# Cardinality of regular spaces admitting only constant continuous functions

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2015/11/4

**Abstract**

We show that an infinite cardinal number $\kappa$ is the cardinality of some connected regular topological space $X$ if and only if $\kappa \geq \omega_1$; such $X$ can be separable if and only if $\omega_1 \leq \kappa \leq 2^{\mathfrak{c}}$; $X$ can be both first countable and separable if and only if $\omega_1 \leq \kappa \leq \mathfrak{c}$; and $X$ can be first countable if and only if $\kappa \geq \omega_1$. The main tools used in our investigation come from the analysis of several popular constructions of a regular topological space which is not completely regular. In particular, this work contains a concise and self-contained presentation of the examples of Mysior [My], Thomas [Th], and those that can be constructed by the, so called, Jones' counterexample machine [Jo] (compare [Ru, pp. 27-28]). Our exposition is based on extracting a common core of these constructions.

We describe one of the simplest examples of a regular space which is not completely regular. It is of the first uncountable cardinality $\omega_1$. We show that this example can be modified, by a variation of a construction of Herrlich [He], to a regular space $Y$ of the same cardinality such that any continuous function from $Y$ into any Hausdorff space $Z$ with a countable pseudo-character is constant. Since this includes the case of $Z = [0,1]$, $Y$ is connected and not completely regular. Such space $Y$ can be separable. Moreover, if we are interested only in $Z = \mathbb{R}$, then $Y$ can be also first countable.

## 1 Introduction

Throughout this paper all considered topological spaces will have more than one element.

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Mathematical Reviews subject classification: Primary: 54G15, 54A25; Secondary: 54C05
Key words: cardinal invariants, regularity, continuity, separability

Let $Y$ be a connected topological space. What is the smallest possible cardinality of $Y$? The answer clearly depends on the additional properties that the space $Y$ is to have. If we assume only that $Y$ is Hausdorff, the answer is $\omega$: a countable Hausdorff connected space was constructed in 1925 by Urysohn [Ur] and in 1953 by Bing [Bi]. (Such a space clearly cannot be finite, as every finite Hausdorff space is discrete. In Bing's example the space is also second countable.) On the other hand, a completely regular connected space must have cardinality greater than or equal to continuum $\mathfrak{c}$, as it can be mapped onto the closed interval $[0,1]$. Niemytzki plane is an example of such a space which is not normal.

So, what happens if we assume that a connected space $Y$ is regular? Then $Y$ can no longer be countable, since a countable regular space is normal, so completely regular. (See e.g. [Mu, exercise 2, p. 212].) The goal of this paper is to show that this is the only restriction on the cardinality of $Y$, that is, that there exist regular connected spaces of any uncountable cardinality. In particular, under the negation of the Continuum Hypothesis, $Y$ can have cardinality strictly less than continuum. The spaces $Y$ that we construct have the property that there are no non-constant continuous functions from $Y$ to $\mathbb{R}$. In particular, any such $Y$ is connected and not completely regular. Actually, our primary example of $Y$ has a stronger property that any continuous function from $Y$ into any Hausdorff space $Z$ with a *countable pseudo-character* (i.e., with every singleton being a $G_\delta$-set) is constant. Such a $Y$ can be separable. If we are interested only in $Z=\mathbb{R}$, then $Y$ can be also first countable.

Our construction is based on the following theorem, proved in Section 3.1, which is a variation of a result of Herrlich [He]. (See also Gartner [Ga].) However, Herrlich's transformation (of $Y$ into $Y^*$), unlike ours, preserves neither first countability nor separability.

