Restricted continuity and a theorem of Luzin

Krzysztof Chris Ciesielski[a,b] and Joseph Rosenblatt[c] *

[a]Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310

[b]Department of Radiology, MIPG, University of Pennsylvania, Blockley Hall – 4th Floor, 423 Guardian Drive, Philadelphia, PA 19104-6021

[c]Department of Mathematics University of Illinois, Urbana, IL 61801

2014/5/5

Abstract

Let $P(X,\mathcal{F})$ denotes the property: *For every function $f\colon X\times\mathbb{R}\to\mathbb{R}$, if $f(x,h(x))$ is continuous for every function $h\colon X\to\mathbb{R}$ from $\mathcal{F}$, then f is continuous.* In this paper we investigate the assumptions of a theorem of Luzin [13], which states that $P(\mathbb{R},\mathcal{F})$ holds for $X=\mathbb{R}$ and $\mathcal{F}$ being the class $C(X)$ of all continuous functions from X to $\mathbb{R}$. The question for which topological spaces $P(X,C(X))$ holds was investigated by Dalbec in [5]. Here, we examine $P(\mathbb{R}^n,\mathcal{F})$ for different families $\mathcal{F}$. In particular, we notice that: (1) $P(\mathbb{R}^n,\text{“}C^1\text{”})$ holds, where “C^1” is the family of all functions in $C(\mathbb{R}^n)$ having continuous directional derivatives, allowing infinite values; (2) this result is the best possible, since $P(\mathbb{R}^n,D^1)$ is false, where D^1 is the family of all differentiable functions (no infinite derivatives allowed).

We notice, that if $\mathcal{D}$ is the family of the graphs of functions from $\mathcal{F}\subseteq C(X)$, then $P(X,\mathcal{F})$ is equivalent to the property $P^*(X,\mathcal{D})$: *For every function $f\colon X\times\mathbb{R}\to\mathbb{R}$, if $f\restriction D$ is continuous for every $D\in\mathcal{D}$, then f is continuous.* Note that if $\mathcal{D}$ is the family of all lines in in $\mathbb{R}^n$, then, for $n\geq 2$, $P^*(\mathbb{R}^n,\mathcal{D})$ is false, since there are discontinuous linearly continuous functions on $\mathbb{R}^n$. In this direction, we prove that: (1) there exists a Baire class 1 function $h\colon\mathbb{R}^n\to\mathbb{R}$ such that $P^*(\mathbb{R}^n,T(h))$ holds, where $T(H)$ stands for all possible translations of $H\subset\mathbb{R}^n\times\mathbb{R}$; (2) this result is the best possible, since $P^*(\mathbb{R}^n,T(h))$ is false for any $h\in C(\mathbb{R}^n)$. We also notice that $P^*(\mathbb{R}^n,T(Z))$ holds for any Borel $Z\subseteq\mathbb{R}^n\times\mathbb{R}$ either of positive measure or of second category. Finally, we give an example of a perfect nowhere dense $Z\subseteq\mathbb{R}^n\times\mathbb{R}$ of measure zero for which $P^*(\mathbb{R}^n,T(Z))$ holds.

1 Background

The standard way we teach calculus follows, in its outline, the path of the historical development of real analysis: we start with teaching the theory of functions of one variable, $f\colon\mathbb{R}\to\mathbb{R}$; only after this theory is mastered do we turn our attention to the theory of multivariable functions $g\colon\mathbb{R}^n\to\mathbb{R}^m$, generalizing the one-variable results. But how easily can this generalization be made? Even if we restrict our attention just to the continuity, the transition is not that simple. True, the dimension of the range of a function is not a problem, as $f=\langle f_1,\ldots,f_m\rangle\colon\mathbb{R}^n\to\mathbb{R}^m$ is continuous if, and only if, every coordinate function $f_i\colon\mathbb{R}^n\to\mathbb{R}$ is continuous. However, we do not know any analogous simple-minded reduction of the dimension of the domain of a function, when studying the continuity of $f\colon\mathbb{R}^n\to\mathbb{R}$. The prehistory of this problem can be traced to the 1821 mathematical analysis textbook of Cauchy [2], where the author, working with the set of $\mathscr{R}$ of reals with infinitesimals, proves that $f\colon\mathscr{R}^2\to\mathscr{R}$ is continuous if, and only if,

(SC) f is continuous with respect to each variable separately, that is, the mappings $\mathscr{R}\ni t\mapsto f(x,t)\in\mathscr{R}$ and $\mathscr{R}\ni t\mapsto f(t,y)\in\mathscr{R}$ are continuous for every $x,y\in\mathscr{R}$.

*Mathematics Subject Classification: Primary 26B05; Secondary 26B35.
Key words: continuity, separate continuity, translations

The fact, that this result is false when $\mathscr{R}$ is replaced with the standard set $\mathbb{R}$ of real numbers (with no infinitesimals) was not observed for several decades. A first counterexample to Cauchy's claim, due to E. Heine (see [14]), appeared in the 1870 calculus text of J. Thomae [17]. The following well-known counterexample, included in many calculus books,

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{for } \langle x,y\rangle \neq \langle 0,0\rangle \text{ and } f(0,0)=0 \tag{1}$$

comes from the 1884 treatise on calculus by Genocchi and Peano [9]. The class of functions satisfying (SC), called *separately continuous*, was studied by many prominent mathematicians: Volterra (see Baire [1, p. 95]), Baire (1899, see [1]), Lebesgue (1905, see [12, pp. 201-202]), and Hahn (1919, see [10]). In particular, these studies led to the introduction of Baire's classification of functions.

Of course, the function (1) while discontinuous, is continuous when restricted to any straight line. Lebesgue [12] gave an example showing that the continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point is not insured by the continuity at this point along the graphs of all analytic functions. These results show that, to insure the continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$, it is not enough to test the continuity of the restriction of f to all straight lines or to all graphs of analytic functions. Along these lines, the major break through is the 1955 result of Rosenthal [16]:

($*$) For any function $f\colon \mathbb{R}^2 \to \mathbb{R}$, if its restriction is continuous when restricted to every graph of continuously differentiable functions, from x to y or from y to x, then f is continuous. However, the implication is false when considering only the restrictions to the graphs of twice continuously differentiable functions.

In particular, to verify the continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$, it is enough to test for its continuity along the (one-dimensional) graphs of C^1 (i.e., continuously differentiable) functions from x to y or from y to x; however testing for its continuity along the graphs of twice continuously differentiable functions is not enough.

