LINEABILITY, SPACEABILITY, AND ADDITIVITY CARDINALS FOR DARBOUX-LIKE FUNCTIONS

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Abstract. We introduce the concept of *maximal lineability cardinal number*, $\mathrm{m}\mathcal{L}(M)$, of a subset M of a topological vector space and study its relation to the cardinal numbers known as: additivity $A(M)$, homogeneous lineability $\mathcal{HL}(M)$, and lineability $\mathcal{L}(M)$ of M. In particular, we will describe, in terms of $\mathcal{L}$, the lineability and spaceability of the families of the following Darboux-like functions on $\mathbb{R}^n$, $n \geq 1$: extendable, Jones, and almost continuous functions.

1. Preliminaries and background

The work presented here is a contribution to a recent ongoing research concerning the following general question: *For an arbitrary subset M of a vector space W, how big can be a vector subspace V contained in $M \cup \{0\}$?* The current state of knowledge concerning this problem is described in the very recent survey article [8]. So far, the term *big* in the question was understood as a cardinality of a basis of V; however, some other measures of bigness (i.e., in a category sense) can also be considered.

Following [1, 29] (see, also, [17]), given a cardinal number μ we say that $M \subset W$ is *μ-lineable* if $M \cup \{0\}$ contains a vector subspace V of the dimension $\dim(V) = \mu$. Consider the following *lineability* cardinal number (see [4]):

$$\mathcal{L}(M) = \min\{\kappa \colon M \cup \{0\} \text{ contains no vector space of dimension } \kappa\}.$$

Notice that $M \subset W$ is μ-lineable if, and only if, $\mu < \mathcal{L}(M)$. In particular, μ is the maximal dimension of a subspace of $M \cup \{0\}$ if, and only if, $\mathcal{L}(M) = \mu^+$. The number $\mathcal{L}(M)$ need not be a cardinal successor (see, e.g., [1]); thus, the maximal dimension of a subspace of $M \cup \{0\}$ does not necessarily exist.

If W is a vector space over the field K and $M \subset W$, let

$$\mathrm{st}(M) = \{w \in W \colon (K \setminus \{0\})w \subset M\}.$$

Notice that

$$\text{if } V \text{ is a subspace of } W, \text{ then } V \subset M \cup \{0\} \text{ if, and only if, } V \subset \mathrm{st}(M) \cup \{0\}. \tag{1}$$

In particular,

$$\mathcal{L}(M) = \mathcal{L}(\mathrm{st}(M)). \tag{2}$$

Recall also (see, e.g., [19]) that a family $M \subset W$ is said to be *star-like* provided $\mathrm{st}(M) = M$. Properties (1) and (2) explain why the assumption that M is star-like appears in many results on lineability.

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A simple use of Zorn's lemma shows that any linear subspace V_0 of $M \cup \{0\}$ can be extended to a maximal linear subspace V of $M \cup \{0\}$. Therefore, the following concept is well defined.

Definition 1.1 (maximal lineability cardinal number)**.** Let M be any arbitrary subset of a vector space W. We define

$$\mathrm{m}\mathcal{L}(M) = \min\{\dim(V)\colon V \text{ is a maximal linear subspace of } M \cup \{0\}\}.$$

Although this notion might seem similar to that of maximal-lineability and maximal-spaceability (introduced by Bernal-González in [7]) they are, in general, not related.

In any case, (1) implies that $\mathrm{m}\mathcal{L}(M) = \mathrm{m}\mathcal{L}(\mathrm{st}(M))$.

Remark 1.2. It is easy to see that $\mathcal{HL}(M) = \mathrm{m}\mathcal{L}(M)^+$, where $\mathcal{HL}(M)$ is a homogeneous lineability number defined in [4]. (This explains why $\mathcal{HL}$ is always a successor cardinal, as shown in [4].) Clearly we have

$$\mathcal{HL}(M) = \mathrm{m}\mathcal{L}(M)^+ \le \mathcal{L}(M).$$

The inequality may be strict, as shown in [4].

For $M \subset W$ we will also consider the following *additivity* number (compare [4]), which is a generalization of the notion introduced by T. Natkaniec in [25, 26] and thoroughly studied by the first author [11–15] and F.E. Jordan [23] for $V = \mathbb{R}^{\mathbb{R}}$ (see, also, [20]):

$$A(M,W) = \min\big(\{|F|\colon F \subset W \ \&\ (\forall w \in W)(w + F \not\subset M)\} \cup \{|W|^+\}\big),$$

where $|F|$ is the cardinality of F and $w + F = \{w + f\colon f \in F\}$. Most of the times the space W, usually $W = \mathbb{R}^{\mathbb{R}}$, will be clear by the context. In such cases we will often write $A(M)$ in place of $A(M,W)$.

We are mostly interested in the topological vector spaces W. We say that $M \subset W$ is *μ-spaceable* with respect to a topology τ on W, provided there exists a τ-closed vector space $V \subset M \cup \{0\}$ of dimension μ. In particular, we can consider also the following *spaceability* cardinal number:

$$\mathcal{L}_\tau(M) = \min\{\kappa\colon M \cup \{0\} \text{ contains no } \tau\text{-closed subspace of dimension } \kappa\}.$$

Notice that $\mathcal{L}(M) = \mathcal{L}_\tau(M)$ when τ is the discrete topology.[1]

In what follows, we shall focus on spaces $W = \mathbb{R}^X$ of all functions from $X = \mathbb{R}^n$ to $\mathbb{R}$ and consider the topologies τ_u and τ_p of uniform and pointwise convergence, respectively. In particular, we write $\mathcal{L}_u(M)$ and $\mathcal{L}_p(M)$ for $\mathcal{L}_{\tau_u}(M)$ and $\mathcal{L}_{\tau_p}(M)$, respectively. Clearly

$$\mathcal{L}_p(M) \le \mathcal{L}_u(M) \le \mathcal{L}(M).$$

Recall also a series of definitions that shall be needed throughout the paper.

