Separately continuous functions on $\mathbb{R}^n$: from Cauchy mistake, through Lebesgue contributions, to contemporary results

Krzysztof Chris Ciesielski[a,b,*]

[a]Department of Mathematics, West Virginia University,

Morgantown, WV 26506-6310

[b]Department of Radiology, MIPG, University of Pennsylvania, Blockley Hall – 4th Floor,

423 Guardian Drive, Philadelphia, PA 19104-6021

Abstract

We will briefly describe a history of study of functions from $\mathbb{R}^n$ to $\mathbb{R}$ that are continuous with respect to each variable or, more generally, when restricted to different surfaces of co-dimension 1. In particular, we present a new elementary example of a discontinuous function of n-variables whose restriction to any hyperplane is continuous.

It is well known, that a mapping $g = \langle g_i \rangle_i \colon \mathbb{R} \to \mathbb{R}^n$ is continuous if, and only if, its coordinate functions $g_i \colon \mathbb{R} \to \mathbb{R}$ are continuous. For the functions $f \colon \mathbb{R}^n \to \mathbb{R}$ of n-variables, there is no analogous characterization. More precisely, a notion for f that is analogous to that of continuity of the coordinates g_i of g, is the *separate continuity* of f, defined as a continuity of f with respect of each variable (i.e., $f(x_0, \ldots, x_{n-1})$ is continuous with respect to each x_i, other x_j's being fixed.) Cauchy, in his 1821 book *Cours d'analyse* [Ca], incorrectly claimed that separate continuity of $f \colon \mathbb{R}^2 \to \mathbb{R}$ indeed characterizes the continuity of f, that is, that separate continuity of f implies its continuity. This mistake was not corrected for several decades.

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E-mail: KCies@math.wvu.edu; web page: http://www.math.wvu.edu/~kcies.

A first counterexample to this Cauchy's claim, due to E. Heine (see [Pi]), appeared in the 1870 calculus text of J. Thomae [Th]. A simpler and better known example of a discontinuous separately continuous function comes from 1884 treatise on calculus by Genocchi and Peano [GP], defined as

$$g_2(x_0, x_1) = \frac{(x_0)(x_1)^2}{(x_0)^2 + (x_1)^4} \tag{1}$$

for $\langle x_0, x_1\rangle \neq \langle 0, 0\rangle$ and $g_2(0, 0) = 0$. Actually, g_2 has even stronger property than separate continuity: it has a continuous restriction to every straight line, a property known as *linear continuity*. Clearly, g_2 is discontinuous when restricted to a parabola $x_0 = (x_1)^2$. Thus, linear continuity of $f : \mathbb{R}^2 \to \mathbb{R}$, a property stronger than separate continuity, still does no imply continuity.

The early contributions to the theory of separately continuous functions came also from Volterra (cited by Baire, see [BN]), Baire (1899, citation in [Ke] and [BN]), and Hahn (1919, citation in [Ke]). A question on the regularity of separately continuous functions, initiated by Baire 1899 (for the functions of two variables), was settled in 1905 by Lebesgue (see [Le] or [Ke]), who proved that every separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is of $(n-1)^{\text{st}}$ Baire class and that the Baire class cannot be lowered in this result.

Lebesgue, in 1905 paper [Le], and Scheefer, in 1890 [Sc], gave also examples showing that the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ along all analytic paths (i.e., isometric copies of graphs of analytic functions) does not implies continuity of f. The study of continuities of functions defined in terms of paths, was culminated by the following 1955 result of A. Rosenthal [Ro].

Theorem 1 *If $f : \mathbb{R}^n \to \mathbb{R}$ has continuous restriction to any isometric copy of a graph of $\mathcal{C}^1$ (i.e., continuously differentiable) function $h : \mathbb{R} \to \mathbb{R}$, then f is continuous. On the other hand, there are discontinuous functions $g : \mathbb{R}^2 \to \mathbb{R}$ with continuous restrictions to any isometric copy of a graph of any twice differentiable function.*

Yet a different direction of study concerning separately continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ concerns the structure of the sets $D(f)$ of their points of discontinuity. Here, the ultimate result is the following 1943 theorem of Kershner [Ke].

Theorem 2 *For any set $D \subset \mathbb{R}^n$ there is a separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with $D(f) = D$ if, and only if, D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

However, at present, there is no elegant characterization of the sets $D(f)$ for the class of linearly continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$. The partial results here are as follows.

