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SETS OF DISCONTINUITIES FOR FUNCTIONS CONTINUOUS ON FLATS

Abstract

For families $\mathcal{F}$ of flats (i.e., affine subspaces) of $\mathbb{R}^n$, we investigate the classes of $\mathcal{F}$-continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$, whose restrictions $f \restriction F$ are continuous for every $F \in \mathcal{F}$. If $\mathcal{F}_k$ is the class of all k-dimensional flats, then $\mathcal{F}_1$-continuity is known as linear continuity; if $\mathcal{F}_k^+$ stands for all $F \in \mathcal{F}_k$ parallel to vector subspaces spanned by coordinate vectors, then $\mathcal{F}_1^+$-continuous maps are the separately continuous functions, that is, those which are continuous in each variable separately. For the classes $\mathcal{F} = \mathcal{F}_k^+$, we give a full characterization of the collections $\mathcal{D}(\mathcal{F})$ of the sets of points of discontinuity of $\mathcal{F}$-continuous functions. We provide the structural results on the families $\mathcal{D}(\mathcal{F}_k)$ and give a full characterization of the collections $\mathcal{D}(\mathcal{F}_k)$ in the case when $k \geq n/2$. In particular, our characterization of the class $\mathcal{D}(\mathcal{F}_1)$ for $\mathbb{R}^2$ solves a 60 year old problem of Kronrod.

1 Introduction

When teaching undergraduates about the pitfalls inherent to the limit concept in $\mathbb{R}^2$ as opposed to its safer 1-dimensional counterpart, it is standard practice to mention a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such as

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } \langle x,y\rangle \neq \langle 0,0\rangle, \\ 0 & \text{if } \langle x,y\rangle = \langle 0,0\rangle. \end{cases} \tag{1}$$

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It has the property that, for any fixed $x_0, y_0 \in \mathbb{R}$, the maps $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are continuous. Yet, the function f from (1) is discontinuous at the origin. In other words, this function f is continuous in each variable separately but discontinuous.

Another easily introduced pathology is highlighted by the following function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } \langle x,y\rangle \neq \langle 0,0\rangle, \\ 0 & \text{if } \langle x,y\rangle = \langle 0,0\rangle. \end{cases} \tag{2}$$

The restriction of this function to any line in the plane is continuous. However, this function, too, is discontinuous at the origin. In general, we say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with the property that its restriction to any line parallel to a coordinate axis is continuous (such as function (1)) is a *separately continuous* function. Similarly, we say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with the property that its restriction to *any* line is continuous (such as function (2)) is a *linearly continuous* function. Separately and linearly continuous functions which are discontinuous have been studied throughout the last two centuries, ever since Cauchy, in 1827, made an erroneous statement precluding their existence, see e.g. [2]. For a full historical background on the research on these functions, see [1, 2, 3, 4, 14, 15], or the forthcoming monograph [16].

Both classes of generalized continuities described above are the particular examples of what we call *restriction continuities*. More specifically, for any family $\mathcal{S}$ of subsets of $\mathbb{R}^n$ (or, more generally, of a topological space X). we say that a function f from $\mathbb{R}^n$ (or X) to $\mathbb{R}$ is $\mathcal{S}$-continuous provided its restriction $f\restriction S$ is continuous for every $S \in \mathcal{S}$. Hence, when $\mathcal{S}$ is the set of all lines in $\mathbb{R}^n$ parallel to the coordinate axes, $\mathcal{S}$-continuous functions coincide with separately continuous functions, while for the family $\mathcal{S}$ of all lines in $\mathbb{R}^n$, the notion of $\mathcal{S}$-continuity coincides with that of linear continuity. It is worth noticing the following.

Proposition 1.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the collections of subsets of $\mathbb{R}^n$ such that for every $S \in \mathcal{S}_1$ there exists a $T \in \mathcal{S}_2$ with $S \subset T$. Then any $\mathcal{S}_2$-continuous function is $\mathcal{S}_1$-continuous.*

The $\mathcal{S}$-continuity for $\mathcal{S}$ consisting of the graphs of functions were investigated in [3] and [5]. In this paper we study the $\mathcal{S}$-continuous functions on $\mathbb{R}^n$, when $\mathcal{S}$ is a class of k-flats ($k \le n$), that is, of k-dimensional affine subspaces of $\mathbb{R}^n$. Recall that a k-flat is a subset of $\mathbb{R}^n$ isometric to $\mathbb{R}^k$. (In particular, for $k = 0$ we will identify $\mathbb{R}^k$ with $\{0\}$.) We use a term *right k-flat* for any k-flat parallel to a vector subspace of $\mathbb{R}^n$ spanned by k-many coordinate vectors. The family of all k-flats is denoted as $\mathcal{F}_k$, while $\mathcal{F}_k^+$ will stand for the family of all right k-flat. In this notation, the class of $\mathcal{F}_1^+$-continuous ($\mathcal{F}_1$-continuous)

$f\colon \mathbb{R}^n \to \mathbb{R}$ is identical with the class of separately (linearly, respectively) continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. Clearly, $\mathcal{F}_n = \mathcal{F}_n^+ = \{\mathbb{R}^n\}$, so $\mathcal{F}_n^+$-continuity is the standard continuity, while $\mathcal{F}_0^+ = \mathcal{F}_0$ is the class of all singletons, so that every function $f\colon \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{F}_0$-continuous. So, we concentrate on the cases when $0 < k < n$.

Proposition 1.2. *For every $n \geq 2$,*

$$\begin{array}{ccccccc} \mathcal{F}_n\text{-continuity} & \Longrightarrow & \mathcal{F}_{n-1}\text{-continuity} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1\text{-continuity} \\ \Updownarrow & & \Downarrow & & & & \Downarrow \\ \mathcal{F}_n^+\text{-continuity} & \Longrightarrow & \mathcal{F}_{n-1}^+\text{-continuity} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1^+\text{-continuity} \end{array}$$

None of the implications can be reversed, as follows from Corollary 2.6.

The $\mathcal{F}_k^+$-continuous functions on $\mathbb{R}^n$ are fairly well documented in the literature. They have been studied in connection with the theory of Sobolev spaces, see e.g. [1]. It is perhaps instructive to think of $\mathcal{F}_k^+$-continuous functions as those which are continuous when looked at in any k variables separately.

One of the most fruitful and most frequently pursued avenues of research on the separately and linearly continuous functions is the study of the size and structure of the sets of discontinuities of these functions (see e.g., [9], [1], [17], and [18]). Here we pursue this line of research for the classes of $\mathcal{F}_k^+$-continuous and $\mathcal{F}_k$-continuous functions.

For a function f, the set of points at which f is discontinuous, *the discontinuity set of f*, is denoted as $D(f)$. We also use the notation

$$\mathcal{D}_{k,n}^+ = \{D(f) \subset \mathbb{R}^n\colon \text{ function } f\colon \mathbb{R}^n \to \mathbb{R} \text{ is } \mathcal{F}_k^+\text{-continuous}\},$$

$$\mathcal{D}_{k,n} = \{D(f) \subset \mathbb{R}^n\colon \text{ function } f\colon \mathbb{R}^n \to \mathbb{R} \text{ is } \mathcal{F}_k\text{-continuous}\}.$$

Only the obvious inclusions, implied by Proposition 1.2, are between these families, as shown in Corollary 2.6.

The structure of sets of discontinuity of separately and linearly continuous continuous functions has been described by several theorems presented below. The first of these is a theorem of Kershner [9], which completely characterizes the family $\mathcal{D}_{1,n}^+$.

Theorem 1.1. *For any set $D \subset \mathbb{R}^n$, there exists a separately continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $D = D(f)$ if, and only if, D is an F_σ-set and all of the orthogonal projections of D onto the coordinate $(n-1)$-dimensional hyperplanes are of first category.*

In 1944, the year after Kershner's result was published, A.S. Kronrod attended a course by Luzin (at this point, a rare opportunity), which prompted him to begin a research program in Moscow aimed at developing a geometric theory of real functions of two variables [10]. As a part of this program he asked for a characterization of the sets $\mathcal{D}_{1,2}$. The first progress in this direction was Slobodnik's result [18], stated below, which gives a necessary condition for a set to be in $\mathcal{D}_{1,n}$.

Theorem 1.2. *If $D \in \mathcal{D}_{1,n}$, then $D = \bigcup_{i<\omega} D_i$, where each D_i is isometric to the graph of a Lipschitz function $f_i \colon H_i \to \mathbb{R}$, with H_i being a nowhere dense subset of a hyperplane in $\mathbb{R}^n$. Furthermore, the orthogonal projections of D onto any hyperplane is of the first category and if $p \in D_i$, then the central projection through p of D_i onto any hyperplane in $\mathbb{R}^n \setminus \{p\}$ is nowhere dense in that hyperplane.*

The following natural "lower bound" counterpart to Slobodnik's theorem, found by the authors [4], gives a sufficient condition for a set to be in $\mathcal{D}_{1,n}$ and shows that Slobodnik's result is close to a characterization.