In the above characterization, in terms of the graphs of C^1 functions, is it necessary to include also the graphs of functions from y to x? The answer is negative, if we restrict our attention to the graphs of continuous functions, as implied by the following theorem of Luzin [13]: *For every function $f\colon \mathbb{R}^2 \to \mathbb{R}$, if $f(x, h(x))$ is continuous for every continuous $h\colon X \to \mathbb{R}$, then f is continuous.*

The goal of this paper is to further study of how small $\mathcal{F}$ can be, where $\mathcal{F}$ is a family of functions of one variable, so that the continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$ can be insured by testing only the restrictions of f to the graphs of functions from $\mathcal{F}$. In particular, we show that such characterization holds for the class of "C^1" but is false for the class D^1 of differentiable functions.

We also investigate problem of whether there exists a single continuity testing function $g\colon \mathbb{R} \to \mathbb{R}$, for the functions of two variables, where we take the graph of g together with all its translations. We show that indeed, such a function g exists in the class of Baire class one function (i.e., g is a limit of continuous functions). However, no graph of a continuous function can be enough for such testing.

2 Preliminaries and notation

For the topological spaces W and Z and a family $\mathcal{D}$ of subsets of W, we say that a function $f\colon W \to Z$ is $\mathcal{D}$-continuous, provided the restriction $f \restriction D$ of f to D is continuous for every $D \in \mathcal{D}$. We investigate a question:

(Q) For which families $\mathcal{D}$, does $\mathcal{D}$-continuity imply continuity?

For W being a product $X \times Y$ and $\mathcal{D} = \{X \times \{y\} \colon y \in Y\} \cup \{\{x\} \times Y \colon x \in X\}$, $\mathcal{D}$-continuity is known as the *separate continuity* and was studied intensively. (See e.g. [14, 15, 3, 4] and the literature cited therein.) Of course, for most spaces, separate continuity of $f\colon X \times Y \to Z$ does not imply its continuity. For $W = \mathbb{R}^n$ and $\mathcal{D}$ being the family of all straight lines in $\mathbb{R}^n$, $\mathcal{D}$-continuity is known as *linear continuity*. (See e.g. [15, 3, 4].) Once again, linear continuity of $f\colon \mathbb{R}^n \to \mathbb{R}$ does not imply its continuity for $n > 1$.

In this paper we investigate (Q) for $W = X \times Y$ and $\mathcal{D}$ consisting of functions $h\colon X \to Y$, identified with their graphs. We will concentrate on the case when $X = \mathbb{R}^n$ and $Y = Z = \mathbb{R}$, but state the results in a more general setting, whenever it is possible. Notice the following straightforward relation between the continuities of $f \restriction h$ and of a function $f_h\colon X \to Z$ given as $f_h(x) = f(x, h(x))$.

Fact 1 *Let $f\colon X \times Y \to Z$ and $h\colon X \to Y$.*

(a) *If $f_h(x) = f(x, h(x))$ is continuous, then so is $f \restriction h$.*

(b) *If h and $f \restriction h$ are continuous, then so is f_h.*

(c) *f_h need not be continuous if h is discontinuous, even when f is continuous.*

PROOF. (a) $f \restriction h$ is a composition of two continuous functions, since $(f \restriction h)(x, y) = (f_h \circ \pi)(x, y)$, where $\pi\colon X \times Y \to X$ is the projection onto the first coordinate.

(b) $f_h = (f \restriction h) \circ \langle id, h\rangle$ is a composition of two continuous functions, where $id(x) = x$.

(c) Let $f(x, y) = y$. Then f is continuous and $f_h = h$. ■

Fact 1 shows that the assumption "$f \restriction h$ is continuous" is weaker than "f_h is continuous;" however, they are equivalent, when h is continuous. In Sections 3 and 4 we will consider only families consisting of continuous functions h, making the distinction irrelevant. In Section 5, we consider discontinuous functions h in which case only a weaker assumptions "f_h is continuous" makes sense.

In what follows, we use a convention that if a sequence $\bar{w} = \langle w_k \in W\colon k < \omega\rangle$ is convergent in W, then its limit is the first element, w_0, of the sequence. The same will be true for the subsequences $\langle w_{k_i}\colon i < \omega\rangle$ of $\bar{w}$, that is, we always impose that $w_{k_0} = w_0$ is its limit. In addition, a one-to-one sequence $\bar{w}$ will be identified with the set $\{w_k\colon k < \omega\}$ of its values. Let $\mathcal{S}(W)$ be the family of all one-to-one sequences $\{w_k \in W\colon k < \omega\}$ converging in W. (So, $\lim_{k\to\infty} w_k = w_0$.) The following two simple facts are used in the sequel. We include their proofs here to keep the paper self-contained.

Fact 2 *Let W be Hausdorff and $\mathcal{S} \subset \mathcal{S}(W)$. If $\mathcal{D}$ is a family of subsets of W such that*

(A) *for every subsequence $S \in \mathcal{S}(W)$ of an $S_0 \in \mathcal{S}$ there exists a $D \in \mathcal{D}$ such that $D \cap S \in \mathcal{S}(W)$,*

then every $\mathcal{D}$-continuous function $f\colon W \to Z$ is $\mathcal{S}$-continuous.

PROOF. Assume that f is $\mathcal{D}$-continuous and fix an $S_0 = \{s_k\colon k < \omega\} \in \mathcal{S}$. We need to show that $f \restriction S_0$ is continuous. As W is Hausdorff, every $s_k \in S$ with $k > 0$ is an isolated point in S_0. So, it is enough to show that $f \restriction S_0$ is continuous at s_0, that is, that $\lim_{k\to\infty} f(s_k) = f(s_0)$. By way of contradiction, assume that this is not the case. Then, there exist an open neighborhood U of $f(s_0)$ and a subsequence $S = \{s'_k\colon k < \omega\} \in \mathcal{S}(W)$ of S_0 such that $f(s_k) \notin U$ for all $k > 0$. Take a $D \in \mathcal{D}$ such that $D \cap S \in \mathcal{S}(W)$. Then $f \restriction (D \cap S)$ is discontinuous, as $f^{-1}(U)$ contains only the limit point s_0 of $D \cap S$. But this contradicts the continuity of $f \restriction D$, guaranteed by $\mathcal{D}$-continuity of f. ■

Obviously, if W is metric (or, more generally, sequential), then $\mathcal{S}(W)$-continuity of $f\colon W \to Z$ implies its continuity. In the case when $W = X \times Y$, we will rely on the same implication for the following subfamily $\mathcal{S}_f(C)$ of $\mathcal{S}(X \times Y)$, when $C \subset X \times Y$ is a dense in $X \times Y$: $\mathcal{S}_f(C)$ is the family of all sequences $S = \{\langle x_k, y_k\rangle\colon k < \omega\} \in \mathcal{S}(X \times Y)$ such that $\langle x_k, y_k\rangle \in C$ for all $0 < k < \omega$ and S is a partial function from X to Y, that is, $x_i \neq x_j$ for all $i < j < \omega$. We will write $\mathcal{S}_f$ for $\mathcal{S}_f(X \times Y)$ when X and Y are clear from the context.