Theorem 3 (Slobodnik [Sl]) *If $D \subset \mathbb{R}^n$ is the set of discontinuity points of some linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, then D admits a representation $D = \bigcup_{i=1}^{\infty} D_i$, where each D_i is isometric to the graph of a Lipschitz function $\phi_i\colon K_i \to \mathbb{R}$ with K_i being a compact nowhere dense subset of $\mathbb{R}^{n-1}$.*

The above theorem describes a necessary structure of sets $D(f)$. Some sufficient conditions for a set to be in form $D(f)$ are given below.

Theorem 4 (Ciesielski-Glatzer 2012 [CG2]) *If D is a restriction of a convex function $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to a compact nowhere dense subset of $\mathbb{R}^{n-1}$, then $D = D(f)$ for some linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$.*

For $n = 2$ the results remains true when ϕ is required to be continuously twice differentiable, instead of being convex.

Interestingly, the sets $D(f)$ can be big, even if the family of curves is as big as it make sense, according to Rosenthal's Theorem 1.

Theorem 5 (Ciesielski-Glatzer 2012 [CG1]) *There exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ whose restriction to any isometric copy of a graph of twice differentiable function is continuous and such that the set $D(f)$ is perfect of positive one dimensional Hausdorff measure.*

Lets go back to function g_2 from (1). It is very easy to transform this example to a discontinuous linearly function $g_n\colon \mathbb{R}^n \to \mathbb{R}$ for $n > 2$: just put $g_n(x_0, x_1, \ldots, x_{n-1}) = g_2(x_0, x_1)$. For $n = 2$, the function g_n has also continuous restriction to every hyperplane in $\mathbb{R}^n$ of co-dimension 1. However, g_n does not have this property for $n > 2$. Nevertheless, below we show that, for every $n \geq 2$, there exists a simple example of a function $f_n\colon \mathbb{R}^n \to \mathbb{R}$ with such property, that is, discontinuous but with continuous restrictions $f_n \restriction H$ to any hyperplane H in $\mathbb{R}^n$ of co-dimension 1. Although the definition of f_n follows general format of g_2, the exponents needed to be chosen very carefully, to make the presented proof simple.

For the origin θ in $\mathbb{R}^n$ we put $f_n(\theta) = 0$ and for $\vec{x} = \langle x_0, x_1, \ldots, x_{n-1}\rangle \neq \theta$ we define

$$f_n(\vec{x}) = \frac{x_0 (x_0)^{4^0} (x_1)^{4^1} \cdots (x_{n-1})^{4^{n-1}}}{(x_0)^{2^n} + (x_1)^{2^{n+1}} + \cdots + (x_{n-1})^{2^{n+(n-1)}}} = \frac{x_0 \prod_{i=0}^{n-1} (x_i)^{2^{2i}}}{\sum_{i=0}^{n-1} (x_i)^{2^{n+i}}}.$$

For example, for $n=4$ this gives $f_4(x_0,x_1,x_2,x_3)=\frac{(x_0)(x_0)(x_1)^4(x_2)^{16}(x_3)^{64}}{(x_0)^{16}+(x_1)^{32}+(x_2)^{64}+(x_3)^{128}}$, while $f_2(x_0,x_1)=\frac{(x_0)(x_0)(x_1)^4}{(x_0)^4+(x_1)^8}=\frac{(x_0)^2(x_1)^4}{(x_0)^4+(x_1)^8}$, a simple modification of the Genocchi-Peano example: $f_2(x_0,x_1)=g_2((x_0)^2,(x_1)^2)$.

The function f_n is discontinuous on a path $\vec{p}(t)=\langle t^{2^n},t^{2^{n-1}},\ldots,t^{2^2},t^{2^1}\rangle$ since, for $t\neq 0$,

$$f_n(\vec{p}(t))=\frac{t^{2^n}t^{2^n}t^{2^{n+1}}\cdots t^{2^{n+(n-1)}}}{t^{2^{2n}}+t^{2^{2n}}+\cdots+t^{2^{2n}}}=\frac{t^{2^{2n}}}{nt^{2^{2n}}}=\frac{1}{n}$$

is constant, while $f_n(\vec{p}(0))=f_n(\theta)=0\neq\frac{1}{n}$.