Theorem 1.3. *If $D \subset \mathbb{R}^n$ can be written as $D = \bigcup_{i=1}^{\infty} D_i$, where each D_i is isometric to the graph of a convex function $f_i \colon H_i \to \mathbb{R}$ and H_i is a closed nowhere dense subset of $\mathbb{R}^{n-1}$, then there exists a linearly continuous function $f \colon \mathbb{R}^n \to \mathbb{R}$, such that $D(f) = D$.*

However, in the same paper [4, proposition 2.5], the authors showed that there are nowhere dense compact subsets of graphs of Lipschitz functions which are not in $\mathcal{D}_{1,n}$.

The paper is organized as follows. In Section 2 we state our main theorems and derive from them some corollaries. In Section 3 we overview the tools to be used in the following proofs. Sections 4, 5, and 6 are devoted, respectively, to the proofs of our three main Theorems: 2.1, 2.2, and 2.3. We close, in Section 7, by discussing a few related results.

2 The results

2.1 Characterization of the family $\mathcal{D}^+_{k,n}$

The format of our characterization theorem of $\mathcal{D}^+_{k,n}$ was motivated by the following result [1], proved for the case of $n > 1$:

> An F_σ-set D is in $\mathcal{D}^+_{n-1,n}$ if, and only if, $D \subset D_1 \times D_2 \times \ldots \times D_n$, where each D_i is a first category subset of $\mathbb{R}$.

Theorem 2.1. *A set $D \subset \mathbb{R}^n$ is in $\mathcal{D}^+_{k,n}$ if, and only if, D is an F_σ-set whose orthogonal projection on any right $(n-k)$-flat is of first category.*

The proof of Theorem 2.1 will be presented in Section 4. (Note, that for $k = 0$ the result is trivially satisfied, since in this case each of the condition can hold only for $D = \emptyset$.) Below, we discuss some of its consequences.

Proposition 2.1. *For any $0 < k \leq n$, there exists a $D \in \mathcal{D}^+_{k-1,n} \setminus \mathcal{D}^+_{k,n}$ of positive n-dimensional Lebesgue measure.*

PROOF. Let $K \subset \mathbb{R}$ be a compact, first category set with positive Lebesgue measure (i.e., a "fat" Cantor set). Let $D = K^k \times \mathbb{R}^{n-k}$ and note that D has positive measure in $\mathbb{R}^n$. Moreover, by Theorem 2.1, $D \in \mathcal{D}^+_{k-1,n}$, since for any $F \in \mathcal{F}^+_{n-k+1}$ the projection $\pi_F[D]$ is of the form $K^j \times \mathbb{R}^{n-k+1-j}$ for some $j > 0$, so that $\pi_F[D]$ is of first category in F. Finally, one again by Theorem 2.1, $D \notin \mathcal{D}^+_{k,n}$ since for $F = \{0\}^k \times \mathbb{R}^{n-k} \in \mathcal{F}^+_{n-k}$, $\pi_F[D] = F$ is not of first category in F. ■

We can also easily derive the following result about the sets of discontinuity for $\mathcal{F}_k$-continuous functions.

Corollary 2.2. *For any $D \in \mathcal{D}_{k,n}$ and any $F \in \mathcal{F}_{n-k}$, the projection $\pi_F[D]$ is of first category in F.*

PROOF. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{F}_k$-continuous function with $D = D(f)$. Choose a perpendicular coordinate system for $\mathbb{R}^n$ such that k-many of the axes are perpendicular to F, while the remaining $(n-k)$-many axes are parallel to F. Since f is $\mathcal{F}_k$-continuous, f is also $\mathcal{F}^+_k$-continuous in this coordinate system. Hence, by Theorem 2.1, $\pi_F[D]$ is of first category in F. ■

2.2 $\mathcal{F}_k$-continuity and the structure of sets in $\mathcal{D}_{k,n}$

Although the $\mathcal{F}_k$-continuous functions are a natural refinement of linear continuity, this paper marks their first appearance in the literature. Therefore, we start here with several examples of such functions, the first of which constitutes a generalization of the example (2).

Example 2.3. For every $n \geq 2$, the following function $f_n \colon \mathbb{R}^n \to \mathbb{R}$, constructed by the first author in [2], is $\mathcal{F}_{n-1}$-continuous and discontinuous precisely at the origin

$$f_n(x_0, x_1, \dots, x_{n-1}) = \begin{cases} \frac{x_0 \prod_{i=0}^{n-1} (x_i)^{2^{2i}}}{\sum_{n=0}^{n-1} (x_i)^{2^{n+i}}} & \text{if } \langle x_0, x_1, \dots, x_{n-1} \rangle \neq \langle 0, 0, \dots, 0 \rangle, \\ 0 & \text{if } \langle x_0, x_1, \dots, x_{n-1} \rangle = \langle 0, 0, \dots, 0 \rangle. \end{cases}$$

In particular, $f_3\colon \mathbb{R}^3 \to \mathbb{R}$ is defined as

$$f_3(x,y,z) = \begin{cases} \frac{x^2y^4z^{16}}{x^8+y^{16}+z^{32}} & \text{if } \langle x,y,z\rangle \neq \langle 0,0,0\rangle \\ 0 & \text{if } \langle x,y,z\rangle = \langle 0,0,0\rangle. \end{cases}$$

Note that if we define the path $p(t) = \langle t^{2^n}, t^{2^{n-1}}, \ldots, t^{2^2}, t^{2^1}\rangle$ for $t \in [0,1]$, then along this path $f_n(p(t)) = \frac{1}{n}$ for $t \neq 0$, while $f_n(p(0)) = 0$.

A more general example of this kind is given by the following result.

Proposition 2.4. *For every $k < n$ and any compact nowhere dense $K \subset \mathbb{R}$, the set $\{0\}^k \times K \times \mathbb{R}^{n-k-1}$ belongs to $\mathcal{D}_{k,n}$. In particular, $\mathcal{D}_{k,n}$ contains the sets of positive $(n-k)$-Hausdorff measure.*[1]

PROOF. For $k = 0$ the statement is clearly true, since any function from $\mathbb{R}^n$ to $\mathbb{R}$ is $\mathcal{F}_0$-continuous.

First, we prove, by induction on $n = 1, 2, 3, \ldots$, that

(I_n) the statement is true for $k = n-1$.

For $n = 1$ this is true, since then $k = n - 1 = 0$. So, assume that (I_n) holds for some n. We need to show (I_{n+1}).

By (I_n), there exists an $\mathcal{F}_{n-1}$-continuous $g\colon \mathbb{R}^n \to \mathbb{R}$ with $D(g) = \{\theta\}$, where $\theta = \langle 0, 0, \ldots, 0\rangle$. We can assume that $g(\theta) = 0$. So, there is a sequence $s = \langle s_m \in \mathbb{R}^n \colon m < \omega\rangle$ converging to θ such that $\lim_{m\to\infty} |g(s_m)| > 0$. In particular, there exists a $c > 0$ such that $|g(s_m)| > c$ for every $m < \omega$. Let $S = \{s_m \in \mathbb{R}^n \colon m < \omega\}$. Choose distinct points $y_j \in \mathbb{R} \setminus K$, $j < \omega$, such that K is the set of accumulation points of $\{y_j \colon j < \omega\}$. Choose distinct points $x_j \in S$, $j < \omega$, such that

$$\frac{\|x_j\|}{\operatorname{dist}(y_j, K)} < 2^{-j} \text{ for every } j < \omega.$$

Choose numbers $\varepsilon_j \in (0, 2^{-j})$ such that the sets $B_j = B(\langle x_j, y_j\rangle, \varepsilon_j)$ are pairwise disjoint and that

$$0 < \frac{\|a_j\|}{\operatorname{dist}(b_j, K)} < 2^{-j} \text{ for every } \langle a_j, b_j\rangle \in \mathbb{R}^n \times \mathbb{R} \text{ from } B_j. \tag{3}$$

Choose continuous maps $f_j \colon \mathbb{R}^{n-1} \times \mathbb{R} \to [0,1]$, $j < \omega$, such that $\operatorname{supp}(f_j) \subset B_j$ and $f_j(x_j, y_j) = 1$. Define $\hat{g}\colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ via formula $\hat{g}(x,y) = g(x)$, put $f = \hat{g} \cdot \sum_{j<\omega} f_j$, and notice that f is the desired function.

[1] For $k = 0$ and $k = n-1$, the set $\{0\}^k \times K \times \mathbb{R}^{n-k-1}$ is interpreted as $K \times \mathbb{R}^{n-1}$ and as $\{0\}^{n-1} \times K$, respectively.