Fact 3 *Let X, Y, and Z be metric spaces such that X has no isolated points. Let C be a dense subset of $X \times Y$. If function $f\colon X \times Y \to Z$ is $\mathcal{S}_f(C)$-continuous, then it is continuous.*

PROOF. Suppose $f\colon X \times Y \to Z$ is $\mathcal{S}_f(C)$-continuous and let $\mathcal{S} = \mathcal{S}(X \times Y)$. It is enough to show that f is $\mathcal{S}$-continuous. To see this, fix an $S = \{s_k\colon k < \omega\} \in \mathcal{S}(X \times Y)$. We need to show that $f \restriction S$ is continuous, that is, that $\lim_{k\to\infty} f(s_k) = f(s_0)$.

Indeed, since C is dense in $X \times Y$ and X has no isolated points, for every $k < \omega$ there exists a sequence $\{\langle x_i^k, y_i^k\rangle : i < \omega\} \in \mathcal{S}_f(C)$ with $\langle x_0^k, y_0^k\rangle = s_k$. So, $\lim_{i\to\infty} f(x_i^k, y_i^k) = f(s_k)$. Let ρ and d be the metrics on $X \times Y$ and Z, respectively. For every $k > 0$ we can choose an $i_k < \omega$ such that $\rho(\langle x_{i_k}^k, y_{i_k}^k\rangle, s_k) < 2^{-k}$ and $d(f(x_{i_k}^k, y_{i_k}^k), f(s_k)) < 2^{-k}$. Moreover, we can choose the i_k's to ensure that the sequence $\langle x_{i_k}^k\rangle_{k<0}$ is one-to-one, where we put $\langle x_{i_0}^0, y_{i_0}^0\rangle = s_0$. Since the inequalities $\rho(\langle x_{i_k}^k, y_{i_k}^k\rangle, s_k) < 2^{-k}$ ensure that $\lim_{k\to\infty}\langle x_{i_k}^k, y_{i_k}^k\rangle = \lim_{k\to\infty} s_k = s_0 = \langle x_{i_0}^0, y_{i_0}^0\rangle$, we have $\{\langle(x_{i_k}^k, y_{i_k}^k\rangle : k < \omega\} \in \mathcal{S}_f(C)$, so $\lim_{k\to\infty} f((x_{i_k}^k, y_{i_k}^k) = f(x_{i_0}^0, y_{i_0}^0) = f(s_0)$. But the inequalities $d(f(x_{i_k}^k, y_{i_k}^k), f(s_k)) < 2^{-k}$ ensure that $\lim_{k\to\infty} f(s_k) = \lim_{k\to\infty} f(x_{i_k}^k, y_{i_k}^k) = f(s_0)$, completing the proof. ■

3 $\mathcal{D}$-continuity for $\mathcal{D} \subset C(X)$

First notice that the above three facts immediately imply the following result, which for $X = Z = \mathbb{R}$ is due to Luzin [13] and is a special case of a theorem of Dalbec [5].

Proposition 4 *Let X and Z be metric spaces such that X has no isolated points. If $f : X \times \mathbb{R} \to Z$ is such that $f_h(x) = f(x, h(x))$ is continuous for every continuous function $h : X \to \mathbb{R}$, then f is continuous.*

PROOF. By Fact 3 it is enough to show that f is $\mathcal{S}_f$-continuous. So, take an $S \in \mathcal{S}_f$. Then S is a partial function which, by Tietze extension theorem, can be extended to a continuous function $h : X \to \mathbb{R}$. Since $f(x, h(x))$ is continuous, by Fact 1, so is $f \restriction h$. As $S \subset h$, $f \restriction S$ is continuous. ■

The main goal of this section is to prove that, in the case when $X = \mathbb{R}^n$ and $Z = \mathbb{R}$, the above result holds even if we additionally require the test functions h to be smooth. We say that a function $h \in C(\mathbb{R}^n)$ is "C^1" provided h has all continuous directional derivatives, where the derivatives are allowed to have infinite values.

Theorem 5 *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be such that $f(x, h(x))$ is continuous for every "C^1" function $h : \mathbb{R}^n \to \mathbb{R}$. Then f is continuous. In other words, "C^1"-continuity implies continuity.*

PROOF. Let $\mathcal{D}$ be the family of all "C^1" functions and put $\mathcal{S} = \mathcal{S}_f$. By Facts 2 and 3, it is enough to show that these families satisfy the property (A). So, let $S \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$ be a subsequence of an $S_0 \in \mathcal{S} = \mathcal{S}_f$. Then, there exists a subsequence $S_1 = \{s_k : k < \omega\} \in \mathcal{S}_f$ of S and an $i < n$ such that $T = \{\pi_i(s_k) : k < \omega\} \in \mathcal{S}_f(\mathbb{R})$, where $\pi_i(x) = x(i)$ is the projection onto the i-th coordinate. Rosenthal in [16] proved that T contains a subsequence $T_1 = \{\pi_i(s_{k_j}) : j < \omega\} \in \mathcal{S}_f(\mathbb{R})$ which can be extended to a monotone continuous function $g : \mathbb{R} \to \mathbb{R}$ such that either g is C^1, or it has a continuously differentiable inverse g^{-1} with $(g^{-1})'(x) \neq 0$ for all $x \in \mathbb{R}$. In both cases, such a g is "C^1". Let $h : \mathbb{R}^n \to \mathbb{R}$ be defined via $h(x) = g(\pi_i(x))$. Then h is "C^1" and it contains a subsequence $S_1 = \{s_{k_j} : j < \omega\} \in \mathcal{S}_f$ of S. In particular, $h \cap S \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$, so (A) holds. ■

Next, note that in the statement of Theorem 5, the class "C^1" cannot be replaced with D^1.