Next, fix a hyperplane H in $\mathbb{R}^n$ of co-dimension 1. We need to show that $f_n\restriction H$ is continuous. Clearly, this is so if H does not contain the origin. So, assume that $\theta\in H$. Then, H can be represented by an equation

$$x_k=\sum_{i=0}^{k-1}a_ix_i$$

for some $k\in\{0,\ldots,n-1\}$. Since for $k=0$ this equation reduces to $x_0=0$, making $f_n\restriction H$ constant, we can assume that $k\geq 1$. We need to show that $f_n\restriction H$ is continuous at θ. Let $d=\sum_{i=0}^{n-1}(x_i)^{2^{n+i}}$ be the denominator of f_n. Since $2^{2n-1}=2^{n-1}+2^{n-1}+2^{n-2}+\cdots+2^{n-1+(n-1)}$, we have

$$(f_n(\vec{x}))^{2^{2n-1}}=\frac{\left(x_0\prod_{i=0}^{n-1}(x_i)^{2^{2i}}\right)^{2^{2n-1}}}{d^{2^{n-1}}\prod_{i=0}^{n-1}d^{2^{n-1+i}}}=\left(\frac{x_0}{d}\right)^{2^{2n-1}}\prod_{i=0}^{n-1}\left(\frac{(x_i)^{2^{n+i}}}{d}\right)^{2^{n-1+i}}.$$

Since $\frac{(x_i)^{2^{n+i}}}{d}=\frac{(x_i)^{2^{n+i}}}{\sum_{i=0}^{n-1}(x_i)^{2^{n+i}}}\leq 1$, this leads to $0\leq(f_n(\vec{x}))^{2^{2n-1}}\leq\frac{(x_k)^{2^{n+k}}}{d}$. It remains to show that $\frac{(x_k)^{2^{n+k}}}{d}$ converges to 0 when $\vec{x}\in H$ approaches to θ. But

$$(x_k)^{2^{n+k}}=\left(\sum_{i=0}^{k-1}a_ix_i\right)^{2^{n+k}}\leq\sum_{i=0}^{k-1}|a_i|^{2^{n+k}}\left(|x_i|^{2^{n+i}}\right)^{2^{k-i}}\leq\sum_{i=0}^{k-1}|a_i|^{2^{n+k}}d^{2^{k-i}}.$$

Hence $\frac{(x_k)^{2^{n+k}}}{d}\leq\sum_{i=0}^{k-1}|a_i|^{2^{n+k}}d^{2^{k-i}-1}$ with the right hand side converges to 0 as $d\to 0$, since $2^{k-i}-1>0$. This completes the proof.

References

[BN] J.C. Breckenridge and T. Nishiura, "Partial Continuity, Quasi-Continuity, and Baire Spaces," *Bull. Inst. Math. Acad. Sinica* **4** (1976), 191–203.

[Ca] L.A. Cauchy, *Cours D'analyse de l'Ecole Royale Polytechnique*, Paris, 1821.

[CG1] K. Ciesielski and T. Glatzer, "Functions continuous on twice differentiable curves, discontinuous on large sets," *Real Anal. Exchange*, in print.

[CG2] K.C. Ciesielski and T. Glatzer, "Sets of Discontinuities of Linearly Continuous Functions," submitted.

[GP] A. Genocchi and G. Peano, *Calcolo differentiale e principii di Calcolo*, Torino, 1884.

[Ke] R. Kershner, "The Continuity of Functions of Many Variables," *Trans. Amer. Math. Soc.* **53** (1943), 83–100.

[Le] H. Lebesgue, "Sur les fonctions representable analytiquement," *J. Math. Pure Appl.* **6** (1905), 139–216.

[Pi] Z. Piotrowski, "The genesis of separate versus joint continuity," *Tatra Mountains Math. Publ.* **8** (1996), 113–126.

[Ro] A. Rosenthal, 11On the Continuity of Functions of Several Variables," *Math. Zeitschr.* **63** (1955), 31-38.

[Sc] L. Scheefer, Theorie der Maxima und Minima einer Function von zwei Variabeln, *Math. Ann.* **35** (1890), 541 -567.

[Sl] S.G. Slobodnik, "An Expanding System of Linearly Closed Sets," *Mat. Zametki* **19** (1976), 67-84; English translation *Math Notes* **19** (1976), 39-48.

[Th] J. Thomae, *Abriss einer Theorie der complexen Funktionen*, Halle, 1873. (First edition published in 1870.)