Indeed, f is discontinuous on $\{\theta\} \times K$, since $\{\theta\} \times K$ is in the closure of $T = \{\langle x_j, y_j \rangle \colon j < \omega\}$, $f[T] \cap (-c, c) = \emptyset$, and $f[\{\theta\} \times K] = \{0\}$. Moreover, f is continuous at every $p \in \mathbb{R}^n \setminus (\{\theta\} \times K)$, since every function $\hat{g} \cdot f_j$ is continuous and p has a neighborhood intersecting only finitely many sets $B_j \supset \mathrm{supp}(g \cdot f_j)$. Therefore, $D(f) = \{\theta\} \times K$, as required.

To see that f is $\mathcal{F}_n$-continuous, choose an $F \in \mathcal{F}_n$. If F intersects only finitely many sets B_j, then clearly $f \restriction F$ is continuous. On the other hand, if F intersects infinitely many sets B_j, then, by (3), F intersects $\{\theta\} \times K$ and contains lines forming arbitrary small angles with the line $\{\theta\} \times \mathbb{R}$. So, $\{\theta\} \times \mathbb{R} \subset F$, that is, $F = F' \times \mathbb{R}$ for some $(n-1)$-flat $F' \subset \mathbb{R}^n$. Clearly $f \restriction F$ is continuous at all points not in $\{\theta\} \times \mathbb{R}$. It is also continuous at any $p \in \{\theta\} \times \mathbb{R}$, since for any sequence $\langle \langle c_j, d_j \rangle \in F' \times \mathbb{R} \colon j < \omega \rangle$ converging to p, we have $|f(c_j, d_j)| \leq |g(c_j)|$ and $\lim_{j \to \infty} |g(c_j)| = |g(\theta)| = 0 = f(p)$, as $g \restriction F'$ is continuous. Therefore, f is indeed $\mathcal{F}_n$-continuous, completing the proof.

To finish the proof of the proposition, assume that $0 < k < n-1$. By (I_{k+1}), there exists an $\mathcal{F}_k$-continuous $h \colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ with $D(h) = \{0\}^k \times K$. Define $f \colon \mathbb{R}^n \to \mathbb{R}$ as $f(x_0, \ldots, x_{n-1}) = h(x_0, \ldots, x_k)$. Clearly such an f is $\mathcal{F}_k$-continuous and $D(f) = D(h) \times \mathbb{R}^{n-k-1} = \{0\}^k \times K \times \mathbb{R}^{n-k-1}$, as required. ∎

The next theorem, on the structure of sets in $\mathcal{D}_{k,n}$, is a natural generalization of Slobodnik's result, Theorem 1.2. In its statement, we will use the following terminology.

Let $\mathcal{V}$ be a family of all vector subspaces of $\mathbb{R}^n$ and let $\mathcal{V}_k = \mathcal{V} \cap \mathcal{F}_k$. For a $V \in \mathcal{V}$ let $V^\perp \in \mathcal{V}$ denote the perpendicular complement of V and notice that every $x \in \mathbb{R}^n$ has a unique representation as $x = v + w$, where $\langle v, w \rangle \in V \times V^\perp$. In what follows, we will identify $\langle v, w \rangle \in V \times V^\perp$ with $x = v + w$.

Theorem 2.2. *For every $0 < k < n$ and $D \in \mathcal{D}_{k,n}$ there exists a sequence $\langle f_i \rangle_{i<\omega}$ of Lipschitz functions f_i from $V_i \in \mathcal{V}_{n-k}$ into $V_i^\perp \in \mathcal{V}_k$ whose graphs cover D.*

Theorem 2.2 will be proved in Section 5. Below, we discuss some of its consequences.

Recall that Lipschitz maps cannot raise Hausdorff dimension, see e.g. [7]. In particular, Theorem 2.2 and Corollary 2.2 immediately imply

Corollary 2.5. *Every $D \in \mathcal{D}_{k,n}$ has Hausdorff dimension $\leq n-k$. Moreover, for every f_i from Theorem 2.2, the domain of $D \cap f_i$ is of first category in V_i.*

This stay in contrast with the families $\mathcal{D}^+_{k,n}$, which contain the sets of Hausdorff dimension n, as shown in Proposition 2.1. Notice also that, by

Proposition 2.4, the upper bound $n-k$ of the Hausdorff dimension of sets in $\mathcal{D}_{k,n}$ is achieved.

Corollary 2.6. *For $n \geq 2$, $\mathcal{D}_{0,n}$ is the family of all F_σ-subsets of $\mathbb{R}^n$ and*

$$\begin{array}{ccccccccccc} \{\emptyset\} & = & \mathcal{D}_{n,n} & \subset & \mathcal{D}_{n-1,n} & \subset & \cdots & \subset & \mathcal{D}_{1,n} & \subset & \mathcal{D}_{0,n} \\ & & \| & & \cap & & & & \cap & & \| \\ & & \mathcal{D}^+_{n,n} & \subset & \mathcal{D}^+_{n-1,n} & \subset & \cdots & \subset & \mathcal{D}^+_{1,n} & \subset & \mathcal{D}^+_{0,n} \end{array}$$

Moreover, all indicated inclusions are proper.

Proof. The inclusions follow from Proposition 1.2. The lower row inclusions are strict by Proposition 2.1. The upper row inclusions are strict, since, for every $k<n$, the family $\mathcal{D}_{k,n}$ contains (by Proposition 2.4) a set of Hausdorff dimension $n-k$, while (by Corollary 2.5) $\mathcal{D}_{k+1,n}$ does not contain such a set. This Hausdorff dimension argument also shows that all indicated inclusions between rows are strict. ∎

2.3 Characterization of $\mathcal{D}_{k,n}$ for $k \geq n/2$

Our final main result requires the following notions.

Definition 2.7. The topology on $\mathcal{F}_k$ is generated by a subbase formed by the sets $\mathcal{F}(U)=\{F \in \mathcal{F}_k \colon F \cap U \neq \emptyset\}$, where U is an open set in $\mathbb{R}^n$. We denote the intersection $\mathcal{F}(U_1) \cap \mathcal{F}(U_2) \cap \cdots \cap \mathcal{F}(U_j)$ as $\mathcal{F}(U_1, U_2, \ldots, U_j)$.

Definition 2.8. We define $\mathcal{J}_{k,n}$ as the family of all bounded sets $S \subset \mathbb{R}^n$ for which there is an increasing sequence $\langle \mathcal{L}_i \colon i<\omega \rangle$ of closed subsets of $\mathcal{F}_k$ such that $\bigcup_{i<\omega} \mathcal{L}_i = \mathcal{F}_k$ and, for every $i<\omega$, S is disjoint with the interior $\text{int}(\bigcup \mathcal{L}_i)$ of the set $\bigcup \mathcal{L}_i \subset \mathbb{R}^n$.

Theorem 2.3. *Let $0<k<n$ be such that $k \geq \frac{n}{2}$. A set $D \subset \mathbb{R}^n$ is in $\mathcal{D}_{k,n}$ if, and only if, D is a countable union of compact sets from $\mathcal{J}_{k,n}$.*

Note the theorem provides a characterization of a family $\mathcal{D}_{1,2}$. In particular, Theorem 2.3 provides a solution Kronrod's problem. The proof of Theorem 2.3 will be presented in Section 6.

3 Terminology and preliminaries

We keep our terminology fairly standard and mostly follow [13]. We will denote the open ball in $\mathbb{R}^n$ of radius $r>0$ and centered at z by $B(z,r)$. If z is the origin, θ, then we denote this ball as $B(r)$. The closures of these balls are denoted $B[z,r]$ and $B[r]$, respectively. We also denote the first infinite ordinal number by ω.

When measuring discontinuity of a function $f\colon X \to \mathbb{R}$, with X being metric space, we use the following quantities. The *oscillation* of f on $E \subset X$ is defined as $\operatorname{osc}(f,E) \stackrel{\text{def}}{=} \sup_{x\in E} f(x) - \inf_{x\in E} f(x)$. The *oscillation* of f at x_0 is

$$\operatorname{osc}(f,x_0) = \lim_{\delta\to 0^+} \operatorname{osc}(f, B(x_0,\delta)).$$

The basic results on these quantities can be found in [13]. Most crucially, f is continuous at x_0 if, and only if, $\operatorname{osc}(f,x_0)=0$. We also will use the following notion of *modulus of continuity:*

$$\omega(f,\delta) = \sup\{|f(x)-f(y)| : \|x-y\| < \delta\},$$

where $\|x-y\|$ is the Euclidean distance between x and y. Recall that a function f is uniformly continuous if, and only if, $\lim_{\delta\to 0}\omega(f,\delta)=0$. The modulus of continuity is discussed in [1]. The *support* of a function f, denoted by $\operatorname{supp}(f)$, is the closure of the set $\{x\colon f(x)\neq 0\}$.