Example 6 There exists a D^1-continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is discontinuous on a perfect subset of $\mathbb{R}^2$ of arbitrarily large 1-Hausdorff measure.

PROOF. An example of a D^1-continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ discontinuous at a single point is constructed below, see Corollary 8. A function f with all stated properties can be defined via $f(x, y) = F(y, x)$, where F is a function constructed in by Ciesielski and Glatzer in [3, theorem 4]. ■

4 $T(h)$-continuity for continuous $h\colon \mathbb{R}^n \to \mathbb{R}$

Recall that $T(h)$ consists of all translations of h.

Theorem 7 *If $h\colon \mathbb{R}^n \to \mathbb{R}$ is continuous, then $T(h)$-continuity of an $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ does not imply its continuity.*

PROOF. Fix a continuous function $h\colon \mathbb{R}^n \to \mathbb{R}$. We will find a $T(h)$-continuous function $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which is not continuous. Let $\mathbf{u}$ be an arbitrary unit vector in $\mathbb{R}^n$. For an $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$ let $B(\bar{x}, \delta)$ be the open ball in $\mathbb{R}^n$ centered at $\bar{x}$ and of radius δ and let $\bar{B}(\bar{x}, \delta)$ be its closure. Note that $\bar{B}(\bar{x}, \delta)$ is compact.

Put $y_k = \sum_{i \le k} 4^{-i}$ for $k < \omega$ and let $\hat{y} = \sum_{i<\omega} 4^{-i}$ be the limit of y_k's. Let $\delta_{-1} = 1$ and, by induction on $m < \omega$, choose a positive $\delta_m < \frac{1}{4}\delta_{m-1}(\hat{y} - y_m)^2$ such that

($\bullet$) for any $\bar{x}', \bar{x}'' \in \bar{B}(0\mathbf{u}, m+1)$, if $\|\bar{x}' - \bar{x}''\| \le \delta_m$, then $|h(\bar{x}') - h(\bar{x}'')| < 4^{-(m+2)}$.

Put $x_k = \sum_{i \le k} \delta_i$ for $k < \omega$ and let $\hat{x} = \sum_{i<\omega} \delta_i$ be the limit of x_k's. Notice that $|\bar{x} - x_k| = \sum_{j>k} \delta_j \le \delta_k$ for every $k < \omega$.

For every $k < \omega$ let $f_k\colon \mathbb{R}^n \times \mathbb{R} \to [0,1]$ be a continuous function such that $f_k(x_k\mathbf{u}, y_k) = 1$ and $f_k(\bar{x}, y) = 0$ for all points $\langle \bar{x}, y\rangle$ outside of the set $R_k = B(x_k\mathbf{u}, \delta_k) \times \left(y_k - 4^{-(k+2)}, y_k + 4^{-(k+2)}\right)$.

Notice that the sets R_k are pairwise disjoint, since $y_{k+1} - y_k = 4^{-(k+1)} > 4^{-(k+3)} + 4^{-(k+2)}$. Therefore, $f = \sum_{k<\omega} f_k$ is a well defined function from $\mathbb{R}^n \times \mathbb{R}$ to $[0,1]$. Clearly, f is discontinuous at $\langle \hat{x}\mathbf{u}, \hat{y}\rangle$ and continuous at all other points. To finish the proof, it is enough to show that f is $T(h)$-continuous at $\langle \hat{x}\mathbf{u}, \hat{y}\rangle$.

So, let $\bar{h} = \langle \bar{a}, b\rangle + h$ be a translation of (the graph of) h. Then, $\bar{h}(\bar{x}) = b + h(\bar{x} - \bar{a})$ for every $\bar{x} \in \mathbb{R}^n$. It is enough to show, that $\bar{h}$ intersects at most finitely many sets R_k. Indeed, this is clearly true if $\langle \hat{x}\mathbf{u}, \hat{y}\rangle \notin \bar{h}$. So, assume that $\langle \hat{x}\mathbf{u}, \hat{y}\rangle \in \bar{h}$. Then, $\hat{y} = \bar{h}(\hat{x}\mathbf{u}) = b + h(\hat{x}\mathbf{u} - \bar{a})$. Let $m < \omega$ be such that $\hat{x}\mathbf{u} - \bar{a} \in \bar{B}(0\mathbf{u}, m)$. We will show that $\bar{h} \cap R_k = \emptyset$ for all $k > m+1$.

So, fix such a k and take an $\bar{x} \in B(x_k\mathbf{u}, \delta_k)$. Since $\|\bar{x} - x_k\mathbf{u}\| < \delta_k$, we have $\|\bar{x} - \hat{x}\mathbf{u}\| < 2\delta_k \le \delta_{k-1} \le 1$. Therefore, we have $\hat{x}\mathbf{u} - \bar{a}, \bar{x} - \bar{a} \in \bar{B}(0\mathbf{u}, m+1) \subset \bar{B}(0\mathbf{u}, k)$ and, using ($\bullet$) for $m = k-1$, we get

$$|\hat{y} - \bar{h}(\bar{x})| = |\bar{h}(\hat{x}\mathbf{u}) - \bar{h}(\bar{x})| = |h(\hat{x}\mathbf{u} - \bar{a}) - h(\bar{x} - \bar{a})| < 4^{-(k+1)}.$$

At the same time, $\hat{y} - y_k = \sum_{j>k} 4^{-j} > 4^{-(k+1)} + 4^{-(k+2)}$. Therefore, we have $|\bar{h}(\bar{x}) - y_k| > 4^{-(k+2)}$ and so $\bar{h}(\bar{x}) \notin \left(y_k - 4^{-(k+2)}, y_k + 4^{-(k+2)}\right)$.

In summary, we show that $\langle \bar{x}, \bar{h}(\bar{x})\rangle \notin R_k$ for any $\bar{x} \in B(x_k\mathbf{u}, \delta_k)$, that is, $\bar{h} \cap R_k = \emptyset$, as desired. ■

Notice that if f is as in Theorem 7, $h_1\colon \mathbb{R}^n \to \mathbb{R}$ is a "D^1" function, and $\bar{h}_1 = \langle \bar{a}, b\rangle + h_1$ is a translation of h_1 intersecting infinitely many sets R_k, then the directional derivative $D_{\mathbf{u}}h_1(\hat{x}\mathbf{u} - \bar{a}) = \infty$. In particular, no D^1 function $h_1\colon \mathbb{R}^n \to \mathbb{R}$ intersects more than finitely many R_k's, that is, f is D^1 continuous. This leads to the following corollary.