Definition 1.3. For $X \subseteq \mathbb{R}^n$ a function $f\colon X \to \mathbb{R}$ is said to be

- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset K of X;

[1] Of course, there might be some other topological properties distinguishing between the families M with the same value $\mathcal{L}_\tau(M)$. For example, in [2] it is shown that if M is the family of strongly singular functions in $\mathrm{CBV}[0,1]$, then $\mathcal{L}_u(M) = \mathfrak{c}^+$ and M contains a linear subspace generated by a discrete set of the cardinality $\mathfrak{c}$. Similarly, if M is the family of all nowhere differentiable functions in $C[0,1]$, then $\mathcal{L}_u(M) = \mathfrak{c}^+$, as proven in [28]. However, the linear subspace of M given in [28] is only separable.

- *Darboux* in the sense of Pawlak if $f[L]$ is a connected subset of $\mathbb{R}$ for every arc L of X (i.e., f maps path connected sets into connected sets);
- *almost continuous* (in the sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also a continuous function from X to $\mathbb{R}$;
- a *connectivity* function if the graph of $f \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X;
- *extendable* provided that there exists a connectivity function $F \colon X \times [0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$.

The above classes of functions are denoted by $\mathrm{D}(X)$, $\mathrm{D_P}(X)$, $\mathrm{AC}(X)$, $\mathrm{Conn}(X)$, $\mathrm{Ext}(X)$, and $\mathrm{PC}(X)$, respectively. The class of continuous functions from X into $\mathbb{R}$ is denoted by $\mathrm{C}(X)$. We will drop the domain X if $X = \mathbb{R}$.

Definition 1.4. A function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called

- *everywhere surjective* if $f[G] = \mathbb{R}$ for every nonempty open set $G \subset \mathbb{R}^n$;
- *strongly everywhere surjective* if $f^{-1}(y) \cap G$ has cardinality $\mathfrak{c}$ for every $y \in \mathbb{R}$ and every nonempty open set $G \subset \mathbb{R}^n$; this class was also studied in [13], under the name of $\mathfrak{c}$ strongly Darboux functions;
- *perfectly everywhere surjective* if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}^n$ (i.e., when $f^{-1}(r)$ is a Bernstein set for every $r \in \mathbb{R}$ (compare [10, chap. 7]));
- a *Jones function* (see [22]) if $f \cap F \neq \varnothing$ for every closed set $F \subset \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable.

The classes of these functions are written as $\mathrm{ES}(\mathbb{R}^n)$, $\mathrm{SES}(\mathbb{R}^n)$, $\mathrm{PES}(\mathbb{R}^n)$, and $\mathrm{J}(\mathbb{R}^n)$, respectively. We will drop the domain $\mathbb{R}^n$ if $n = 1$.

Definition 1.5. A function $f \colon \mathbb{R} \to \mathbb{R}$ has:

- the *Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$;
- the *strong Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous;
- the *weak Cantor intermediate value property* if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set C between x and y such that $f[C] \subset (f(x), f(y))$;
- *perfect roads* if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral (i.e., two sided) limit point for which $f \restriction P$ is continuous at x.

The above classes of functions shall be denoted by CIVP, SCIVP, WCIVP, and PR, respectively.

Notice that all classes defined in the above three definitions are star-like.

The text is organized as follows. In Section 2 we study the relations between additivity and maximal lineability numbers. Sections 3 and 4 focus on the set of extendable functions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Surprisingly enough, we shall obtain very different results when moving from $\mathbb{R}$ to $\mathbb{R}^n$. The lineability of some of the above functions have been recently partly studied (see, e.g., [4, 18–20]) but here we shall give definitive answers concerning the lineability and spaceability of several previous studied classes.

2. Relation between additivity and lineability numbers

The goal of this section is to examine possible values of numbers $A(M)$, $\mathrm{m}\mathcal{L}(M)$, and $\mathcal{L}(M)$ for a subset M of a linear space W over an arbitrary field K. We will concentrate on the cases when $\varnothing \neq M \subsetneq W$, since it is easy for the cases $M \in \{\varnothing, W\}$. Indeed, as it can be easily checked, one has $A(\varnothing) = \mathcal{L}(\varnothing) = 1$ and $\mathrm{m}\mathcal{L}(\varnothing) = 0$; $A(W) = |W|^+$, $\mathcal{L}(W) = \dim(W)^+$, and $\mathrm{m}\mathcal{L}(W) = \dim(W)$.

Proposition 2.1. *Let W be a vector space over a field K and let $\varnothing \neq M \subsetneq W$. Then*

(I) $2 \leq A(M) \leq |W|$ *and* $\mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$*;*
(II) *if* $A(\mathrm{st}(M)) > |K|$*, then* $A(\mathrm{st}(M)) \leq \mathrm{m}\mathcal{L}(M)$*.*

In particular, if M is star-like, then $A(M) > |K|$ implies that

(III) $A(M) \leq \mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$*.*

Proof. (I) These inequalities are easy to see.

(II) This can be proved by an easy transfinite induction. Alternatively, notice that A. Bartoszewicz and S. Głąb proved, in [4, corollary 2.3], that if $M \subset W$ is star-like and $A(M) > |K|$, then $A(M) < \mathcal{HL}(M)$. Hence, $A(\mathrm{st}(M)) > |K|$ implies that $A(\mathrm{st}(M)) < \mathcal{HL}(\mathrm{st}(M)) = \mathrm{m}\mathcal{L}(\mathrm{st}(M))^+ = \mathrm{m}\mathcal{L}(M)^+$. Therefore, $A(\mathrm{st}(M)) \leq \mathrm{m}\mathcal{L}(M)$. □

In what follows, we will restrict our attention to the star-like families, since, by Proposition 2.1, other cases could be reduced to this situation. Our next theorem shows that, for such families and under assumption that $A(M) > |K|$, the inequalities (3) constitute all that can be said on these numbers.