The orthogonal projection of a set K onto a k-flat F will be denoted by $\pi_F[K]$. If $K=\{x\}$, we will write this as $\pi_F(x)$. We note that an $(n-1)$-flat is called a *hyperplane*. We say that $k+1$ points in $\mathbb{R}^n$ are in *general position* if no $(k-1)$-flat contains more than k-many of them.

Several times in this paper, we will use the following simple lemma, first published in [3], useful in constructing functions which satisfy various restriction continuities.

Lemma 3.1. *Let $\mathcal{S}$ be a family of subsets of $\mathbb{R}^n$ and $\{U_j\colon j<\omega\}$ be a pointwise finite family of open sets such that the following condition (F) holds.*

(F) The set $\{j<\omega\colon U_j\cap S\neq\emptyset\}$ is finite for every $S\in\mathcal{S}$.

Then for every sequence $\langle f_j\colon j<\omega\rangle$ of continuous functions from $\mathbb{R}^n$ into $\mathbb{R}$ with $\operatorname{supp}(f_i)\subset U_i$ for all $i<\omega$, the function $f \stackrel{\text{def}}{=} \sum_{j<\omega} f_j$ is $\mathcal{S}$-continuous. In addition, if

(a) the diameters of the sets U_j go to 0, as $j\to\infty$,

(b) D is the set of all $z\in\mathbb{R}^n$ for which every open $U\ni z$ intersects infinitely many sets U_j, and

(c) each function f_j is onto $[0,1]$,

then $D=D(f)$. Moreover, if sets U_j are pairwise disjoint, then $\operatorname{osc}(f,z)=1$ for every $z\in D$.

Proof. Clearly, f is $\mathcal{S}$-continuous, as its restriction to any $S \in \mathcal{S}$ is a finite sum of continuous functions. By the same reason, f is continuous at any $z \in \mathbb{R}^n \setminus D$. Finally, it is easy to see that $\operatorname{osc}(f,z) \geq 1$ for any $z \in D$, while, for disjoint U_j's, we must have $\operatorname{osc}(f,z) = 1$. ■

We also need the following fact, frequently used in geometry.

Lemma 3.2. *For any set $\{x_0, x_1, \ldots, x_k\}$ of $(k+1)$-many points in $\mathbb{R}^n$ in general position, there is an $\varepsilon_0 > 0$, such that for every $\langle y_i \rangle_{i=0}^k \in \prod_{i=0}^k B(x_i, \varepsilon_0)$, the points $y_0, y_1, \ldots, y_k$ are distinct and in general position.*

A thorough exposition of the relevant ideas may be found in [12, chapter 1], where results similar to the above lemma are used extensively. Basically, the lemma holds, since the set of all $\langle x_0, x_1, \ldots, x_k \rangle \in (\mathbb{R}^n)^{k+1}$ for which points $x_0, x_1, \ldots, x_k$ fail to be in general position is closed and nowhere dense in $(\mathbb{R}^n)^{k+1}$.

4 Proof of Theorem 2.1

We will use the following two lemmas. The first, is just a convenient presentation of well know result. It will be used in what follows for $Z = \mathbb{R}^k$.

Lemma 4.1. *Let Z be locally compact and σ-compact, and assume that $f \colon Z \times \mathbb{R}^m \to \mathbb{R}$ is separately continuous as a function of $m+1$ variables. If $K \subset Z \times \mathbb{R}^m$ is compact and there is an $\varepsilon > 0$ such that $\operatorname{osc}(f,p) \geq \varepsilon$ for all $p \in K$, then $\pi_{\mathbb{R}^m}[K]$ is nowhere dense in $\mathbb{R}^m$. In particular, $\pi_{\mathbb{R}^m}[D(f)]$ is of first category in $\mathbb{R}^m$.*

Proof. Let $Y \subset Z$ be compact whose interior contains $\pi_Z[K]$ and let $\bar{f}$ be a restriction of f to $Y \times \mathbb{R}^m$. Then $K \subset \{p \in Y \times \mathbb{R}^m \colon \operatorname{osc}(\bar{f},p) \geq \varepsilon\}$. Also, clearly $\bar{f}(\cdot,x)$ is continuous for every $x \in \mathbb{R}^m$ and, for every $y \in Y$, $\bar{f}(y,\cdot)$ is quasi-continuous, as it is separately continuous on $\mathbb{R}^m$, see e.g. [1, theorem 2.4]. Therefore, $\bar{f}$ satisfies the assumptions of [1, corollary 3.8],[2] so the set $\pi_{\mathbb{R}^m}[\{p \in Y \times \mathbb{R}^m \colon \operatorname{osc}(\bar{f},p) \geq \varepsilon\}]$ is nowhere dense in $\mathbb{R}^m$. Thus, so is its subset $\pi_{\mathbb{R}^m}[K]$, as required.

Since $D(f)$ is a countable union of the sets K as in the assumptions, $\pi_{\mathbb{R}^m}[D(f)]$ is of first category in $\mathbb{R}^m$. ■

The following lemma is a variant of [9, lemma 9].

[2]Statement of [1, corollary 3.8]: *Suppose $f \colon X \times Y \to M$, where X is a Baire space, Y is a compact metric space, and M is a metric space. If $f(x,\cdot)$ is continuous for each x in X and $f(\cdot,y)$ is quasicontinuous for each y in a dense subset E of Y, then there is a set A of first category in X such that $D(f) \subset A \times Y$. Indeed, for each $\eta > 0$, $\{x \in X \colon \operatorname{osc}(f,(x,y)) \geq \eta$ for some $y \in Y\}$ is a closed nowhere dense subset of X.*

Lemma 4.2. *If D is a compact subset of $\mathbb{R}^n$ such that its orthogonal projection onto each $F \in \mathcal{F}^+_{n-k}$ is nowhere dense, then there exists an $\mathcal{F}^+_k$-continuous function $f\colon \mathbb{R}^n \to [0,1]$ such that $D(f) = D$ and $\operatorname{osc}(f,z) = 1$ for all $z \in D$.*

Proof. Let $\mathcal{V}^+_{n-k}$ be the family of all linear spaces $V \in \mathcal{F}^+_{n-k}$ and let $\left\{M_i\colon i < \binom{n}{n-k}\right\}$ be its enumeration. Then, by the assumption, D is contained in a finite union $Z = \bigcup_{V\in\mathcal{V}^+_{n-k}} \pi_V^{-1}(\pi_V[D])$, a closed nowhere dense set.

Construct, by induction on $j < \omega$, a sequence $\{U_j \subset \mathbb{R}^n \setminus Z\colon j < \omega\}$ of disjoint non-empty open balls satisfying the conditions (a), (b), and (F) with $\mathcal{S} = \mathcal{F}^+_k$ of Lemma 3.1. This can be done by: choosing a countable dense subset E of D; fixing a sequence $\langle p_j \in E\colon j < \omega\rangle$, so that each point of E occurs infinitely many times; for every $j < \omega$, choosing a ball $U_j \subset B(p_j, 2^{-j})$, so that its closure $\operatorname{cl}(U_j)$ is disjoint with $Z \cup \bigcup_{i<j}\bigcup_{V\in\mathcal{V}^+_{n-k}} \pi_V^{-1}(\pi_V[\operatorname{cl}(U_i)])$. The choice of U_j is possible, since the construction insures that for every $V \in \mathcal{V}^+_{n-k}$ the sets $\pi_V[\operatorname{cl}(U_i)]$, $i < j$, are disjoint with $\pi_V[D]$.

Choose functions f_j as in Lemma 3.1. Then $f = \sum_{j<\omega} f_j$ is as desired. ■

Proof of Theorem 2.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{F}^+_k$-continuous function and let $D = D(f)$. Clearly, D must be F_σ, as this is true for any set of points of discontinuity of a function $f\colon \mathbb{R}^n \to \mathbb{R}$, see e.g. [13]. Let $F \in \mathcal{F}^+_{n-k}$. We need to show that $\pi_F[D]$ is of first category in F.

Since a translation of F does not change this property, we can assume that $F \in \mathcal{V}_{n-k}$. So, F is an $(n-k)$-dimensional linear subspace of $\mathbb{R}^n$. Let $F^\perp$ be a k-dimensional linear subspace of $\mathbb{R}^n$ perpendicular to F. Identify $\mathbb{R}^n$ with $F \times F^\perp$. Then f can be treated as an $\mathcal{F}^+_k$-continuous function on $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Then, by Lemma 4.1, $\pi[D]$ is of first category in $\mathbb{R}^{n-k}$ and so, $\pi_F[D]$ is of first category in F.