Corollary 8 *There exists a discontinuous D^1-continuous function $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.*

Finally notice that if X is a normed vector space over $\mathbb{R}$ and $h\colon X \to \mathbb{R}$, then the family $T(h)$ is well defined and we may ask the following question.

Problem 1 For which normed vector spaces X over $\mathbb{R}$ there exists a continuous $h\colon X \to \mathbb{R}$ such that $T(h)$-continuity of any function $f\colon X \times \mathbb{R} \to \mathbb{R}$ implies its continuity?

By Theorem 7, such an h cannot exist if X is of finite dimension. But the proof of Theorem 7 cannot help us with an infinitely dimensional spaces X, since it utilizes the compactness of the closed balls.

5 $T(h)$-continuity for Baire 1 functions $h\colon \mathbb{R}^n \to \mathbb{R}$

We show here that there exists a Baire class 1 function $h\colon \mathbb{R}^n \to \mathbb{R}$ such that $T(h)$-continuity of any $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ implies its continuity. Although, by Theorem 7, such an h cannot be continuous, we will show that its sets of points of discontinuity can be of the form P^n, where $P \subset \mathbb{R}$ is compact of Lebesgue measure 0. The construction of such an h will be based on the following lemmas. In what follows C_f will denote the set of points of continuity of a function f.

By Fact 3 to ensure that $T(h)$-continuity implies continuity it is enough to make sure that every $T(h)$-continuous function $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{S}_f(C_f)$-continuous and that the set C_f is dense. The density of C_f will be ensured utilizing the following lemma.

Lemma 9 *Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be such that its graph contains mutually perpendicular line segments S_j, $j = 0, \ldots, n$. If $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is $T(h)$-continuous, then C_f dense.*

Proof. Let $\mathcal{I}$ be an isometry of $\mathbb{R}^n \times \mathbb{R}$ such that each segment $\mathcal{I}[S_j]$ is parallel to one of the axis of $\mathbb{R}^n \times \mathbb{R}$. Let $\mathcal{D} = \{\mathcal{I}[E]\colon E \in T(h)\}$ and notice that $\mathcal{D} = T(\mathcal{I}[h])$. Clearly, $(f \circ \mathcal{I}^{-1}) \restriction \mathcal{I}[E]$ is continuous for every $E \in T(h)$. Thus, $f \circ \mathcal{I}^{-1}$ is $\mathcal{D}$-continuous. Since for every axis the set $\mathcal{I}[h]$ contains a segment parallel to it, $f \circ \mathcal{I}^{-1}$ is separately continuous. So, the set G of points of continuity of $f \circ \mathcal{I}^{-1}$ is dense. (See e.g. [11].) In particular, f is continuous on a dense set $\mathcal{I}^{-1}[G]$. ∎

The $\mathcal{S}_f(C_f)$-continuity of a $T(h)$-continuous function will be ensured by the following result.

Lemma 10 *Assume that a set $X \subset \mathbb{R}^n$ has the property:*

(B) *every sequence $S \in \mathcal{S}(\mathbb{R}^n)$ contains a subsequence $S_1 \in \mathcal{S}(\mathbb{R}^n)$ such that $\bigcap_{s \in S_1}(X - s) \neq \emptyset$.*

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be such that $h(x) = 0$ for all $x \in X$ and that the closure $\mathrm{cl}(h)$ of (the graph of) h contains $X \times [-1, 1]$. If $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is $T(h)$-continuous, then it is $\mathcal{S}_f(C_f)$-continuous.

Proof. Let $S_0 \in \mathcal{S}_f(C_f)$ and chose its arbitrary subsequence $S = \{\langle x_k, y_k\rangle\colon k < \omega\} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$. It is enough to find a subsequence $\{\langle x_{k_i}, y_{k_i}\rangle\colon i < \omega\} \in \mathcal{S}(\mathbb{R}^n)$ of S such that $\lim_{i\to\infty} f(x_{k_i}, y_{k_i}) = f(x_0, y_0)$.

To see this, let $\hat{S} = \{x_k\colon k < \omega\}$ and notice that it belongs to $\mathcal{S}(\mathbb{R}^n)$. Then, by (B), it contains a subsequence $\{x_{k_i}\colon i < \omega\} \in \mathcal{S}(\mathbb{R}^n)$ such that $\bigcap_{i<\omega}(X - x_{k_i}) \neq \emptyset$. Let $u \in \bigcap_{i<\omega}(X - x_{k_i})$. Then, $\{x_{k_i}\colon i < \omega\} \subset X - u$. Moreover, $y_{k_i} \in [-1, 1] + y_0$ for all but finitely many i's, so we can assume that $\{y_{k_i}\colon i < \omega\} \subset [-1, 1] + y_0$. Then, $\{\langle x_{k_i}, y_{k_i}\rangle\colon i < \omega\} \subset \langle -u, y_0\rangle + X \times [-1, 1] \subset \langle -u, y_0\rangle + \mathrm{cl}(h)$. Notice that $s_0 = \langle x_{k_0}, y_{k_0}\rangle \in \langle -u, y_0\rangle + h$, since $h(x_{k_0} + u) = 0$, as $x_{k_0} + u \in X$.

For every $i > 0$ chose a sequence $\langle s^i_j \in \langle -u, y_0\rangle + h\colon j < \omega\rangle$ converging to $\langle x_{k_i}, y_{k_i}\rangle \in \langle -u, y_0\rangle + \mathrm{cl}(h)$. Since $\langle x_{k_i}, y_{k_i}\rangle \in C_f$, we can choose an $s_i \in \{s^i_j\colon j < \omega\}$ with the property that $\|s_i - \langle x_{k_i}, y_{k_i}\rangle\| \le 2^{-i}$ and $\|f(s_i) - f(x_{k_i}, y_{k_i})\| \le 2^{-i}$. In particular, $\lim_{i\to\infty} s_i = \lim_{i\to\infty}\langle x_{k_i}, y_{k_i}\rangle = \langle x_{k_0}, y_{k_0}\rangle = s_0$. Since $\{s_i\colon i < \omega\} \subset \langle -u, y_0\rangle + h$ and f is $T(h)$-continuous, this implies that $\lim_{i\to\infty} f(s_i) = f(s_0) = f(x_0, y_0)$. Now $\|f(s_i) - f(x_{k_i}, y_{k_i})\| \le 2^{-i}$ ensures that $\lim_{i\to\infty} f(x_{k_i}, y_{k_i}) = \lim_{i\to\infty} f(s_i) = f(s_0) = f(x_0, y_0)$, completing the proof. ∎

The above considerations can be summarized as follows.