Theorem 2.2. *Let W be an infinite dimensional vector space over an infinite field K and let α, μ, and λ be the cardinal numbers such that $|K| < \alpha \leq \mu < \lambda \leq \dim(W)^+$. Then there exists a star-like $M \subsetneq W$ containing 0 such that $A(M) = \alpha$, $\mathrm{m}\mathcal{L}(M) = \mu$, and $\mathcal{L}(M) = \lambda$.*

The proof of this theorem will be based on the following two lemmas. The first of them shows that the theorem holds when $\alpha = \mu$, while the second shows how such an example can be modified to the general case.

Lemma 2.3. *Let W be an infinite dimensional vector space over an infinite field K and let μ and λ be the cardinal numbers such that $|K| < \mu < \lambda \leq \dim(W)^+$. Then there exists a star-like $M \subsetneq W$ containing 0 such that $A(M) = \mathrm{m}\mathcal{L}(M) = \mu$ and $\mathcal{L}(M) = \lambda$.*

Proof. For $S \subset W$, let $V(S)$ be the vector subspace of W spanned by S.

Let B be a basis for W. For $w \in W$, let $\mathrm{supp}(w)$ be the smallest subset S of B with $w \in V(S)$ and let $c_w\colon \mathrm{supp}(w) \to K$ be such that $w = \sum_{b \in \mathrm{supp}(w)} c_w(b) b$. Let E be the set of all cardinal numbers less than λ and choose a sequence $\langle B_\eta \colon \eta \in E\rangle$ of pairwise disjoint subsets of B such that $|B_0| = \mu$ and $|B_\eta| = \eta$ whenever $0 \neq \eta \in E$. Define

$$M = \mathcal{A} \cup \bigcup_{\eta \in E} V(B_\eta),$$

where

$$\mathcal{A} = \{w \in W\colon c_w(b_0) = c_w(b_1) \text{ for some } b_0 \in \operatorname{supp}(w) \cap B_0,\ b_1 \in \operatorname{supp}(w) \setminus B_0\}.$$

We will show that M is as desired.

Clearly, M is star-like and $0 \in M \subsetneq W$. Also, $\mathcal{L}(M) \geq \lambda$, since for any cardinal $\eta < \lambda$ the set M contains a vector subspace $V(B_\eta)$ with $\dim(V(B_\eta)) \geq \eta$.

To see that $A(M) \geq \mu$, choose an $F \subset W$ with $|F| < \mu$. It is enough to show that $|F| < A(M)$, that is, that there exists a $w \in W$ with $w + F \subset \mathcal{A}$. As $\operatorname{supp}(F) = \bigcup_{v \in F} \operatorname{supp}(v)$ has cardinality at most $|F| + \omega < \mu = |B_0| = |B_\mu| \leq |B \setminus B_0|$, there exist $b_0 \in B_0 \setminus \operatorname{supp}(F)$ and $b_1 \in B \setminus (B_0 \cup \operatorname{supp}(F))$. Let $w = b_0 + b_1$ and notice that $w + F \subset \mathcal{A} \subset M$, since for every $v \in F$ we have $b_0 \in \operatorname{supp}(w + v) \cap B_0$, $b_1 \in \operatorname{supp}(w + v) \setminus B_0$, and $c_{w+v}(b_0) = 1 = c_{w+v}(b_1)$.

Next notice that the inequalities $|K| < \mu \leq A(M)$ and Proposition 2.1 imply that $\mu \leq A(M) \leq \mathrm{m}\mathcal{L}(M)$. Thus, to finish the proof, it is enough to show that $\mathrm{m}\mathcal{L}(M) \leq \mu$ and $\mathcal{L}(M) \leq \lambda$.

To see that $\mathrm{m}\mathcal{L}(M) \leq \mu$, it is enough to show that $V(B_0)$ is a maximal vector subspace of M. Indeed, if V is a vector subspace of W properly containing $V(B_0)$, then there exists a non-zero $v \in V \cap V(B \setminus B_0)$. Choose a $b_0 \in B_0$ and a non-zero $c \in K \setminus \{c_v(b)\colon b \in \operatorname{supp}(v)\}$. Then $cb_0 + v \in V \setminus M$. So, $V(B_0)$ is a maximal vector subspace of M and indeed $\mathrm{m}\mathcal{L}(M) \leq \dim(V(B_0)) = \mu$.

To see that $\mathcal{L}(M) \leq \lambda$, notice that this is obvious for $\lambda = \dim(W)^+$. So, we can assume that $\lambda \leq \dim(W)$ and choose a vector subspace V of W of dimension λ. It is enough to show that $V \setminus M \neq \varnothing$. To see this, for every ordinal $\eta \leq \lambda$ let us define $\hat{B}_\eta = \bigcup\{B_\zeta\colon \zeta \in E \cap \eta\}$. Notice that

$$\text{for every } \eta < \lambda \text{ there is a non-zero } w \in V \text{ with } \operatorname{supp}(w) \cap \hat{B}_\eta = \varnothing.$$

Indeed, if $\pi_\eta\colon W = V(\hat{B}_\eta) \oplus V(B \setminus \hat{B}_\eta) \to V(\hat{B}_\eta)$ is the natural projection, then there exist distinct $w_1, w_2 \in V$ with $\pi_\eta(w_1) = \pi_\eta(w_2)$ (as $|V(\hat{B}_\eta)| < \lambda = \dim(V)$). Then $w = w_1 - w_2$ is as required.