**Theorem 1.** *For any infinite $T_1$ topological space $Y$ with two fixed distinct points $-\infty,\infty\in Y$ there exists a topological space $Y^*$ of the same cardinality such that*

*(i) if $Y$ is first countable, Hausdorff, or regular, then so is $Y^*$;*

*(ii) if $Y\setminus\{\infty\}$ is separable, then so is $Y^*$;*

*(iii) if $Z$ is a topological space such that*

*($H_Z$): $f(-\infty)=f(\infty)$ for every continuous function $f$ from $Y$ into $Z$,*

*then the constant functions are the only continuous functions from $Y^*$ into $Z$.*

Theorem 1 reduces our main task to construction of topological spaces $Y$ of appropriate cardinalities that satisfy the property ($H_Z$) either for every Hausdorff space $Z$ with a countable pseudo-character or, for the first countable spaces $Y$, just ($H_\mathbb{R}$). Notice that any space $Y$ satisfying ($H_\mathbb{R}$) is not *completely Hausdorff*, as points $-\infty$ and $\infty$ cannot be separated by a continuous function from $Y$ into $[0,1]$.

It is worth noticing that for every topological space $X$ the condition ($H_\mathbb{R}$) implies ($H_{\mathbb{R}^\kappa}$) for every cardinal $\kappa$, and so, it also implies ($H_Y$) for every completely regular space $Y$ (as any such $Y$ can be embedded into some $\mathbb{R}^\kappa$). However, as we will show in Section 2.6, it is possible that ($H_\mathbb{R}$) holds, while ($H_Y$) fails for some Hausdorff space of countable pseudo-character.

# 2 Examples of regular not completely regular spaces

## 2.1 Example based on Mysior's construction

We start with the following modification of Mysior's example from [My]. Notice that our version requires no algebraic structure on the space, while such structure is used in [My].

Let $A \subseteq \mathbb{R}$ be such that the intersection $A_k = A \cap [k, k+1)$ is uncountable for every integer $k \in \mathbb{Z}$. Let $\Delta = \{\langle a, a\rangle : a \in A\}$ be the diagonal of $X = A^2$ and define the following sets, displayed in Figure 1:

$$U_k = \{\langle a, b\rangle \in X : a > k\} \quad \text{for } k \in \mathbb{Z},$$

$$\Gamma_a = \{\langle a+\varepsilon, a\rangle \in X : \varepsilon \in [0,3]\} \cup \{\langle a, a-\varepsilon\rangle \in X : \varepsilon \in [0,3]\} \quad \text{for } a \in A.$$

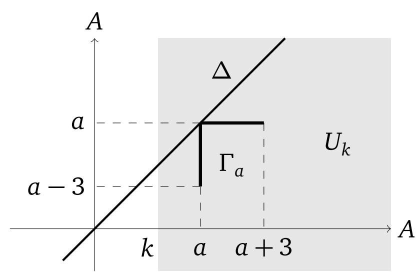


Figure 1: The sets $A^2$, $\Delta$, $\Gamma_a$ (horizontal and vertical segments), and $U_k$ (shaded).

Consider a topology $\mathscr{T}$ on $X = A^2$ generated by a basis consisting of all singletons $\{x\}$ with $x \in X \setminus \Delta$ and all sets $\Gamma_a \setminus F$, where $a \in A$ and $F$ is finite. Clearly $X$ is Hausdorff and zero-dimensional, so, completely regular. Let $\mathscr{E} = \{U_k : k \in \mathbb{Z}\}$.

**Definition 2.** Let $X_{\mathscr{E}} = X \cup \{-\infty, \infty\}$ be endowed with a topology generated by a basis $\mathscr{T} \cup \mathscr{B}^{\mathscr{E}}_{\infty} \cup \mathscr{B}^{\mathscr{E}}_{-\infty}$, where

$$\mathscr{B}^{\mathscr{E}}_{\infty} = \{\{\infty\} \cup U_k : k \in \mathbb{Z}\}, \quad \mathscr{B}^{\mathscr{E}}_{-\infty} = \left\{\{-\infty\} \cup \widetilde{U_k} : k \in \mathbb{Z}\right\},$$

and $\widetilde{U_k}$ is defined as $\bigcup \mathscr{E} \setminus \mathrm{cl}_X(U_k)$.[1]

The space $X_{\mathscr{E}}$ is the goal of this section: it has the same cardinality as $A$ and

**Theorem 3.** *$X_{\mathscr{E}}$ is a regular space satisfying the following property for every Hausdorff space $Z$ with a countable pseudo-character:*

*($H_Z$) $f(-\infty) = f(\infty)$ for every continuous function $f$ from $X_{\mathscr{E}}$ into $Z$.*

*In particular, $X_{\mathscr{E}}$ is connected and not completely regular.*

[1] Of course, $\bigcup \mathscr{E} = X$ for $X = A^2$. But in Theorem 5 we consider $X_{\mathscr{E}}$ assuming only $\bigcup \mathscr{E} \subseteq X$.

