Four Continuities on the Real Line

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The two most commonly used topologies on $\mathbf{R}$ are the ordinary topology and the density topology. The ordinary topology is, of course, more widely known, and has the open intervals as its basic sets. The density topology, on the other hand, appears naturally in deeper studies of the structure of measurable sets. Its open sets are all measurable sets $S \subset \mathbf{R}$ which have the property that if $x \in S$, then x is a Lebesgue density point of S; i.e.,

$$\lim_{h \to 0+} \frac{|(x-h, x+h) \cap S|}{2h} = 1, \qquad \forall x \in S,$$

where $|A|$ is the Lebesgue measure of A. We write $\mathbf{R}_O$ and $\mathbf{R}_D$ for the real line with the ordinary topology and the density topology, respectively.

If $f : \mathbf{R} \to \mathbf{R}$ is a function, then there are four ways f can be continuous, using the ordinary topology and the density topology on the domain and range. We denote these by $\mathbf{C}_{OO}$, $\mathbf{C}_{OD}$, $\mathbf{C}_{DO}$ and $\mathbf{C}_{DD}$, where, for example,

$$\mathbf{C}_{DO} = \{f : \mathbf{R}_D \to \mathbf{R}_O : f \text{ is continuous}\}.$$

Two of these classes have been studied extensively. $\mathbf{C}_{OO}$ is the collection of functions which are continuous in the sense of ordinary calculus and $\mathbf{C}_{DO}$ is the collection of all *approximately continuous* functions [**GNN**]. Less well-known are the *density continuous* functions, $\mathbf{C}_{DD}$.

Approximate continuity appears naturally from studies of differentiation and integration. In particular, it is known that if $f \in \mathbf{C}_{DO}$, then f is in Baire class 1 and has the Darboux property. Like $\mathbf{C}_{OO}$, $\mathbf{C}_{DO}$ is vector space and is closed under pointwise multiplication. The bounded functions in $\mathbf{C}_{DO}$ are a Banach space when given the uniform norm [**B**]. It follows from Theorem 1, given below, that $\mathbf{C}_{OD}$ also shares these properties in a trivial way.

On the other hand, $\mathbf{C}_{DD}$ is not closed under addition, pointwise multiplication [**CL**] or uniform convergence [**O**].

Since the density topology is a refinement of the ordinary topology, the following containments are obvious.

$$\mathbf{C}_{OD} \subset \mathbf{C}_{OO} \subset \mathbf{C}_{DO} \supset \mathbf{C}_{DD} \tag{1}$$

We investigate these relationships further. The relationship between $\mathbf{C}_{OD}$ and $\mathbf{C}_{OO}$ is fully understood.

THEOREM 1. *$f \in \mathbf{C}_{OD}$ if and only if f is constant.*

PROOF: It is easy to see that any constant function is in $\mathbf{C}_{OD}$. To prove the other direction, let $f \in \mathbf{C}_{OD}$ and suppose there is a compact interval I on which f is not constant. Let

$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x).$$

Choose any strictly increasing sequence y_n from (m, M) such that $y_n \to M$. Since $\{y_n\}$ is a set with zero measure, it is closed in $\mathbf{R}_D$. Let w_n be any sequence such that $w_n \in f^{-1}(y_n) \neq \emptyset$ and let x_n be a convergent subsequence of w_n with limit x_0. Because $f \in \mathbf{C}_{OO}$, (by (1)) it must be the case that $f(x_0) = M$. But, $M \notin f^{-1}(\{y_n\})$, so $f^{-1}(\{y_n\})$ is not closed in $\mathbf{R}_O$. This is a contradiction of the assumption that $f \in \mathbf{C}_{OD}$. Therefore, no such interval I exists and f must be constant.

As a result of this theorem, we see that the first containment in (1) is proper.

EXAMPLE 1. *There is an $f \in \mathbf{C}_{OO}$ which is not in $\mathbf{C}_{DD}$.*

PROOF: Choose any sequence y_n which strictly decreases to 0 and let $[a_n, b_n]$ be a sequence of intervals such that

$$0 < b_{n+1} < a_n < b_n, \qquad n = 1, 2, \dots,$$

$a_n \to 0$ and

$$\lim_{h \to 0+} \frac{|\bigcup_{n \geq 0} [a_n, b_n] \cap (0, h)|}{h} = 1. \tag{2}$$

Define $f(x) = y_n$ when $x \in [a_n, b_n]$, $f(x) = 0$ when $x \leq 0$ and let f be strictly monotone and continuous on each interval $[b_{n+1}, a_n]$.

It is easy to see that f is continuous and that

$$f^{-1}(\{y_n\}) = \bigcup_{n \geq 1} [a_n, b_n].$$

The set $\{y_n\}$ is closed in $\mathbf{R}_D$. From (2) and the observation that $0 \notin \bigcup_{n \geq 1} [a_n, b_n]$, it is clear that $\bigcup_{n \geq 1} [a_n, b_n]$ is not closed in $\mathbf{R}_D$. This shows that $f \notin \mathbf{C}_{DD}$.