Now, choose a non-zero $w_1 \in V$ with $\operatorname{supp}(w_1) \cap B_0 = \operatorname{supp}(w_1) \cap \hat{B}_1 = \varnothing$. Then, $w_1 \notin \mathcal{A}$ and if $\operatorname{supp}(w_1) \not\subset \hat{B}_\lambda = \bigcup_{\eta \in E} B_\eta$, then also $w_1 \notin \bigcup_{\eta \in E} V(B_\eta)$, and we have $w_1 \in V \setminus M$. Therefore, we can assume that $\operatorname{supp}(w_1) \subset \hat{B}_\lambda = \bigcup_{\eta < \lambda} \hat{B}_\eta$. Let $\eta < \lambda$ be such that $\operatorname{supp}(w_1) \subset \hat{B}_\eta$ and choose a non-zero $w_2 \in V$ with $\operatorname{supp}(w_2) \cap \hat{B}_\eta = \varnothing$. Then $w = w_2 - w_1 \in V \setminus M$ (since $w \notin \mathcal{A}$, being non-zero with $\operatorname{supp}(w) \cap B_0 = \varnothing$, and $w \notin \bigcup_{\zeta \in E} V(B_\zeta)$, as its support intersects both $\hat{B}_\eta$ and $B \setminus \hat{B}_\eta$). □

Lemma 2.4. *Let W, W_0, and W_1 be the vector spaces over an infinite field K such that $W = W_0 \oplus W_1$. Let $M \subsetneq W_0$ and*

$$\mathcal{F} = M + W_1 = \{m + w\colon m \in M \ \&\ w \in W_1\}.$$

Then

(I) *If M is star-like, then $\mathcal{F}$ is also star-like.*
(II) $A(\mathcal{F}, W) = A(M, W_0)$.
(III) *If $0 \in M$, then* $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1)$.
(IV) *If $0 \in M$ and* $\dim(W_1) < \mathcal{L}(M)$, *then* $\mathcal{L}(\mathcal{F}) = \mathcal{L}(M) + \dim(W_1)$.

Proof. In the following, let $\pi_0\colon W = W_0 \oplus W_1 \to W_0$ be the canonical projection.

(I) Let $x \in \mathcal{F}$ and $\lambda \in K \setminus \{0\}$. Since M is star-like and $\pi_0(x) \in M$, we have that $\pi_0(\lambda x) = \lambda\pi_0(x) \in M$, and hence $\lambda x \in M + W_1 = \mathcal{F}$.

(II) Let us see that $A(M, W_0) \leq A(\mathcal{F}, W)$. To this end, let $\kappa < A(M, W_0)$. We need to prove that $\kappa < A(\mathcal{F}, W)$. Indeed, if $F \subset W$ and $|F| = \kappa$, then $|\pi_0[F]| \leq |F| = \kappa$. So, there exists a $w_0 \in W_0$ such that $\pi_0[w_0 + F] = w_0 + \pi_0[F] \subset M$, that is, $w_0 + F \subset M + W_1 = \mathcal{F}$. Therefore, $\kappa < A(\mathcal{F}, W)$.

To see that $A(\mathcal{F}, W) \leq A(M, W_0)$ let $\kappa < A(\mathcal{F}, W)$. We need to show that $\kappa < A(M, W_0)$. Indeed, let $F \subset W_0$ be such that $|F| = \kappa$. Since $|F| < A(\mathcal{F}, W)$, there is a $w \in W$ with $w + F \subset \mathcal{F}$. Then $\pi_0(w) \in W_0$ and $\pi_0(w) + F = \pi_0[w + F] \subset \pi_0[\mathcal{F}] = M$, so indeed $\kappa < A(M)$.

(III) First notice that it is enough to show that

$$V \text{ is a maximal vector subspace of } \mathcal{F} \text{ if, and only if, } V = V_0 + W_1, \text{ where } V_0 \text{ is a maximal vector subspace of } M. \tag{3}$$

Indeed, if V is a maximal vector subspace of $\mathcal{F}$ with $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(V)$, then, by (3), $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(V) = \dim(V_0) + \dim(W_1) \geq \mathrm{m}\mathcal{L}(M) + \dim(W_1)$. Conversely, if V_0 is a maximal vector subspace of M with $\mathrm{m}\mathcal{L}(M) = \dim(V_0)$, then we have $\mathrm{m}\mathcal{L}(M) + \dim(W_1) = \dim(V_0) + \dim(W_1) = \dim(V_0 + W_1) \geq \mathrm{m}\mathcal{L}(\mathcal{F})$.

To see (3), take a maximal vector subspace V of $\mathcal{F}$. Notice that $W_1 \subset V$, since $V \subset V + W_1 \subset \mathcal{F} + W_1 = \mathcal{F}$ and so, by maximality, $V + W_1 = V$. In particular, $V = V_0 + W_1 \subset \mathcal{F} = M + W_1$, where $V_0 = \pi_0[V]$. Thus, V_0 is a vector subspace of M. It must be maximal, since for any its proper extension $\hat{V}_0 \subset M$, the vector space $\hat{V}_0 + W_1 \subset \mathcal{F}$ would be a proper extension of V.

Conversely, if V_0 is a maximal vector subspace of M, then $V = V_0 + W_1$ is a vector subspace of $\mathcal{F}$. If cannot have a proper extension $\hat{V} \subset \mathcal{F}$, since then the vector space $\pi_0[\hat{V}] \subset M$ would be a proper extension of V_0.

(IV) To see that $\mathcal{L}(\mathcal{F}) \leq \dim(W_1) + \mathcal{L}(M)$, choose a vector space $V \subset \mathcal{F}$. We need to show that $\dim(V) < \dim(W_1) + \mathcal{L}(M)$. Indeed, $V_1 = V + W_1$ is a vector subspace of $\mathcal{F} + W_1 = \mathcal{F}$ and $\dim(V) \leq \dim(V_1) = \dim(W_1) + \dim(\pi_0[V_1])$, since $V_1 = W_1 \oplus \pi_0[V_1]$. Therefore, $\dim(V) \leq \dim(W_1) + \dim(\pi_0[V_1]) < \dim(W_1) + \mathcal{L}(M)$, since $\dim(W_1) < \mathcal{L}(M)$ and $\dim(\pi_0[V_1]) < \mathcal{L}(M)$, as $\pi_0[V_1]$ is a vector subspace of $M = \pi_0[\mathcal{F}]$. So, $\mathcal{L}(\mathcal{F}) \leq \dim(W_1) + \mathcal{L}(M)$.