To prove the converse implication, fix an F_σ subset D of $\mathbb{R}^n$ whose orthogonal projection on any $F \in \mathcal{F}^+_{n-k}$ is of first category. In particular, there exists a sequence $\langle D_j\colon j < \omega\rangle$ of compact sets such that $D = \bigcup_{j<\omega} D_j$. Now, each D_j satisfies the assumptions of Lemma 4.2. Therefore, for every $j < \omega$, there exists an $\mathcal{F}^+_k$-continuous function $f_j\colon \mathbb{R}^n \to [0,1]$ with the property that $D(f_j) = D_j = \{z \in \mathbb{R}^n\colon \operatorname{osc}(f_j, z) = 1\}$. Define $f = \sum_{j<\omega} 3^{-j} f_j$. Then f is $\mathcal{F}^+_k$-continuous, as a uniform limit of such functions. Moreover, f is discontinuous precisely on D, since for any $z \in D_i \setminus \bigcup_{j<i} D_j$ we have $\operatorname{osc}(\sum_{j\le i} 3^{-j} f_j, z) = 3^{-i}$, while the range of $\sum_{i<j<\omega} 3^{-j} f_j$ is contained in $\left[0, 3^{-i}\right)$. ■

5 Proof of Theorem 2.2

The following lemma translates the formulation of Theorem 2.2: it implies that it is enough to prove that any $D \in \mathcal{D}_{k,n}$ can be covered by countably many sets K with some nice projection properties. In what follows, $K - K$ refers to the set $\{p - q \colon p, q \in K\}$.

Lemma 5.1. *Assume that $K \subset \mathbb{R}^n$, $V \in \mathcal{V}_{n-k}$, and $C > 0$ have the property that $\|z\| \leq C\|\pi_V(z)\|$ for every $z \in K - K$. Then K is contained in a graph of a Lipschitz function from V into $V^\perp$.*

Proof. Let $g = \{\langle \pi_V(z), \pi_{V^\perp}(z)\rangle \colon z \in K\}$ and notice that g is a Lipschitz function from $\pi_V[K]$ into $V^\perp$, since for every $z_0, z_1 \in K$

$$\begin{aligned}
\|\pi_{V^\perp}(z_0) - \pi_{V^\perp}(z_1)\|^2 &= \|\pi_{V^\perp}(z_0 - z_1)\|^2 \\
&= \|z_0 - z_1\|^2 - \|\pi_V(z_0 - z_1)\|^2 \\
&\leq C^2\|\pi_V(z_0 - z_1)\|^2 - \|\pi_V(z_0 - z_1)\|^2 \\
&= (C^2 - 1)\|\pi_V(z_0 - z_1)\|^2 \\
&= (C^2 - 1)\|\pi_V(z_0) - \pi_V(z_1)\|^2.
\end{aligned}$$

Since every partial Lipschitz function from $\mathbb{R}^k$ into $\mathbb{R}^m$ can be extended to an entire Lipschitz function (see e.g. [6, p. 80]), the result follows. ■

The following lemma is a generalization of [18, lemma 2].

Lemma 5.2. *For every $0 < k < n$, $D \in \mathcal{D}_{k,n}$, and $V \in \mathcal{V}_{n-k+1}$, there exists a countable partition $\mathcal{K}$ of D with the following property.*

(†) *For every $K \in \mathcal{K}$ there exist a $c_K > 0$ and a perpendicular decomposition $\langle L_K, W_K\rangle \in \mathcal{V}_1 \times \mathcal{V}_{n-k}$ of V such that $\|\pi_{L_K}(z)\| \leq c_K\|\pi_{W_K}(z)\|$ for every $z \in K - K$.*

Proof. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{F}_k$-continuous function with $D(f) = D$. Then, there exists a countable cover $\mathcal{P}$ of D by the compact sets such that for every $P \in \mathcal{P}$ there is an $\varepsilon > 0$ for which $\operatorname{osc}(f, z) \geq \varepsilon$ for every $z \in P$. It is enough to show that every $P \in \mathcal{P}$ admits a countable $\mathcal{K}_P$ of P satisfying property (†). So, fix a $P \in \mathcal{P}$ and an associated ε.

Choose an arbitrary perpendicular decomposition $\langle L, W\rangle \in \mathcal{V}_1 \times \mathcal{V}_{n-k}$ of V, fix a non-zero $u \in L$, and for $v \in L$ let $a_v \in \mathbb{R}$ be such that $v = a_v u$. Define function $g \colon (V^\perp \times L) \times W \to \mathbb{R}^n$ via $g(\langle y, v\rangle, w) = y + v + a_v w = y + a_v(u + w)$. Of course, the restriction $\bar{g}$ of g to $V^\perp \times (L \setminus \{0\}) \times W$ establishes a homeomorphism between this set and $\mathbb{R}^n \setminus L^\perp$.

For every $p \in \mathbb{R}^n$, let $f_p \colon (V^\perp \times L) \times W \to \mathbb{R}$ be defined as $f(p+g(d))$ for d in the domain. Notice that $f_p(\cdot, w)$ is continuous, since $p + g[(V^\perp \times L) \times \{w\}]$ is a k-flat, a p-translation of a vector space spanned by $V^\perp$ and a vector $u+w \in V$. Moreover, $f_p(\langle y, v\rangle, \cdot)$ is separately continuous, upon identification of W with $\mathbb{R}^{n-k}$.

Let us define $S_p = \{d \colon \pi_L(g(d)) \neq 0 \ \&\ p + g(d) \in P\}$. Then, $\operatorname{osc}(f_p, d) = \operatorname{osc}(f, p + g(d)) \geq \varepsilon$ for every $d \in S_p$, since such d belongs to the domain of the homeomorphism $\bar{g}$. Moreover, it is easy to see that S_p is bounded, so its closure is compact. Thus, by Lemma 4.1, there exists a dense open set $U_p \subset W$ such that $\pi_W[S_p] \cap U_p = \emptyset$.

Now, consider a countable basis $\mathcal{B}$ of W formed by the open balls $B(w, c)$, $c > 0$. For every such ball let $P_{w,c} = \{p \in P \colon B(w, c) \subset U_p\}$. We claim that $K = P_{w,c}$ satisfies (†) with $c_K = \frac{\|u+w\|}{c}$, L_K spanned by $u+w$, and W_K being the perpendicular complement of L_K in W.

Indeed, let $z = q - p$, where $p, q \in P_{w,c}$. If $\pi_L(z) = 0$, then clearly the inequality holds. So, assume that $\pi_L(z) \neq 0$. Then, $z = g(d)$ for some $d \in S_p$. In particular, $\pi_W(d) \notin B(w, c)$, that is, $\|\pi_W(d) - w\| \geq c$.

Now, $z = g(d) = \pi_{V^\perp}(d) + a_{\pi_L(d)} u + a_{\pi_L(d)} \pi_W(d)$. Therefore,

$$\pi_{L_K}(z) = a_{\pi_L(d)}(u + w) \ \ \&\ \ \pi_{W_K}(z) = a_{\pi_L(d)}(\pi_W(d) - w),$$

as $\pi_{W_K}(z) = \pi_W(z) - \pi_{L_K}(z) = (a_{\pi_L(d)} u + a_{\pi_L(d)} \pi_W(d)) - a_{\pi_L(d)}(u + w)$. So,

$$\begin{aligned} \|\pi_{L_K}(z)\| &= |a_{\pi_L(d)}| \|u + w\| \\ &\leq |a_{\pi_L(d)}| \|u + w\| \frac{\|\pi_W(d) - w\|}{c} \\ &= \frac{\|u + w\|}{c} \|\pi_{W_K}(z)\|, \end{aligned}$$

as required. ■

Proof of Theorem 2.2. Let $D \in \mathcal{D}_{k,n}$. We will prove the following property by induction on $\ell \leq k$:

(I_ℓ) There exists a countable partition $\mathcal{P}_\ell$ of D such that for every $P \in \mathcal{P}_\ell$ there exist $V_P \in \mathcal{V}_{n-\ell}$ and $C_P > 0$ such that $\|z\| \leq C_P \|\pi_{V_P}(z)\|$ for any $z \in P - P$.

For $\ell = 0$ the property (I_ℓ) is satisfied with $\mathcal{P} = \{D\}$, $V_D = \mathbb{R}^n$, and $C_D = 1$. So, assume that for some $\ell < k$ the property (I_ℓ) holds. We need to show $(I_{\ell+1})$.

So, fix a $P \in \mathcal{P}_\ell$ and let $V \in \mathcal{V}_{n-k+1}$ be contained in V_P. By Lemma 5.2, there exists a partition $\mathcal{K}_P$ of D such that for every $K \in \mathcal{K}_P$ there exist a

$c_K^P > 0$ and a perpendicular decomposition $\langle L_K^P, W_K^P\rangle \in \mathcal{V}_1 \times \mathcal{V}_{n-k}$ of V such that $\|\pi_{L_K^P}(z)\| \le c_K^P \|\pi_{W_K^P}(z)\|$ for every $z \in K - K$.