The application of Theorem 1 to the space $Y = X_{\mathscr{E}}$ from Theorem 3 produces a regular space $Y^*$ of cardinality $|A|$ (i.e., of arbitrary cardinality between $\omega_1$ and $\mathfrak{c}$) that satisfies $(H_Z)$ for every Hausdorff space $Z$ with a countable pseudo-character. Latter, in Theorem 7, we show, that such space $Y^*$ can actually have an arbitrary uncountable cardinality.

In what follows we will be mainly interested in $X_{\mathscr{E}}$ of cardinality $\omega_1$, that is, when $|A| = \omega_1$. But it should be noted that for $A = \mathbb{R}$ the space $X_{\mathscr{E}}$ constitutes, after some natural identifications, the example of Mysior from [My]. Thus, the argument for Theorem 3 closely resembles one used in [My]. It is based on the following result.

**Proposition 4.** *For every Hausdorff space $Z$ with a countable pseudo-character*

*$(C^*_Z)$ every continuous $f : X \to Z$ is constant on $\Delta \setminus S$ for some countable $S \subset \Delta$.*

Although it is possible to compress the proof of Proposition 4 to a single paragraph, in a way similar to that used in [My], we believe it deserves a bit more scrutiny and we present it in detail in Section 3.2.

*Proof of Theorem 3.* It is easy to see that the regularity of $X$ implies the regularity $X_{\mathscr{E}}$. (Compare the property (E) from Section 2.2.) The argument is completed by noticing that $(C^*_Z)$ implies $(H_Z)$: if $f : X_{\mathscr{E}} \to Z$ is continuous, then application of $(C^*_Z)$ to the restriction of $f$ to $X$ gives $f[\Delta \setminus S] = \{z\}$ for some $z \in Z$ and this ensures that $f(-\infty) = z = f(\infty)$. □

A slightly different, more general and detailed, argument for Theorem 3 is presented in Section 2.2.

## 2.2 The core properties needed to prove Theorem 3

The goal of this section is to extract the properties of the example from Section 2.1 that are sufficient to prove Theorem 3. The properties we extract are satisfied also by the examples of Thomas [Th] and those that can be constructed by the Jones' counterexample machine [Jo]. We will use the flexibility of the obtained results to construct several other examples, as indicated in the abstract.

In what follows, for a fixed set $X$ and an uncountable set $D \subseteq X$, we use the symbol $\mathscr{I}_D$ to denote the $\sigma$-ideal consisting of all $T \subseteq X$ such that $T \cap \Delta$ is at most countable. If $\mathscr{I} = \mathscr{I}_\Delta$, then the family $\mathscr{E}$ from Section 2.1 clearly satisfies:

$(J)$ $U_k \notin \mathscr{I}$ and $\widetilde{U_k} = \bigcup \mathscr{E} \setminus \mathrm{cl}_X(U_k) \notin \mathscr{I}$ for every $k \in \mathbb{Z}$;

$(E)$ $\mathscr{E}$ is a $\subset$-decreasing family of sets open in $X$ such that: $\bigcap \mathscr{E} = \emptyset$, $\bigcup \mathscr{E}$ is closed in $X$, and for every $k \in \mathbb{Z}$ there exist $k', k'' \in \mathbb{Z}$ such that $\mathrm{cl}_X(U_{k'}) \subseteq U_k$ and $\mathrm{cl}_X(U_k) \subseteq U_{k''}$.

