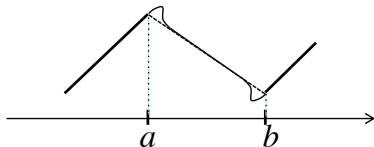


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

g must satisfy: $g \upharpoonright [a, b]$ is C^1 ,

$$(*) \quad D^+g(a) = \tilde{f}'(a) - \frac{\tilde{f}(b) - \tilde{f}(a)}{b-a} \quad \text{and} \quad D^-g(b) = \tilde{f}'(b) - \frac{\tilde{f}(b) - \tilde{f}(a)}{b-a}.$$

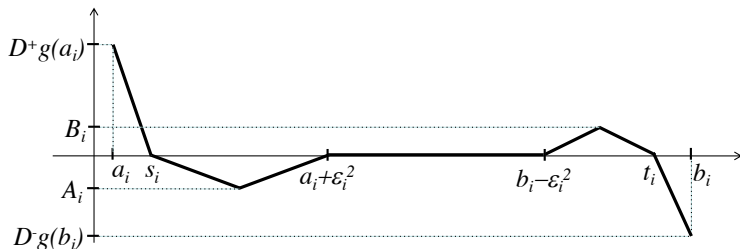
Format of a modifier g

$\{(a_i, b_i) : 1 \leq i \leq \kappa\}$ enumeration of components of Q^c ; need

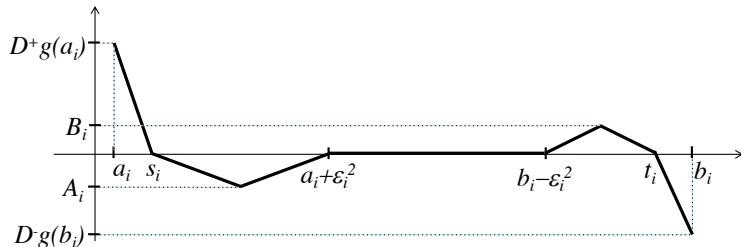
$$(*) \quad D^+g(a_i) = f'(a_i) - \frac{f(b_i) - f(a_i)}{b_i - a_i}, \quad D^-g(b_i) = f'(b_i) - \frac{f(b_i) - f(a_i)}{b_i - a_i}$$

and g is **small enough** to preserve differentiability at Q .

$g \upharpoonright [a_i, b_i]$ defined as $g(x) = \int_{a_i}^x h_i(r) dr$, where h_i is as in figure



How to ensure that g works?



For $\ell_i = \min\{1, b_i - a_i\}$ let $\varepsilon_i \in (0, 3^{-i}\ell_i)$ be such that

- (a) $\left| f'(a_i) - \frac{f(x) - f(a_i)}{x - a_i} \right| < 3^{-i}$ for every $x \in Q \cap [a_i - \varepsilon_i, a_i]$;
- (b) $\left| f'(b_i) - \frac{f(x) - f(b_i)}{x - b_i} \right| < 3^{-i}$ for every $x \in Q \cap (b_i, b_i + \varepsilon_i]$.

Pick $s_i \in (a_i, a_i + \varepsilon_i^2)$ s.t. $\int_{a_i}^{s_i} |h_i(r)| dr = \frac{1}{2}|h_i(a_i)|(s_i - a_i) < \varepsilon_i^2$;

A_i so that $\int_{a_i}^{a_i + \varepsilon_i^2} h_i(r) dr = 0$; similarly for t_i and B_i . **Prove it works.**

That is all!

Thank you for your attention!