To see that $\dim(W_1) + \mathcal{L}(M) \leq \mathcal{L}(\mathcal{F})$, choose a $\kappa < \dim(W_1) + \mathcal{L}(M)$. We need to show that $\kappa < \mathcal{L}(\mathcal{F})$, that is, that there exists a vector subspace V of $\mathcal{F}$ with $\dim(V) \geq \kappa$. First, notice that $\dim(W_1) < \mathcal{L}(M)$ and $\kappa < \dim(W_1) + \mathcal{L}(M)$ imply that there exists a $\mu < \mathcal{L}(M)$ such that $\kappa \leq \dim(W_1) + \mu < \dim(W_1) + \mathcal{L}(M)$. (For finite values of $\mathcal{L}(M)$, take $\mu = \max\{\kappa - \dim(W_1), 0\}$; for infinite $\mathcal{L}(M)$, the number $\mu = \max\{\kappa, \dim(W_1)\}$ works.) Choose a vector subspace V_0 of M with $\dim(V_0) \geq \mu$. Then the vector subspace $V = V_0 + W_1 = V_0 \oplus W_1$ of $\mathcal{F}$ is as desired, since we have $\dim(V) = \dim(W_1) + \dim(V_0) \geq \dim(W_1) + \mu \geq \kappa$. □

Proof of Theorem 2.2. Represent W as $W_0 \oplus W_1$, where $\dim(W_0) = \lambda$ and $\dim(W_1) = \mu$. Use Lemma 2.3 to find a star-like $M \subsetneq W_0$ containing 0 such that $A(M, W_0) = \mathrm{m}\mathcal{L}(M) = \alpha$ and $\mathcal{L}(M) = \lambda$. Let $\mathcal{F} = M + W_1 \subsetneq B$. Then, by Lemma 2.4, $\mathcal{F} \ni 0$ is star-like such that $A(\mathcal{F}) = A(M, W_0) = \alpha$, $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1) = \alpha + \mu = \mu$, and $\mathcal{L}(\mathcal{F}) = \mathcal{L}(M) + \dim(W_2) = \lambda + \alpha = \lambda$, as required. □

A. Bartoszewicz and S. Głąb have asked [4, open question 1] whether the inequality $A(\mathcal{F})^+ \geq \mathcal{HL}(\mathcal{F})$ (which is equivalent to $A(\mathcal{F}) \geq \mathrm{m}\mathcal{L}(\mathcal{F})$) holds for any family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. Of course, for the star-like families $\mathcal{F}$ with $A(\mathcal{F}) > \mathfrak{c}$, a positive answer to this question would mean that, under these assumptions, we have $A(\mathcal{F}) = \mathrm{m}\mathcal{L}(\mathcal{F})$. Notice that Theorem 2.2 gives, in particular, a negative answer to this question.

We do not have a comprehensive example, similar to that provided by Theorem 2.2, for the case when $A(M) \leq |K|$. However, the machinery built above, together with the results from [4], lead to the following result.

Theorem 2.5. *Let W be a vector space over an infinite field K with $\dim(W) \geq 2^{|K|}$. If $2 \leq \kappa \leq |W|$, there exists a star-like family $\mathcal{F} \subsetneq W$ containing 0 such that $A(\mathcal{F}) = \kappa$ and $\mathrm{m}\mathcal{L}(\mathcal{F}) = \dim(W)$ (so that $\mathcal{L}(\mathcal{F}) = \dim(W)^+$).*

Proof. Represent W as $W = W_0 \oplus W_1$, where $\dim(W_0) = 2^{|K|}$ and $\dim(W_1) = \dim(W)$. If $2 \leq \kappa \leq |K|$, then, by [4, Theorem 2.5], there exists a star-like family $M \subset W_0$ such that $A(M, W_0) = \kappa$. Notice that the family constructed in that result contains 0. Then, by Lemma 2.4, the family $\mathcal{F} = M + W_1$ satisfies that $A(\mathcal{F}) = A(M, W_0) = \kappa$ and $\mathrm{m}\mathcal{L}(\mathcal{F}) = \mathrm{m}\mathcal{L}(M) + \dim(W_1) = \dim(W)$. □

3. Spaceability of Darboux-like functions on $\mathbb{R}$

Recall (see, e.g., [12, chart 1] or [11]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$. The next theorem, strengthening the results presented in the table from [8, page 14], determines fully the lineability, $\mathcal{L}$, and spaceability, $\mathcal{L}_p$, numbers for these classes.

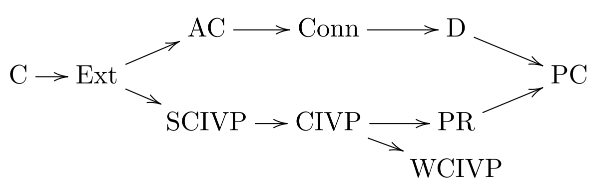

Figure 1. Relations between the Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$. Arrows indicate strict inclusions.

Theorem 3.1. $\mathcal{L}_p(\mathrm{Ext}) = (2^{\mathfrak{c}})^+$. *In particular, all Darboux-like classes of functions from Figure 1, except* C, *are $2^{\mathfrak{c}}$-spaceable with respect to the topology of pointwise convergence.*

Proof. In [15, corollary 3.4] it is shown that there exists an $f \in \mathrm{Ext}$ and an F_σ first category set $M \subset \mathbb{R}$ such that

$$\text{if } g \in \mathbb{R}^{\mathbb{R}} \text{ and } g \restriction M = f \restriction M, \text{ then } g \in \mathrm{Ext}. \tag{4}$$

It is easy to see that for any real number $r \neq 0$ the function rf satisfies the same property.