More specifically, since the segments forming sets $\Gamma_a$ have length 3, we can take $k' = k+3$ and $k'' = k-3$. Moreover, for every Hausdorff space $Z$ with a countable pseudo-character, $\mathscr{E}$ satisfies also:

$(C_Z)$ for every continuous $f: X \to Z$, if $z \in Z$ is such that $f^{-1}(W)$ contains some $U_k \in \mathscr{E}$ for every open $W \ni z$, then $X \setminus f^{-1}(W) \in \mathscr{I}$ for every such $W$.

Indeed, let $f$ and $z$ be as in $(C_Z)$. By $(C_Z^*)$, which holds by Proposition 4, there exists a countable $S \subset \Delta$ such that $f[\Delta \setminus S] = \{z'\}$ for some $z' \in Z$. We must have $z = z'$, since otherwise the assumption on $z$ fails for $W = Z \setminus \{z'\}$, as then $f^{-1}(W) \subset X \setminus (\Delta \setminus S) \in \mathscr{I}_\Delta = \mathscr{I}$ so $f^{-1}(W)$ cannot contain any $U_k$ because, by $(E)$, $U_k \notin \mathscr{I}$. Thus, for every open $W \ni z$, we have $X \setminus f^{-1}(W) \subset X \setminus (\Delta \setminus S) \in \mathscr{I}_\Delta = \mathscr{I}$, as needed.

In the rest of this section we assume that the space $X$, the ideal $\mathscr{I}$ of subsets of $X$, and the family $\mathscr{E} = \{U_k \subset X : k \in \mathbb{Z}\}$ are arbitrary. In such setting, we say that

- $\mathscr{E}$ is a *$Z$-entanglement* (for $X$ w.r.t. $\mathscr{I}$) provided $(E)$, $(J)$, and $(C_Z)$ hold;
- $\mathscr{E}$ is a *strong entanglement* (for $X$ w.r.t. $\mathscr{I}$) provided it is a $Z$-entanglement (for $X$ w.r.t. $\mathscr{I}$) for every Hausdorff space $Z$ of a countable pseudo-character.

Notice that the family $\mathscr{E}$ from Section 2.1 is a strong entanglement. In particular, Theorem 3 is a consequence of the next theorem.

**Theorem 5.** *Let $X$ be a topological space with a family $\mathscr{E}$ satisfying $(E)$ and let $X_{\mathscr{E}}$ be from Definition 2. If $X$ has any of the following properties:*

*Hausdorff property, regularity, first countability, separability,*

*then so does $X_{\mathscr{E}}$. Also, for any Hausdorff space $Z$, if $\mathscr{E}$ is a $Z$-entanglement, then the property $(H_Z)$ holds.*

*Proof.* The preservation of separability is obvious, while the preservation of first countability is insured by the fact that the families $\mathscr{B}^{\mathscr{E}}_{\infty}$ and $\mathscr{B}^{\mathscr{E}}_{-\infty}$ are local bases in $X_{\mathscr{E}}$ for $\infty$ and $-\infty$, respectively. The preservation of Hausdorff property and regularity follows easily from $(E)$. For example, regularity of $X_{\mathscr{E}}$ at $\infty$ follows from the fact that any set $\{\infty\} \cup U_k$ contains $\mathrm{cl}_{X_{\mathscr{E}}}(\{\infty\} \cup U_{k'})$ for some $k' \in \mathbb{Z}$ and at $-\infty$ from the fact that any set $\{-\infty\} \cup \widetilde{U_k} = \{\infty\} \cup \bigcup \mathscr{E} \setminus \mathrm{cl}_X(U_k)$ contains $\mathrm{cl}_{X_{\mathscr{E}}}\left(\{\infty\} \cup \widetilde{U_{k''}}\right)$ for a $k'' \in \mathbb{Z}$ for which $\mathrm{cl}(U_k) \subseteq U_{k''}$.

To see that $(H_Z)$ holds, fix a continuous $f: X_{\mathscr{E}} \to Z$ and, for a contradiction, assume that $f(-\infty) \neq f(\infty)$. Since $Z$ is Hausdorff, there exist disjoint open sets $W_+, W_- \subseteq Z$ with $f(\infty) \in W_+$ and $f(-\infty) \in W_-$.