Then the partition $\mathcal{P}_{\ell+1} = \{P \cap K \colon P \in \mathcal{P}_\ell \ \&\ K \in \mathcal{K}_P\}$ satisfies $(I_{\ell+1})$, with $V_K^P \in \mathcal{V}_{n-\ell-1}$ being a subspace of V_P perpendicular to L_K^P. Indeed, for any point $z \in (P \cap K) - (P \cap K) \subset (P - P) \cap (K - K)$ we have the inequalities $\|\pi_{L_K^P}(z)\| \le c_K^P \|\pi_{W_K^P}(z)\| \le c_K^P \|\pi_{V_K^P}(z)\|$, as $W_K^P \subset V_K^P$. Therefore, $\|\pi_{V_P}(z)\|^2 = \|\pi_{L_K^P}(z)\|^2 + \|\pi_{V_K^P}(z)\|^2 \le ((c_K^P)^2 + 1)\|\pi_{V_K^P}(z)\|^2$. In particular, $\|z\| \le C_P \|\pi_{V_P}(z)\| \le C_P \sqrt{(c_K^P)^2 + 1}\|\pi_{V_K^P}(z)\|$. In other words, $(I_{\ell+1})$ is satisfied with $C_K^P = C_P \sqrt{(c_K^P)^2 + 1}$. This finishes the inductive proof of (I_ℓ)'s.

The theorem is concluded by noticing that the partition given by (I_k) is as required, as implied by Lemma 5.1. ■

6 Proof of Theorem 2.3

Recall, that the topology on $\mathcal{F}_k$ is defined by a subbasis formed by the sets $\mathcal{F}(U) = \{F \in \mathcal{F}_k \colon F \cap U \neq \emptyset\}$, where U is an open subset of $\mathbb{R}^n$, and that $\mathcal{F}(U_0, U_1, \ldots, U_j)$ is defined as $\bigcap_{i \le j} \mathcal{F}(U_i)$. Our proof will require few facts about this topology on $\mathcal{F}_k$.

Fact 6.1. *If $F \in \mathcal{F}_k$ and the points $x_0, x_1, \ldots, x_k \in F$ are in general position, then the sets*

$$\{\mathcal{F}(B(x_0, r), B(x_1, r), \ldots, B(x_k, r)) \colon r > 0\}$$

form a basis at F.

Proof. For $r > 0$ let $\mathcal{F}(r) \stackrel{\text{def}}{=} \mathcal{F}(B(x_0, r), B(x_1, r), \ldots, B(x_k, r)$ and notice that if $r_1 < r_2$, then $\mathcal{F}(r_1) \subset \mathcal{F}(r_2)$.

The result follows from the following property:

(A) For every open subset $U \subset \mathbb{R}^n$, if $U \cap F \neq \emptyset$, then there exists an $r > 0$ such that $\mathcal{F}(r) \subset \mathcal{F}(U)$.

Indeed, any open set containing F contains a subset of the form $\bigcap_{i=1}^j \mathcal{F}(U_i)$, where each $U_i \subset \mathbb{R}^n$ is an open and intersects F. By (A), for every $i \le j$ there exists an $r_i > 0$ such that $F \in \mathcal{F}(r_i) \subset \mathcal{F}(U_i)$. Put $r = \min_{i \le j} r_i$. Then $F \in \mathcal{F}(r) \subset \bigcap_{i \le j} \mathcal{F}(U_i)$, as required.

We will finish the proof by showing that (A) holds. So, fix an open $U \subset \mathbb{R}^n$ with $U \cap F \neq \emptyset$ and let $x \in U \cap F$.

Since all k-flats are affine sets, the points of F are precisely those which can be expressed as an affine combination of $(k+1)$-many points in F in general position, see for instance [8] or [11, Section 1.2]. In particular, there exist $\beta_0, \beta_1, \ldots, \beta_k \in \mathbb{R}$ such that $\sum_{i=0}^{k} \beta_i = 1$ and $x = \sum_{i=1}^{k} \beta_i x_i$.

Define a function $g\colon (\mathbb{R}^n)^{k+1} \to \mathbb{R}^n$ by setting

$$g(z_0, z_1, \ldots, z_k) = \sum_{i=0}^{k} \beta_i z_i.$$

Note that g is continuous and that $g(x_0, x_1, \ldots, x_k) = x \in U \cap F$. So, there is an $r > 0$ such that if $\langle z_i \rangle_{i=0}^{k} \in \prod_{i=0}^{k} B(x_i, r)$, then $g(z_0, z_1, \ldots, z_k) \in U$. It is enough to show that $\mathcal{F}(r) \subset \mathcal{F}(U)$.

Indeed, let $F' \in \mathcal{F}(r)$. Then, there exists a $\langle z_i \rangle_{i=0}^{k} \in \prod_{i=0}^{k} (F' \cap B(x_i, r))$. As $z = g(z_0, z_1, \ldots, z_k)$ is an affine combination of points $z_0, z_1, \ldots, z_n \in F'$, z belongs to F'. Also, the choice of r insures that $z \in U$. Thus, $z \in F' \cap U$ and so, $F' \in \mathcal{F}(U)$, as required. ■

Fact 6.2. *If Z is a closed subset of $\mathcal{F}_k$, then $\bigcup Z$ is a closed subset of $\mathbb{R}^n$.*

PROOF. Let $\langle z_0^i \colon i < \omega \rangle$ be a sequence of points in $\bigcup Z$ with the property that $\lim_{i \to \infty} z_0^i = z_0 \in \mathbb{R}^n$. We will show that $z_0 \in \bigcup Z$.

For each $i < \omega$ there exists a k-flat $F_i \in Z$ with $z_0^i \in F_i$. Choose the points $z_1^i, \ldots, z_k^i \in F_i$ such that $\|z_j^i - z_\ell^i\| = 1$ for all $j < \ell \le k$. Since the sequence $\langle z_0^i, \ldots, z_k^i \rangle_{i<\omega}$ is bounded, choosing subsequence, if necessary, we can ensure that it converges to a point $\langle z_0, \ldots, z_k \rangle \in \mathbb{R}^{k+1}$. Clearly we have $\|z_j - z_\ell\| = 1$ for all $j < \ell \le k$. In particular, the points $z_0, \ldots, z_k$ are in general position. Thus, they all belong to some k-flat F and, by Fact 6.1, the family $\{\mathcal{F}(B(z_0, r), B(z_1, r), \ldots, B(z_k, r)) \colon r > 0\}$ forms a basis at F. Since every set $\mathcal{F}(B(z_0, r), B(z_1, r), \ldots, B(z_k, r))$ contains some $F_i \in Z$, F is in the closure of Z, that is, $F \in Z$. In particular, $z_0 \in F \subset \bigcup Z$, as required. ■

Now that some basic facts about our topology have been laid out, consider the family $\mathcal{J}_{k,n}$ discussed earlier. We prove the following structural result about $\mathcal{J}_{k,n}$ which will be essential to the proof of Theorem 2.3.

Fact 6.3. *If $S \in \mathcal{J}_{k,n}$, then $\mathrm{cl}(S) \in \mathcal{J}_{k,n}$ is nowhere dense.*

PROOF. Let $\{\mathcal{L}_i \colon i < \omega\}$ be an increasing sequence of closed subsets of $\mathcal{F}_k$ which justifies that S belongs to $\mathcal{J}_{k,n}$. Notice that

$$\mathbb{R}^n = \bigcup \mathcal{F}_k = \bigcup_{i<\omega} \bigcup \mathcal{L}_i = \bigcup_{i<\omega} \mathrm{int}\left(\bigcup \mathcal{L}_i\right) \cup \bigcup_{i<\omega} \left(\bigcup \mathcal{L}_i \setminus \mathrm{int}\left(\bigcup \mathcal{L}_i\right)\right).$$

Define $G = \bigcup_{i<\omega} \mathrm{int}(\bigcup \mathcal{L}_i)$. Then G is an open in $\mathbb{R}^n$, being the union of open sets. Moreover, by Fact 6.2, each set $\bigcup \mathcal{L}_i \setminus \mathrm{int}\,(\bigcup \mathcal{L}_i)$ is closed nowhere dense in $\mathbb{R}^n$. Since clearly $\mathbb{R}^n \setminus G$ is a subset of $\bigcup_{i<\omega}(\bigcup \mathcal{L}_i \setminus \operatorname{int}(\bigcup \mathcal{L}_i))$, $\mathbb{R}^n \setminus G$ is of first category in $\mathbb{R}^n$. So, being closed, it is nowhere dense. As $\operatorname{cl}(S)$ is disjoint with $G = \bigcup_{i<\omega} \operatorname{int}(\bigcup \mathcal{L}_i)$, it is nowhere dense and belongs to $\mathcal{J}_{k,n}$. ∎

Our characterization of $\mathcal{D}_{k,n}$, for $k \geq \frac{n}{2}$, follows from the next three lemmas.