Notice also that there exists a family $\{ h_\xi \in \mathbb{R}^{\mathbb{R}} \colon \xi < \mathfrak{c} \}$ of increasing homeomorphisms such that the sets $M_\xi = h_\xi[M]$, $\xi < \mathfrak{c}$, are pairwise disjoint. (See, e.g., [15, lemma 3.2].) It is easy to see that each function $f_\xi = f \circ h_\xi^{-1}$ satisfies (4) with the set M_ξ. Increasing one of the sets M_ξ, if necessary, we can also assume that $\{M_\xi\colon \xi<\mathfrak{c}\}$ is a partition of $\mathbb{R}$. Let $\vec{f}=\langle f_\xi\restriction M_\xi\colon \xi<\mathfrak{c}\rangle$ and define

$$V(\vec{f})=\left\{\bigcup_{\xi<\mathfrak{c}} t(\xi)(f_\xi\restriction M_\xi)\colon t\in\mathbb{R}^{\mathfrak{c}}\right\}. \tag{5}$$

It is easy to see that $V(\vec{f})$ is $2^{\mathfrak{c}}$-dimensional τ_p-closed linear subspace of Ext. □

As the cardinality of the family $\mathcal{B}$or of Borel functions from $\mathbb{R}$ to $\mathbb{R}$ is $\mathfrak{c}$, Theorem 3.1 easily implies that Ext $\setminus\,\mathcal{B}$or is $2^{\mathfrak{c}}$-lineable: $\mathcal{L}(\text{Ext}\setminus\mathcal{B}\text{or})=(2^{\mathfrak{c}})^+$. Actually, we have an even stronger result:

Proposition 3.2. $\mathcal{L}_p(\text{Ext}\cap\text{SES}\setminus\mathcal{B}\text{or})=(2^{\mathfrak{c}})^+$.

Proof. The function $f\restriction M$ satisfying (4) may also have the property that

$$M \text{ is } \mathfrak{c}\text{-dense in } \mathbb{R} \text{ and } f\restriction M \text{ is SES non-Borel.} \tag{6}$$

Indeed, this can be ensured by enlarging M by a $\mathfrak{c}$-dense first category set $N\subset\mathbb{R}\setminus M$ and redefining f on N so that $f\restriction N$ is non-Borel and SES.

Now, if f satisfies both (4) and (6) and $\vec{f}=\langle f_\xi\restriction M_\xi\colon \xi<\mathfrak{c}\rangle$ is defined as in Theorem 3.1, then the space $V(\vec{f})$ given in (5) is as required. □

Notice also that $\text{Ext}\cap\text{PES}=\text{PR}\cap\text{PES}=\varnothing$. In particular, the space V from Proposition 3.2 is disjoint with PES.

Remark 3.3. Clearly, Theorem 3.1 implies that Ext is $2^{\mathfrak{c}}$-lineable. This result has been also independently proved by T. Natkaniec in [27]. The idea used in [27] is similar, but the technique is different from that used in the proof of Theorem 3.1. The similar technique was also used in the recent papers [3, 5].

Recall, that it is known that $\mathcal{L}(\text{AC}\setminus\text{Ext})=(2^{\mathfrak{c}})^+$. See [19] or [8, page 14]. However, we do not know what the exact values of the following cardinals are.

Problem 3.4. Determine the following numbers:

$$\mathcal{L}_p(\mathcal{F}\setminus\mathcal{G}),\ \mathcal{L}_u(\mathcal{F}\setminus\mathcal{G}),\ \text{and}\ \mathcal{L}(\mathcal{F}\setminus\mathcal{G})$$

for $\mathcal{F}\in\{\text{Conn}\setminus\text{AC},\text{D}\setminus\text{Conn},\text{PC}\setminus\text{D}\}$ and $\mathcal{G}\in\{\text{SCIVP},\text{CIVP},\text{PR}\}$.

Recall (see [15] or [11]) that for every $\mathcal{F}\in\{\text{Ext},\text{AC},\text{Conn},\text{D}\}$ we have $A(\mathcal{F})\ge\mathfrak{c}^+$ and so, by Proposition 2.1,

$$\mathfrak{c}^+\le A(\mathcal{F})\le \text{m}\mathcal{L}(\mathcal{F})<\mathcal{L}(\mathcal{F})\le(2^{\mathfrak{c}})^+. \tag{7}$$

In particular, under the generalized continuum hypothesis GCH we have $A(\mathcal{F})=\text{m}\mathcal{L}(\mathcal{F})=2^{\mathfrak{c}}$ and $\text{m}\mathcal{L}(\mathcal{F})^+=\mathcal{L}(\mathcal{F})=(2^{\mathfrak{c}})^+$. However, without the GCH the situation is less clear. Of course, by Theorem 3.1, the value of $\mathcal{L}(\mathcal{F})$ is determined to be $(2^{\mathfrak{c}})^+$, reducing the inequalities of (7) to $\mathfrak{c}^+\le A(\mathcal{F})\le\text{m}\mathcal{L}(\mathcal{F})\le 2^{\mathfrak{c}}$. At the same time, it is consistent with ZFC that $A(\mathcal{F})<2^{\mathfrak{c}}$. (See [13] or [11].) In such situation, the exact position of the number $\text{m}\mathcal{L}(\mathcal{F})$ between $A(\mathcal{F})$ and $2^{\mathfrak{c}}$ is unclear, leading to the following problem.

Problem 3.5. Let $\mathcal{F}\in\{\text{Ext},\text{AC},\text{Conn},\text{D}\}$. Is it consistent with the axioms of set theory ZFC that $A(\mathcal{F})<\text{m}\mathcal{L}(\mathcal{F})$? What about the consistency of $\text{m}\mathcal{L}(\mathcal{F})<2^{\mathfrak{c}}$?