Lemma 11 *Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be such that for some bounded open $U \subset \mathbb{R}^n$:*

(i) *$h \restriction (R^n \setminus U)$ is continuous and contains $n + 1$ mutually perpendicular line segments, and*

(ii) *U contains a set X with the property (B) and h satisfies the assumptions of Lemma 10.*

Then any $T(h)$-continuous function $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be $T(h)$-continuous. Then, by (ii) and Lemma 10, it is $\mathcal{S}_f(C_f)$-continuous and, by (i) and Lemma 10, C_f is dense. So, by Fact 3, f is indeed continuous. ∎

The last fact needed for the construction of h is the following result.

Theorem 12 *There exists a compact perfect $P \subset \mathbb{R}$ of measure zero such that $X = P^n$ satisfies (B). Moreover, (B) is satisfied by any Borel set $X \subset \mathbb{R}^n$ which is either of positive measure or of the second category.*

Before we prove the theorem, we show, in the next two corollaries, how to use it to construct the desired function h. The first of these gives a weaker result, but its proof relies on the simplest part of Theorem 12.

Corollary 13 *There exists a Baire class 2 function $h\colon \mathbb{R}^n \to \mathbb{R}$ such that the $T(h)$-continuity of an $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ implies its continuity.*

Proof. Let U be an open ball in $\mathbb{R}^n$ and choose a countable dense subset D of U. By Theorem 12 the set $X = U \setminus D$ satisfies the property (B). Put $h(x) = 0$ for all $x \in X$ and define $h \restriction D$ such that $\mathrm{cl}(h \restriction D) = \mathrm{cl}(U) \times [-1, 1]$. This ensures (ii) of Lemma 11. Extend h to $\mathbb{R}^n$ so that (i) of Lemma 11 is satisfied and h is continuous on the complement of $U \times (-1, 1)$. Then, h is of Baire class 2 and $T(h)$-continuity implies continuity. ∎

Corollary 14 *There exists a Baire class 1 function $h\colon \mathbb{R}^n \to \mathbb{R}$ such that the $T(h)$-continuity of an $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ implies its continuity. Moreover, $C_h = P^n$, where $P \subset \mathbb{R}$ is compact nowhere dense. In addition, we can assume that P is of Lebesgue measure 0.*

Proof. By Theorem 12, there exists a compact nowhere dense $P \subset \mathbb{R}$ for which $X = P^n$ satisfies (B). Actually, any P of positive measure has this property, but we can choose P also to be of Lebesgue measure 0.

Let U be an open ball containing P^n. Define $h(x) = 0$ for $x \in P^n$ and

$$h(x) = \sin\left(\frac{1}{\mathrm{dist}(x, P^n)}\right)$$

for $x \in \mathrm{cl}(U) \setminus P^n$, where $\mathrm{dist}(x, P^n)$ is the distance of x from P^n.

Extend h to $\mathbb{R}^n$ ensuring that (i) of Lemma 11 is satisfied and h is continuous on the complement of $U \times (-1, 1)$. Clearly h is of Baire class 1. Also, the definition of h on U ensures that (ii) of Lemma 11 is satisfied. So, h is as desired. ∎

Proof of Theorem 12. If X is Borel of second second category, then there exists an open ball $U = B(x, \varepsilon)$ and a first category set M such that $B \setminus M \subset X$. Chose an $S \in \mathcal{S}(\mathbb{R}^n)$ and let $S_1 = \{x_k\colon k < \omega\} \in \mathcal{S}(\mathbb{R}^n)$ be its subsequence such that $\|x_k - x_0\| < \varepsilon/2$ for all $k < \omega$. Notice that $\bigcap_{k<\omega}(X - x_k) \supset B(x - x_0, \varepsilon/2) \setminus \bigcup_{k<\omega}(M - x_k) \neq \emptyset$. So, X satisfied (B).

Next, assume that X has a positive measure. Let $F \subset E$ be of finite positive measure and choose an $S = \{v_k\colon k < \omega\} \in \mathcal{S}(\mathbb{R}^n)$. Then $\langle \chi_{F+v_k} \rangle_k$ converges in L_1-norm to χ_{F+v_0} as $k \to \infty$. So, there is a subsequence $\{v_{k_i}\colon i < \omega\} \in \mathcal{S}(\mathbb{R}^n)$ such that $\langle \chi_{F+v_{k_i}} \rangle_i$ that converges a.e. to $\chi_{F+v_{k_0}}$. Hence, for a.e. $x \in F$, we must have $\chi_{F+v_{k_i}}(x + v_{k_0}) = 1$ for sufficiently large i. That is, $(F + v_{k_0}) \cap \bigcup_{K \geq 1} \bigcap_{i \geq K}(F + v_{k_i})$ is a set of full measure in $F + v_{k_0}$. In particular, there exists a K such that $(F + v_{k_0}) \cap \bigcap_{i \geq K}(F + v_{k_i}) \neq \emptyset$. Then $S_1 = \{v_{k_0}\} \cup \{v_{k_i}\colon i \geq K\} \in \mathcal{S}(\mathbb{R}^n)$ justifies (B).

Finally, we construct a perfect $P \subset \mathbb{R}$ of measure 0 that satisfies (B) for $n = 1$. Then, an easy induction shows that P^n satisfies (B). Actually, the set P we use is a known example of a compact set of measure zero of Hausdorff dimension 1 and was studied, e.g., in [7] and [6].

Let $K = \{2, 3, 4, \ldots\}$. For every number $k \in K$ consider the group $G_k = \{0, \ldots, k-1\}$ with addition modulo k. Moreover, for every sequence $\mathcal{A} = \langle A_k \subset G_k\colon k \in K \rangle$ of non-empty sets define

$$P(\mathcal{A}) = \left\{ \sum_{k \in K} \frac{a_k}{k!}\colon \langle a_k \rangle_k \in \prod_{k \in K} A_k \right\}.$$

In what follows, we will consider sets $P(\mathcal{A})$ only for the families $\mathcal{A} = \langle A_k \rangle_{k\in K}$ for which: $k-1 \notin A_k$ for all $k \in K$, and A_k has at least two elements for all but finitely many $k \in K$. Then, each such $P(\mathcal{A})$ is nowhere dense, compact, perfect subset of $[0,1]$ of measure zero.