Lemma 6.4. *Let k and n be integers, with $n > k \geq \frac{n}{2}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{F}_k$-continuous and let $\delta, \epsilon > 0$ be given. If $B = B[N]$ for some $N \in (0,\infty)$, then the set $Z_{\delta,\epsilon} \stackrel{\text{def}}{=} \{F \in \mathcal{F}_k \colon \omega(f \restriction (F \cap B), \delta) \leq \epsilon\}$ is closed in $\mathcal{F}_k$.*

Proof. Let ϵ, $\delta > 0$ be given and let $Z = Z_{\delta,\epsilon}$. It is enough to show that the complement of Z, $Z^c = \mathcal{F}_k \setminus Z$, is open. So, fix an $F \in Z^c$. We will find an open neighborhood $\mathcal{W}$ of F disjoint with Z.

Let $V \in \mathcal{V}_k$ be such that $F = x_0 + V$ for some $x_0 \in F$ and let $W = V^\perp$. Choose the points $x_0, x_1, \ldots, x_k \in F$ in general position. The set $\mathcal{W}$ will be of the form $\mathcal{F}(r) = \mathcal{F}(B(x_0,r), B(x_1,r), \ldots, B(x_k,r))$ for some $r > 0$. Invoking Lemma 3.2, we can choose an $r > 0$ small enough so that for any $y = \langle y_i \rangle_{i \leq k} \in \prod_{i\leq k} B(x_i, r)$ the points from $Y = \{y_i \colon i \leq k\}$ are in general position. In particular, there is a unique $F(y) \in \mathcal{F}_k$ containing Y.

Our proof of the lemma is based on the following claim.

Claim 6.5. *For every $z \in F$ and $y = \langle y_i \rangle_{i\leq k} \in \prod_{i \leq k} B(x_i, r)$, the intersection $(z + W) \cap F(y)$ contains a unique point $h_z(y)$. Moreover, the mapping $h_z \colon \prod_{i\leq k} B(x_i, r) \to \mathbb{R}^n$ is continuous.*

First, notice that the claim implies the lemma. To see this, we will show that, decreasing r, if necessary, $\mathcal{F}(r) \subset Z^c$. So, choose an arbitrary k-flat from $\mathcal{F}(r)$. It is of the form $F(y)$ for some $y = \langle y_i \rangle_{i \leq k} \in \prod_{i\leq k} B(x_i, r)$. We will show that $F(y) \in Z^c$ provided r small enough.

As $F \in Z^c$, we have $\omega(f \restriction (F \cap B), \delta) > \epsilon$. So there are $z_0, z_1 \in F \cap B$ for which $||z_1 - z_0|| < \delta$ and $|f(z_1) - f(z_0)| > \epsilon$. Since $f \restriction F$ is continuous, we can choose $z_0, z_1 \in \operatorname{int}(B) = B_0$.

Choose an $\varepsilon_0 > 0$ such that $||z_1 - z_0|| + 2\varepsilon_0 < \delta$ and $|f(z_1) - f(z_0)| - 2\varepsilon_0 > \epsilon$. For $i < 2, z_i + W$ is contained in a k-flat (as it has dimension $n - k \leq k$). Therefore, function $f \restriction (z_i + W)$ is continuous at z_i and so, there exists a $\delta_0 \in (0, \varepsilon_0)$ such that $|f(z_i) - f(z)| < \varepsilon_0$ whenever $z \in (z_i + W) \cap B(z_i, \delta_0)$. Since functions h_{z_i} are continuous, we can decrease r so that $\|h_{z_i}(y) - z_i\| = \|h_{z_i}(y) - h_{z_i}(x_0, \ldots, x_k)\| < \delta_0$ whenever $y \in \prod_{i\leq k} B(x_i, r)$. For $i < 2$ define

$z_i' \stackrel{\text{def}}{=} h_{z_i}(y) \in F(y)$ and note that

$$\begin{aligned}||z_0' - z_1'|| &= ||h_{z_0}(y) - h_{z_1}(y)|| \\ &\le ||h_{z_0}(y) - z_0|| + ||z_0 - z_1|| + ||h_{z_1}(y) - z_1|| \\ &< \delta_0 + \delta - 2\varepsilon_0 + \delta_0 < \delta - 2\varepsilon_0 + 2\varepsilon_0 = \delta.\end{aligned}$$

Furthermore,

$$\begin{aligned}|f(z_0') - f(z_1')| &= |f(z_0') - f(z_0) + f(z_0) - f(z_1) + f(z_1) - f(z_1')| \\ &= |-[f(z_1) - f(z_0)] - [f(z_0) - f(z_0')] - [f(z_1') - f(z_1)]| \\ &\ge |f(z_1) - f(z_0)| - |f(z_0) - f(z_0')| - |f(z_1) - f(z_1')| \\ &> \epsilon + 2\varepsilon_0 - \varepsilon_0 - \varepsilon_0 = \epsilon.\end{aligned}$$

The last two computations justify the statement $\omega(f\restriction(F(y) \cap B), \delta) > \epsilon$. Hence, indeed $F(y) \in Z^c$ as required.

The above argument reduces the proof of the lemma to that of Claim 6.5. So, we proceed to prove the claim. For this, fix y and z as in the assumptions of Claim 6.5. First notice that $\langle \pi_F(y_i)\rangle_{i\le k} \in \prod_{i\le k} B(x_i, r)$. Thus, the points $\pi_F(y_0), \ldots, \pi_F(y_k)$ are in general position. In particular, z has a unique representation as $z = \sum_{i\le k} \alpha_i(y)\pi_F(y_i)$, where $\sum_{i\le k} \alpha_i(y) = 1$. We shall show that $(z + W) \cap F(y) = \left\{\sum_{i\le k} \alpha_i(y) y_i\right\}$, that is, that $h_z(y) = \sum_{i\le k} \alpha_i(y) y_i$.

Indeed, every $p \in F(y)$ has a unique representation as $p = \sum_{i\le k} \alpha_i y_i$ with $\sum_{i\le k} \alpha_i = 1$. This p belongs to $z + W$ if, and only if,

$$\sum_{i\le k} \alpha_i(y)\pi_F(y_i) = z = \pi_F(p) = \pi_F\left(\sum_{i\le k} \alpha_i y_i\right) = \sum_{i\le k} \alpha_i \pi_F(y_i).$$

In particular, p belongs to $z + W$ if, and only if, $\alpha_i = \alpha_i(y)$ for every $i \le k$ if, and only, if $p = \sum_{i\le k} \alpha_i(y) y_i$. In other words, the intersection $(z+W)\cap F(y)$ indeed contains a unique point $h_z(y)$:

$$h_z(y) = \sum_{i=0}^{k} \alpha_i(y) y_i = y_0 + \sum_{i=1}^{k} \alpha_i(y)(y_i - y_0).$$

To finish the proof, it is enough to show that $h_z(y)$ is continuous, that is, that each function $\alpha_i(y)$ is continuous for $0 < i \le k$.

Now, $z - \pi_F(y_0) = \sum_{i=1}^{k} \alpha_i(y)\pi_F(y_i - y_0)$ and the vectors $v_i = \pi_F(y_i - y_0) = \pi_F(y_i) - \pi_F(y_0)$ are linearly independent for $i = 1, \ldots, k$. So, we can choose the vectors v_i, $k < i \leq n$, such that the family $\beta = \{v_i : 0 < i \leq n\}$ forms a basis for $\mathbb{R}^n$. Then, the numbers $\alpha_i(y)$ constitute the coordinates of $z - \pi_F(y_0)$ with respect to this basis β. Thus, $[\alpha_1(y) \cdots \alpha_n(y)]^T = A^{-1}(z - \pi_F(y_0))^T$, where A is the change of basis matrix which takes points in standard coordinates and gives their coordinates in the coordinate system induced by the basis β. (Thus, the i-th column of A constitutes of the coordinates of v_i with respect to the standard coordinate system.) Since all terms in $A^{-1}(z - \pi_F(y_0))^T$ are continuous with respect to y, so are the functions $\alpha_i(y)$. ∎

Lemma 6.6. *For every compact $K \in \mathcal{J}_{k,n}$, there is an $\mathcal{F}_k$-continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $K = D(f) = \{x : \operatorname{osc}(f, x) = 1\}$.*

Proof. Let $\{\mathcal{L}_i : i < \omega\}$ be a sequence which justifies the inclusion of K in $\mathcal{J}_{k,n}$. Let E be a countable dense subset of K and let $\langle p_i : i < \omega\rangle$ be a sequence of elements of E enumerated so that every element of E occurs in the sequence infinitely many times. We construct, by induction on $i < \omega$, a sequence $\langle D_i \subset \mathbb{R}^n \setminus K : i < \omega\rangle$ of disjoint closed balls of positive radius. For every $i < \omega$, there is a point $q_i \in B(p_i, 2^{-i}) \setminus (K \cup \bigcup \mathcal{L}_i \cup \bigcup_{j<i} D_j)$. We can find such a point because K is nowhere dense and $p_i \notin \operatorname{int}(\bigcup \mathcal{L}_i)$. Choose D_i to be a closed ball centered at q_i and disjoint from $K \cup \bigcup \mathcal{L}_i \cup \bigcup_{j<i} D_j$. Let $f_i\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous surjection vanishing identically outside of D_i. Now define $f = \sum_{i=1}^{\infty} f_i$. We claim f is as desired.