It is worth to mention, that the formula (7) is also true when $\mathcal{F}$ is the class $\mathcal{SZ}$ of the Sierpiński-Zygmund functions. Once again, it is consistent with ZFC that $A(\mathcal{SZ}) < 2^{\mathfrak{c}}$, as proved in [14]. However, in contrast with Theorem 3.1, it is also consistent with ZFC that $\mathcal{L}(\mathcal{SZ}) < (2^{\mathfrak{c}})^{+}$. (See [21]; compare also [6].)

4. Spaceability of Darboux-like functions on $\mathbb{R}^n$, $n \geq 2$

Recall (see, e.g., [12, chart 2] or [11]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}^n$ to $\mathbb{R}$ for $n \geq 2$.

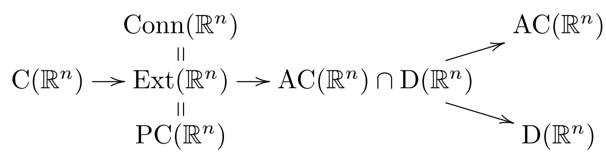

Figure 2. Relations between the Darboux-like classes of functions from $\mathbb{R}^n$ to $\mathbb{R}$, $n \geq 2$. Arrows indicate strict inclusions.

The proof of the next theorem will be based on the following result [16, Proposition 2.7]:

Proposition 4.1. *Let $n > 0$ and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Then for any $x_0 \in \mathbb{R}^n$ and any open set W in $\mathbb{R}^n$ containing x_0, there exists an open set $U \subseteq W$ such that $x_0 \in U$ and the restriction of f to $\operatorname{bd}(U)$ is continuous. Moreover, given any $\varepsilon > 0$, the set U can be chosen so that $|f(x_0) - f(y)| < \varepsilon$ for every $y \in \operatorname{bd}(U)$.*

Theorem 4.2. *For $n \geq 2$, $\mathcal{L}_p(\mathrm{Ext}(\mathbb{R}^n)) = \mathcal{L}_u(\mathrm{Ext}(\mathbb{R}^n)) = \mathcal{L}(\mathrm{Ext}(\mathbb{R}^n)) = \mathfrak{c}^+$. In particular, the classes $\mathrm{C}(\mathbb{R}^n)$ and $\mathrm{Ext}(\mathbb{R}^n)$ are $\mathfrak{c}$-spaceable with respect to the pointwise convergence topology τ_p but are not $\mathfrak{c}^+$-lineable.*

Proof. First, notice that $\mathcal{L}_p(\mathrm{C}(\mathbb{R}^n)) = \mathfrak{c}^+$ is justified by the space C_0 of all continuous functions linear on the interval $[k, k+1]$ for every integer $k \in \mathbb{Z}$. Indeed, C_0 is linearly isomorphic to $\mathbb{R}^{\mathbb{Z}}$.

Now, since $\mathfrak{c}^+ = \mathcal{L}_p(\mathrm{C}(\mathbb{R}^n)) \leq \mathcal{L}_p(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathcal{L}_u(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathcal{L}(\mathrm{Ext}(\mathbb{R}^n))$, it is enough to show that $\mathcal{L}(\mathrm{Ext}(\mathbb{R}^n)) \leq \mathfrak{c}^+$, that is, that $\mathrm{Ext}(\mathbb{R}^n)$ is not $\mathfrak{c}^+$-lineable. To see this, by way of contradiction, assume that there exists a vector space $V \subset \mathrm{Ext}(\mathbb{R}^n)$ of cardinality greater than $\mathfrak{c}$. Fix a countable dense set $D \subset \mathbb{R}^n$ and let $\langle\langle x_k, \varepsilon_k\rangle \colon k < \omega\rangle$ be an enumeration of $D \times \{2^{-m} \colon m < \omega\}$. By Proposition 4.1, for every function $f \in \mathrm{Ext}(\mathbb{R}^n)$ and $k < \omega$ we can choose an open neighborhood U_k^f of x_k of the diameter at most ε_k such that $f \restriction \operatorname{bd}(U_k^f)$ is continuous. Consider the mapping $V \ni f \mapsto T_f = \langle f \restriction \operatorname{bd}(U_k^f) \colon k < \omega\rangle$. Since its range has cardinality $\mathfrak{c}$, there are distinct $f_1, f_2 \in V$ with $T_{f_1} = T_{f_2}$. In particular, $f = f_1 - f_2 \in V$ is equal zero on the set $M = \bigcup_{k<\omega} \operatorname{bd}(U_k^{f_1})$. Notice that the complement M^c of M is zero-dimensional. We will show that f is not extendable, by showing that it does not satisfy Proposition 4.1.

Indeed, since $f_1 \neq f_2$, there is an $x \in \mathbb{R}^n$ with $f(x) \neq 0$. Let $\varepsilon = |f(x)|$ and let W be any bounded neighborhood of x. Then, there is no set U as required by Proposition 4.1.

To see this, notice that for any open set $U \subseteq W$ with $x \in U$, its boundary is of dimension at least 1. In particular, $M \cap \mathrm{bd}(U) \neq \varnothing$ and, for $y \in M \cap \mathrm{bd}(U)$, we have $|f(x) - f(y)| = |f(x)| = \varepsilon$. □

Theorem 4.2 determines the values of the numbers $\mathcal{L}_p(\mathcal{F})$, $\mathcal{L}_u(\mathcal{F})$, and $\mathcal{L}(\mathcal{F})$ for $\mathcal{F} \in \{\mathrm{C}(\mathbb{R}^n), \mathrm{Ext}(\mathbb{R}^n), \mathrm{Conn}(\mathbb{R}^n), \mathrm{PR}(\mathbb{R}^n)\}$ and $n \geq 2$. In the remainder of this section we will examine these cardinal numbers for the remaining classes from the diagram in Figure 2. For this, we will need the following fact, improving a recent result of the second author. (See [18, Theorem 2.2].)