Let $\mathcal{A}^0 = \langle A^0_k \rangle_{k\in K}$, where $A^0_k = \{0,\ldots,k-2\}$. We show that $P = P(\mathcal{A}^0)$ has the property (B). For this, it is enough to prove that

- every sequence $\langle s_m \in [0,1]\rangle_{m<\omega}$ converging to 0 contains a subsequence $\langle s_{m_i}\rangle_{i<\omega}$ with the property that $P(\mathcal{A}^0) \cap \bigcap_{i<\omega}(-s_{m_i} + P(\mathcal{A}^0)) \neq \emptyset$.

To see this, we construct, by induction on $n < \omega$, the subsequence $\langle s_{m_n}\rangle_{n<\omega}$ and the families $\mathcal{A}^n = \langle A^n_k \rangle_{k\in K}$ such that the following inductive conditions hold.

(a_n) $A^j_k \subset A^i_k$ for every $i < j \le n$ and $k \in K$.

(b_n) For every $k \in K$, $k > 3$, the set A^n_k has at least $\max\{2, k-2-3n\}$ elements and $k-2 \notin A^n_k$ provided $n > 0$.

(c_n) $P(\mathcal{A}^n) \subset P(\mathcal{A}^0) \cap \bigcap_{i<n}(-s_{m_i} + P(\mathcal{A}^0))$.

Now for $n = 0$ the conditions are satisfied. So, assume that they hold for some $n < \omega$. We need to find an m_n (greater than m_{n-1} for $n > 0$) and $P(\mathcal{A}^{n+1})$ preserving the inductive conditions.

For this, find a $\hat{k} \in K$ such that $\hat{k} - 2 - 3n > 4$. Choose m_n large enough so that $s_{m_n} < \sum_{k>\hat{k}} \frac{k-1}{k!}$. We also assume that $s_{m_n} + \max P(\mathcal{A}^0) < 1$. Find $\langle t_k\rangle_k \in \prod_{k\in K} G_k$ such that $s_{m_n} = \sum_{k\in K} \frac{t_k}{k!}$. Notice, that $t_k = 0$ for $k \le \hat{k}$. So, $s_{m_n} = \sum_{k>\hat{k}} \frac{t_k}{k!}$. We need to find an appropriate $\mathcal{A}^{n+1}$.

Notice that, by (c_n),

$$P(\mathcal{A}^0) \cap \bigcap_{i<n}(-s_{m_i} + P(\mathcal{A}^0)) \supset P(\mathcal{A}^n) \cap (-s_{m_n} + P(\mathcal{A}^0)).$$

Thus, to ensure (c_{n+1}), it is enough to make sure that

$$P(\mathcal{A}^{n+1}) \subset P(\mathcal{A}^n) \cap (-s_{m_n} + P(\mathcal{A}^0)).$$

Now, take an $x \in P(\mathcal{A}^n)$, that is, $x = \sum_{k\in K} \frac{x_k}{k!}$ for some $\langle x_k\rangle_k \in \prod_{k\in K} A^n_k$. The question is how to restrict the choices of x so that they are also in $-s_{m_n} + P(\mathcal{A}^0)$. Of course, this is the case exactly when $x + s_{m_n} \in P(\mathcal{A}^0)$. Since $x + s_{m_n} \in [0,1)$, there exists a $\langle a_k\rangle_k \in \prod_{k\in K} G_k$ with $x + s_{m_n} = \sum_{k\in K} \frac{a_k}{k!}$. Thus, we need to examine for which x's

$$\sum_{k\in K} \frac{x_k}{k!} + \sum_{k\in K} \frac{t_k}{k!} = \sum_{k\in K} \frac{a_k}{k!} \tag{2}$$

belongs to $P(\mathcal{A}^0)$, that is, when $a_k \neq k-1$. For this, notice that

(†) either $a_k = x_k + t_k$ modulo k, or $a_k = x_k + t_k + 1$ modulo k.

For $k \ge \hat{k}$, let X_k consist of the possible values $x_k \in A_k$ such that $a_k = k-1$ in (†), and in addition also the value $k-2$. Then, X_k has at most 3 elements. For $k \ge \hat{k}$, let $A^{n+1}_k = A^n_k \setminus X_k$, and for $k < \hat{k}$, let $A^{n+1}_k = A^n_k$.

This choice clearly ensures (a_{n+1}) and (b_{n+1}). Now, to see (c_{n+1}) we need to show that if we choose x from $P(\mathcal{A}^{n+1})$, then $a_k \neq k-1$ for every $k \in K$. For $k > \hat{k}$, we have definitely arranged that $a_k \neq k-1$ by using (†) and by arranging that $A^{n+1}_k \cap X_k = \emptyset$. For $k \le \hat{k}$, we have $t_k = 0$. If $k = \hat{k}$, then $a_k = x_k$ or $a_k = x_k + 1$. Since we have arranged that $k-2 \notin A^{n+1}_k$, we have $x_k < k-2$ and so $a_k \neq k-1$. If $k < \hat{k}$, then $a_k = x_k$ is the part of (†) that holds and so $a_k = x_k < k-1$ by the inductive hypotheses. ■

Although the set $P = P(\mathcal{A}^0) \subset [0,1]$ constructed above Lebesgue has measure zero, it is big in a sense that it has Hausdorff dimension 1. (See below.) On the other hand, the following example shows that there exist perfect sets $P \subset [0,1]$ of arbitrary large Hausdorff dimension $s < 1$ which fail to have the property (B).

Example 15 For every $n \geq 2$ the set $P_n = \left\{\sum_{k=1}^{\infty} \frac{a_k}{n^k} \colon a_k \in \{0,\dots,n-2\} \text{ for all } k\right\}$ does not have the property (B). In particular, the Cantor ternary set $C = 2P_3$ does not satisfy (B).

Proof. Choose the numbers $\{r_k^m \in \{0,\dots,n-1\} \colon k, m < \omega\}$ randomly, independently, with each value of r_j^i having the same probability. For $m > 0$ let $s_m = \sum_{k=m}^{\infty} \frac{a_{m,k}}{n^k}$, where $a_{m,k} = n-1$ when k is even and $a_{m,k} = r_k^m$ when k is odd. Then $\lim_{m\to\infty} s_m = 0$, so, for $s_0 = 0$, $S = \{-s_m \colon m < \omega\} \in \mathcal{S}(\mathbb{R})$. But (B) fails for $X = P_n$ and S, since

- $P_n \cap \bigcap_{m\in M}(P_n - s_m) = \emptyset$ for every $M \subset \{1,2,3,\dots\}$ of size n.