The construction ensures that $\operatorname{osc}(f, p) = 1$ if, and only if, $p \in K$ and that $\operatorname{osc}(f, p) = 0$ otherwise. Since every element of $\mathcal{F}_k$ belongs to some $\mathcal{L}_i$, we may appeal to Lemma 3.1 to see that f is $\mathcal{F}_k$ continuous. ∎

The last of the lemmas is similar in character, purpose, and proof to [1, theorem 3.4].

Lemma 6.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{F}_k$-continuous and let N and k be the natural numbers such that $n > k \geq \frac{n}{2}$. Then $K_N = \{p \in B[N] : \operatorname{osc}(f, p) \geq \frac{1}{N}\}$ belongs to $\mathcal{J}_{k,n}$.*

Proof. For $j = 1, 2, 3, \ldots$, let $\mathcal{L}_j = Z_{\frac{1}{j}, \frac{1}{4N}}$, where we use $B = B[N+1]$ in the definition of $Z_{\frac{1}{j}, \frac{1}{4N}}$. Let $K = K_N$. We claim that $\langle \mathcal{L}_j \rangle_j$ justifies the inclusion of $K \in \mathcal{J}_n$.

Clearly, $\mathcal{L}_j \subset \mathcal{L}_{j+1}$ and by Lemma 6.4, each $\mathcal{L}_j$ is closed. We also have $\bigcup_j \mathcal{L}_j = \mathcal{F}_k$ since for every $F \in \mathcal{F}_k$, the function $f \restriction (F \cap B)$ is uniformly continuous. So, in order to finish, we need only show that $K \cap \operatorname{int}(\bigcup \mathcal{L}_j) = \emptyset$ for every j.

To see this, fix j and $p \in B[N] \cap \operatorname{int}(\bigcup \mathcal{L}_j)$. Our proof will be complete if we can show $p \notin K$. To do this, we show that $\operatorname{osc}(f, p) < \frac{1}{N}$. This portion of the proof is quite similar to that of [1, theorem 3.4].

Let F_0 be the k-flat through p parallel to the k-flat spanned by the first k coordinate axes, and let F_1 be the $(n-k)$-flat through p perpendicular to F_0. Since f is k-continuous and $n-k \leq k$, the functions $f\restriction F_0$ and $f\restriction F_1$ are continuous. Hence, we can find a $\delta > 0$ with $\delta < \frac{1}{2j}$ so that if $r \in F_0 \cup F_1$, and $||r-p|| < \delta$, then $|f(r)-f(p)| < \frac{1}{4N}$.

Decreasing δ, if necessary, we can assume that $B(p,\delta) \subset B \cap \operatorname{int}(\bigcup \mathcal{L}_j)$. To finish the proof, it is enough to show that $|f(p)-f(q)| < \frac{1}{2N}$ for every $q \in B(p,\delta)$, since then the oscillation of f at p will be less than $\frac{1}{N}$, so that $p \notin K$.

So, fix a $q \in B(p,\delta) \subset \bigcup \mathcal{L}_j$. Then, there exists a k-flat $F \in \mathcal{L}_j$ containing q. Since the convex hull of $F_0 \cup F_1$ equals $\mathbb{R}^n$, there are $r_0 \in F_0 \cap F$, $r_1 \in F_1 \cap F$, and $\alpha_0, \alpha_1 \in [0,1]$ such that $q = \alpha_0 r_0 + \alpha_1 r_1$ and $\alpha_0 + \alpha_1 = 1$. Notice that either $\|r_0 - p\| < \delta$ or $\|r_1 - p\| < \delta$, since otherwise

$$\|q-p\| = \|\alpha_0(r_0-p) + \alpha_1(r_1-p)\| \geq \alpha_0\|r_0-p\| + \alpha_1\|r_1-p\| \geq \delta,$$

contradicting the choice of q. Choose $r \in \{r_0, r_1\} \subset F_0 \cap F_1$ with $\|r-p\| < \delta$. Then, $|f(p)-f(r)| < \frac{1}{4N}$. Moreover, $\|r-q\| < 2\delta < \frac{1}{j}$ and $r, q \in F \in \mathcal{L}_j = Z_{\frac{1}{j},\frac{1}{4N}}$. So, $|f(r)-f(q)| \leq \omega\left(f\restriction(F \cap B), \frac{1}{j}\right) \leq \frac{1}{4N}$. Therefore,

$$|f(p)-f(q)| \leq |f(p)-f(r)| + |f(r)-f(q)| < \frac{1}{4N} + \frac{1}{4N} = \frac{1}{2N},$$

finishing the proof. ■

Proof of Theorem 2.3. If $D = D(f)$ for some $\mathcal{F}_k$-continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, then $D = \bigcup_{0<i<\omega} K_i$ where $K_i = \left\{p \in B[i]\colon \operatorname{osc}(f,p) \geq \frac{1}{i}\right\}$. So, by Lemma 6.7, each of these K_i belong to $\mathcal{J}_{k,n}$.

Conversely, assume that $D = \bigcup_{i<\omega} K_i$, where each K_i is compact and belongs to $\mathcal{J}_{k,n}$. Then, by Lemma 6.6, for every $i < \omega$ there exists an $\mathcal{F}_k$-continuous function $f_i\colon \mathbb{R}^n \to [0,1]$ with $D(f_i) = K_i$ and $\operatorname{osc}(f_i, p) = 1$ for all points $p \in K_i$. Then, the function $f = \sum_{i<\omega} 3^{-i} f_i$ is $\mathcal{F}_k$-continuous and $D(f) = D$. ■

7 Discussion

Our proof of Theorem 2.3 does not work for $k < \frac{n}{2}$. In fact, our proof of Lemma 6.4 depends heavily on the fact that f is continuous on $(n-k)$-flats. In particular, the following example shows, that the conclusion of the lemma may be false for $k < \frac{n}{2}$.

Example 7.1. Let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be linearly continuous such that $g(0,0) = 1$, while there exists a sequence $\langle p_i \in \mathbb{R}^2 \rangle_{i<\omega}$ converging to $\langle 0,0\rangle$ such that $g(p_i) = 0$ for all $i < \omega$. For example, if f is given by (2), then $g(x,y) = 1 - f(x,y)$ has this property. Define $h\colon \mathbb{R}^3 \to \mathbb{R}$ as $h(x,y,z) = zg(x,y)$. Then h is clearly linearly (so, $\mathcal{F}_1$-) continuous. However, the set $Z_{2,1}$ for this function is not closed, as it contains all vertical lines through points p_i, but it does not contain their limit, the z-axis.

While this example does not preclude the existence of a version of Theorem 2.3 that would work for $k < \frac{n}{2}$, it emphasizes the difficulties.

An inspection of our the results presented in Section 6 yields some information about the structure of the sets in $\mathcal{D}_{k,n}$, $k \geq \frac{n}{2}$.

Corollary 7.2. *If n and k are integers with $k \geq \frac{n}{2}$, the sets $\mathcal{J}_{k,n}$ are ideals.*

Proof. Clearly any subset of a set $S \in \mathcal{J}_{k,n}$ also belongs to $\mathcal{J}_{k,n}$. We need only show that if K_1 and K_2 are elements of $\mathcal{J}_{k,n}$, then $K_1 \cup K_2 \in \mathcal{J}_{k,n}$. By Fact 6.3, we may assume that K_1 and K_2 are compact. Hence, by Theorem 2.3, there are $\mathcal{F}_k$-continuous functions $f_1, f_2\colon \mathbb{R}^n \to [0,1]$ such that $D(f_i) = K_i = \{z\colon \operatorname{osc}(f_i,z) = 1\}$. Then the function $f_1 + \frac{1}{2}f_2$ is $\mathcal{F}_k$-continuous and $D(f) = K_1 \cup K_2$. Then, by applying Theorem 2.3 again, we see that $K_1 \cup K_2 \in \mathcal{J}_{k,n}$. ∎

Although we are unable to characterize the sets $\mathcal{D}_{k,n}$ for all $k < n$, we are able to derive a sufficient condition for membership in $\mathcal{D}_{k,n}$. In particular, the following theorem gives us a tool for constructing discontinuity sets of $\mathcal{F}_k$-continuous functions without explicitly constructing the functions themselves.

Theorem 7.1. *If S is a countable union of compact members of $\mathcal{J}_{k,n}$, then $S \in \mathcal{D}_{k,n}$.*

Proof. Note that the "sufficiency" part of our proof of Theorem 2.3 depended only upon Lemma 6.6 which holds regards of k and n. ∎

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