Proposition 4.3. $\mathcal{L}_p(J(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$ *for every* $n \geq 1$*. In particular, the families* $J(\mathbb{R}^n)$*,* $\mathrm{PES}(\mathbb{R}^n)$*,* $\mathrm{SES}(\mathbb{R}^n)$*, and* $\mathrm{ES}(\mathbb{R}^n)$ *are* $2^{\mathfrak{c}}$*-spaceable with respect to the topology of pointwise convergence.*

Proof. Let $\{M_\xi \colon \xi < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}^n$ into pairwise disjoint Bernstein sets. For every $\xi < \mathfrak{c}$, let $f_\xi \colon M_\xi \to \mathbb{R}$ be such that $f_\xi \cap F \neq \varnothing$ for every closed set $F \subset \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable. (All of this can be easily constructed by transfinite induction. See, e.g., [10].) Notice that

$$\text{if } g \in \mathbb{R}^{\mathbb{R}} \text{ and } g \restriction M_\xi = r\ f_\xi \text{ for some } \xi < \mathfrak{c} \text{ and } r \neq 0, \text{ then } g \in J(\mathbb{R}^n).$$

Now, if $\vec{f} = \langle f_\xi \restriction M_\xi \colon \xi < \mathfrak{c}\rangle$ and $V(\vec{f})$ is given by (5), then $V(\vec{f})$ is $2^{\mathfrak{c}}$-dimensional τ_p-closed linear subspace of $J(\mathbb{R}^n)$. □

Every function in $\mathrm{J}(\mathbb{R}^n)$ is surjective. In particular, the above result implies that the class of surjective functions is $2^{\mathfrak{c}}$-lineable. One could also wonder about the lineability of the family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$, given below.

Remark 4.4. The family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$ is 1-lineable but not 2-lineable.

Proof. Clearly the family is 1-lineable. To see that is not 2-lineable, choose two injective linearly independent functions f and g generating a linear space Z. Take arbitrary $x \neq y$ in $\mathbb{R}^n$ and consider the function $h = f + \alpha g \in Z \setminus \{0\}$, where $\alpha = (f(x) - f(y))/(g(y) - g(x)) \in \mathbb{R}$. Then, we have $h(x) = h(y)$, so Z contains a function which is not one-to-one. □

Other examples of 1-lineable but not 2-lineable sets and, in general, not lineable sets can be found in [8, 9].

Theorem 4.5. *For* $n \geq 2$*,* $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$*. In particular, the class* $\mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$ *is* $2^{\mathfrak{c}}$*-spaceable and* $\mathcal{L}_p(\mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$*.*

Proof. By Proposition 4.3, it is enough to show that $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$. Clearly, $\mathrm{J}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{PES}(\mathbb{R}^n)$ for any $n \geq 1$. Thus, it is enough to show that $\mathrm{PES}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) = \varnothing$ for $n \geq 2$. But this follows immediately from the fact that, under $n \geq 2$, every Bernstein set in $\mathbb{R}^n$ is connected. □

Remark 4.6. Notice that, since $\mathrm{AC}(\mathbb{R}^n) \subset \mathrm{D_P}(\mathbb{R}^n)$, then, for $n \geq 2$, we have $\mathcal{L}_p(\mathrm{D_P}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$. So, $\mathrm{D_P}(\mathbb{R}^n) \setminus \mathrm{D}(\mathbb{R}^n)$ is also $2^{\mathfrak{c}}$-spaceable.

Theorem 4.7. *For* $n \geq 2$*,* $\mathcal{L}_p(\mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)) = (2^{\mathfrak{c}})^+$*. In particular, the class* $\mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)$ *is* $2^{\mathfrak{c}}$*-spaceable.*

Proof. Let $\pi_1 \colon \mathbb{R}^n \to \mathbb{R}$ the projection of $\mathbb{R}^n$ on its first coordinate. Let $W = V(\vec{f}) \subset \mathrm{J}$ be the vector space of cardinality $2^{\mathfrak{c}}$ build in Proposition 4.3. Then the vector space

$$V = \{\, f \circ \pi_1 : f \in W \,\}$$

is obviously contained in $\mathrm{D}(\mathbb{R}^n)$ and has dimension $2^{\mathfrak{c}}$. On the other side, if $f \in W$ then $f \circ \pi_1$ cannot be in $\mathrm{AC}(\mathbb{R}^n)$, because then f would be continuous. (See [24].) This is not possible, because $\mathrm{J} \cap \mathrm{C} = \varnothing$. Therefore, $V \subset \mathrm{D}(\mathbb{R}^n) \setminus \mathrm{AC}(\mathbb{R}^n)$. To finish, let us remark that the space V is also closed by pointwise convergence. □

Remark 4.8. Notice that, in $\mathbb{R}^n$ (for every $n \in \mathbb{N}$), we have that $\mathrm{AC} \setminus \mathrm{Ext}$ is $2^{\mathfrak{c}}$-spaceable (since this class contains the Jones functions). Also, in $\mathbb{R}$, $\mathrm{J} \subset \mathrm{AC} \setminus \mathrm{SCIVP} \subset \mathrm{AC} \setminus \mathrm{Ext}$ and, since $\mathcal{L}_p(\mathrm{J}) = (2^{\mathfrak{c}})^+$, we have (from the previous results) that

$$\mathcal{L}_p(\mathrm{AC} \setminus \mathrm{Ext}) = \mathcal{L}_u(\mathrm{AC} \setminus \mathrm{Ext}) = (2^{\mathfrak{c}})^+.$$

Problem 4.9. For $n \geq 2$, determine the values of the numbers $\mathcal{L}_p(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$, $\mathcal{L}_u(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$, and $\mathcal{L}(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$.

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