By way of contradiction assume that there exists an $x = \sum_{k=1}^{\infty} \frac{a_k}{n^k}$ in $P_n \cap \bigcap_{m\in M}(P_n - s_m)$. This means that $x + s_m = \sum_{k=1}^{\infty} \frac{a_k}{n^k} + \sum_{k=m}^{\infty} \frac{a_{m,k}}{n^k}$ is in P_m for every $m \in M$. In particular, if $m_0 = \max M$, then for every $m \in M$ and $k > m_0$ either $a_k +_m a_{m,k} \neq n-1$ or $1 +_m a_k +_m a_{m,k} \neq n-1$, where $+_m$ is an addition modulo m. For an even k this translates to: either $a_k +_m n-1 \neq n-1$ or $1 +_m a_k +_m n-1 \neq n-1$. This can happen only when either $a_k + n - 1 > n-1$ or $1 + a_k + n - 1 > n-1$, meaning that for $k-1$ we need to consider only the restriction: $1 +_m a_{k-1} +_m a_{m,k-1} \neq n-1$ for all $m \in M$.

Now, the randomness of the choice of numbers r_k^m ensures that there exists an even $k > m_0$ such $\{a_{m,k-1} \colon m \in M\}$ equals $G_n = \{0,\dots,n-1\}$. So, the restriction above leads to: $1 +_m a_{k-1} +_m j \neq n-1$ for all $j \in G_n$, which is clearly impossible. ■

A calculation similar to that from [8, theorem 1.14] shows that P_m has the Hausdorff dimension $\ln(m-1)/\ln(m)$. Also, there is an interval J such that $J \cap P_m$ has still the same Hausdorff dimension and $J \cap P_m \subset P(\mathcal{A}^0)$. So, indeed $P(\mathcal{A}^0)$ has Hausdorff dimension 1.

The examples above concerning the property (B) raise the following question.

Question: *What condition on a compact perfect set $K \subset \mathbb{R}^n$ characterizes property (B)?*

We finish this discussion with the following observation on the sets without the property (B).

Remark 16 If $X \subset \mathbb{R}$ is such that 0 is not in the interior of $X - X$, then (B) fails for X. Indeed, in such a case, there exists a sequence $\langle s_k \rangle_k$ converging to 0 such that $(s_k + X) \cap X = \emptyset$ for all k. In particular, if $X \subset \mathbb{R}$ is compact and linearly independent over the set of rational numbers, then $X - X$ is is nowhere dense, so (B) fails for P.

6 $I(h)$-continuity and some open questions

For a subset X of $\mathbb{R}^n$, let $I(X)$ consists of all isometric copies (rotations, translations, reflection) of X. The following example shows that, in general, $I(h)$-continuity is a stronger property than $T(h)$-continuity.

Example 17 Let $h \colon \mathbb{R} \to \mathbb{Q}$ be such that $h(x) = 0$ for all $x \notin \mathbb{Q} \cap [0,1]$ and that $h \restriction \mathbb{Q} \cap [0,1]$ has a dense graph in $[0,1] \times \mathbb{R}$. Then, any $I(h)$-continuous function $f \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is separately continuous, so the set C_f of points of continuity of f is dense. In particular, by Lemma 10 and Fact 3, f is continuous. Thus, $I(h)$-continuity implies continuity. However, $T(h)$-continuity does not imply continuity, since the characteristic function of $\mathbb{R} \times \mathbb{Q}$ is discontinuous and $T(h)$-continuous. (Any translation of h is either contained in or disjoint with $\mathbb{R} \times \mathbb{Q}$.)

Problem 2 Does there exist a continuous function $h \colon \mathbb{R} \to \mathbb{R}$ for which $I(h)$-continuity of any function $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ implies its continuity?

Problem 3 What can be said about the sets X for which $I(X)$-continuity implies continuity?

Acknowledgments: The authors like to thank dr. Benjamin Weiss for some useful remarks.

References

[1] R. Baire, Sur les fonctions des variables réelles, *Annali di Matematica Pura ed Applicata* **3** (1899), 1–122.

[2] L.A. Cauchy, *Cours D'analyse de l'Ecole Royale Polytechnique*, Paris, 1821.

[3] K.C. Ciesielski, T. Glatzer, Functions continuous on twice differentiable curves, discontinuous on large sets, *Real Anal. Exchange* **37**(2) (2012), 353–361.

[4] K.C. Ciesielski and T. Glatzer, Sets of Discontinuities of Linearly Continuous Functions, *Real Anal. Exchange* **39**(1) (2014), 117–138.

[5] J.P. Dalbec, When does restricted continuity on continuous function graphs imply joint continuity, *Proc. Amer. Math. Soc.* **118**(2) (1993), 669–674.

[6] M. Elekes and J. Steprans, Less than 2^ω many translates of a compact nullset may cover the real line, *Fund. Math* **181** (2004), 89–96.

[7] P. Erdős and S. Kakutani, On a perfect set, *Colloq. Math.* **4** (1957), 195–196.

[8] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.

[9] A. Genocchi and G. Peano, *Calcolo differentiale e principii di Calcolo*, Torino, 1884.

[10] H. Hahn, Über Funktionen mehrerer Veränderlichen, die nach jeder einzelnen Veränderlichen stetig sind, *Math Zeit.* **4** (1919) 306–313.

[11] R. Kershner, The Continuity of Functions of Many Variables, *Trans. Amer. Math. Soc.* **53** (1943), 83–100.

[12] H. Lebesgue, Sur les fonctions représentable analytiquement, *J. Math. Pure Appl.* **6** (1905), 139–212.

[13] N.N. Luzin, Teoria funktsii deistvitel'nogo peremennogo, 2nd ed., National Educational Publishers, Moscow, 1948.

[14] Z. Piotrowski, *The genesis of separate versus joint continuity*, Tatra Mountains Math. Publ. **8** (1996), 113–126.

[15] Z. Piotrowski, *Topics in Separate Vs. Joint Continuity*, monograph, work in progress.

[16] A. Rosenthal, On the Continuity of Functions of Several Variables, *Math. Zeitschr.* **63** (1955), 31–38.

[17] J. Thomae, *Abriss einer Theorie der complexen Funktionen*, Halle, 1873. (First edition published in 1870.)