The following corollary is an easy generalization of Example 1, and is used in the proof of Theorem 2.

Corollary 1. *Given any $f \in \mathbf{C}_{OO}$, $\varepsilon > 0$ and $(a,b) \subset \mathbf{R}$, there exists a $g \in \mathbf{C}_{OO} \setminus \mathbf{C}_{DD}$ such that $g = f$ on $\mathbf{R} \setminus (a,b)$ and $\|f - g\|_\infty < \epsilon$.*

Example 2. *There is an $f \in \mathbf{C}_{DD}$ such that $f \notin \mathbf{C}_{OO}$.*

Proof: Let the sequence of intervals $[a_n, b_n]$ converge monotonically to 0 as in Example 1, but this time choose them so that

$$\lim_{h \to 0+} \frac{|\bigcup_{n \geq 0}[a_n, b_n] \cap (0,h)|}{h} = 0.$$

If $c_n = (a_n + b_n)/2$, then define $f(x) = 0$ when $x \notin \bigcup_{n\geq 0}(a_n, b_n)$, $f(x) = 1$ when $x = c_n$ for some n and let f be continuous and linear on $[a_n, c_n]$ and $[c_n, b_n]$. It is evident that f is not continuous at $x = 0$, but $f \in \mathbf{C}_{DD}$.

The following corollary will be useful in the proof of Theorem 2. It is an obvious consequence of Example 2.

Corollary 2. *Given any $f \in \mathbf{C}_{OO}$, $\varepsilon > 0$ and $(a,b) \subset \mathbf{R}$, there exists a $g \in \mathbf{C}_{DD} \setminus \mathbf{C}_{OO}$ such that $g = f$ on $\mathbf{R} \setminus (a,b)$ and $\|f - g\|_\infty < \epsilon$.*

In light of Theorem 1 and the two examples, we see that (1) can be refined into the form

$$\mathbf{C}_{OD} \subset \mathbf{C}_{OO} \cap \mathbf{C}_{DD} \subset \mathbf{C}_{OO} \subset \mathbf{C}_{DO} \supset \mathbf{C}_{DD} \supset \mathbf{C}_{OO} \cap \mathbf{C}_{DD}.$$

All of these containments are proper and are the only containments between these classes, More can be said.

Theorem 2. *Suppose that $\mathbf{C}$ and $\mathbf{C}'$ are two distinct elements from*

$$\{\mathbf{C}_{OO}, \mathbf{C}_{OD}, \mathbf{C}_{DO}, \mathbf{C}_{DD}, \mathbf{C}_{OO} \cap \mathbf{C}_{DD}\}$$

such that $\mathbf{C}' \subset \mathbf{C}$. If $\mathbf{C}$ and $\mathbf{C}'$ are given the topology generated by uniform convergence, then $\mathbf{C}'$ is a first category subset of $\mathbf{C}$.

Proof: It is easy to see that the piecewise linear functions from $\mathbf{C}_{OO}$ are in $\mathbf{C}_{DD}$. Since the piecewise linear functions are dense in $\mathbf{C}_{OO}$, Theorem 1 implies that $\mathbf{C}_{OD}$ is nowhere dense in $\mathbf{C}_{OO} \cap \mathbf{C}_{DD}$ and consequently nowhere dense in the three other classes.

Corollary 1 and the fact that $\mathbf{C}_{OO}$ is closed in $\mathbf{C}_{DO}$ shows that $\mathbf{C}_{OO}$ is nowhere dense in $\mathbf{C}_{DO}$.

For each open interval I, define C_I to be the set of all functions $f \in \mathbf{C}_{DD}$ such that $g|_I$ is continuous in the ordinary sense. C_I is a closed subspace of $\mathbf{C}_{DD}$. It is known [**CLO**] that if $f \in \mathbf{C}_{DD}$, then there is an open interval I such that $f \in C_I$. Letting I_n be the sequence of all open intervals with rational endpoints, we see

$$\mathbf{C}_{DD} = \bigcup_{n=1}^{\infty} C_{I_n}.$$

Using Corollaries 1 and 2 with $(a,b) = I_n$, we conclude that $C_{I_n} \cap \mathbf{C}_{OO}$ is nowhere dense in both $\mathbf{C}_{DD}$ and $\mathbf{C}_{OO}$. Hence, $\mathbf{C}_{DD} \cap \mathbf{C}_{OO}$ is a first category subset of both $\mathbf{C}_{DD}$ and $\mathbf{C}_{OO}$. Finally, again using Corollary 1, we can also conclude that C_{I_n} is nowhere dense in $\mathbf{C}_{DD}$; i.e., $\mathbf{C}_{DD}$ is first category in itself. This implies that $\mathbf{C}_{DD}$ is first category in $\mathbf{C}_{DO}$.

It remains open whether $\mathbf{C}_{OO} \cap \mathbf{C}_{DD}$ is first category in itself.

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