Let $z = f(\infty)$. Notice that for every open $W$ containing $z$ the set $f^{-1}(W)$ contains an element of $\mathscr{B}^{\mathscr{E}}_{\infty}$, so $(f \restriction X)^{-1}(W) = X \cap f^{-1}(W)$ contain some $U_k \in \mathscr{E}$. In particular, by $(C_Z)$ applied to $f \restriction X$ and $z$, $X \setminus f^{-1}(W_+) \in \mathscr{I}$. So, $X \cap f^{-1}(W_-) \in \mathscr{I}$, as $X \cap f^{-1}(W_-)$ is a subset of $X \setminus f^{-1}(W_+)$. However, the set $f^{-1}(W_-)$ is open in $X_{\mathscr{E}}$ and contains $-\infty$ implying that $\bigcup \mathscr{E} \setminus \mathrm{cl}_X(U_k) = \widetilde{U_k} \subseteq f^{-1}(W_-)$ for some $k \in \mathbb{Z}$. Thus, $\bigcup \mathscr{E} \setminus \mathrm{cl}_X(U_k)$ belongs to $\mathscr{I}$, contradicting $(J)$. □

The examples of regular spaces with strong entanglements can be traced to the papers of Mysior [My] and Thomas [Th]. More specifically, Mysior defines a regular zero-dimensional topology on the upper half-plane $X = \left\{\langle x, y\rangle \in \mathbb{R}^2 : y \geq 0\right\}$. If

$$U_k = \{\langle x, y\rangle \in X : x > k\} \quad \text{for each } k \in \mathbb{Z}, \tag{1}$$

then $\mathscr{E} = \{U_k : k \in \mathbb{Z}\}$ is a strong entanglement for $X$ with respect to $\mathscr{I}_{\mathbb{R}\times\{0\}}$. Analogously, Thomas [Th] defines a regular zero-dimensional topology on the subset $X = S_1 \cup S_2 \cup S_3$ of the upper half-plane. For this topology the family $\mathscr{E} = \{X \cap U_k : k \in \mathbb{Z}\}$, with sets $U_k$ defined by (1), is a strong entanglement for $X$ with respect to $\mathscr{I}_{S_1}$.

## 2.3 Strong entanglements of arbitrary cardinalities

The cardinality of an entanglement can be increased arbitrary by using the following fact, which is easy to check.

**Fact 6.** *Let $X$ be a topological space with a $Z$-entanglement $\mathscr{E}$ w.r.t. some ideal $\mathscr{I}$ and let $\bar{X}$ be the direct sum of $X$ and some other topological space $T$. Then $\mathscr{E}$ is a $Z$-entanglement for $\bar{X}$ w.r.t. the ideal $\mathscr{I}' = \{S \subseteq \bar{X} : S \cap X \in \mathscr{I}\}$.*

The goal of this section is to prove the following theorem.

**Theorem 7.** *The following conditions are equivalent.*

*(i)* *$\kappa$ is an uncountable cardinal.*

*(ii)* *There exists a regular space $\bar{X}$ with a strong entanglement such that $|\bar{X}| = \kappa$.*

*(iii)* *There exists a regular space $Y^*$ of cardinality $\kappa$ such that any continuous function from $Y^*$ into any Hausdorff space $Z$ with a countable pseudo-character is constant.*

*(iv)* *There exists a regular not completely regular space $Y^*$ of cardinality $\kappa$.*

*(v)* *There exists a regular connected space $Y^*$ of cardinality $\kappa$.*

*Proof.* (i)$\Rightarrow$(ii): Fix an uncountable cardinal $\kappa$. Let $X$ be a regular space of cardinality $\omega_1$ constructed in Section 2.1. So, $X$ has a strong entanglement. Let $\bar{X}$ be the direct sum of $X$ and a discrete space of cardinality $\kappa$. Then $\bar{X}$ is regular, has cardinality $\kappa$, and, by Fact 6, has a strong entanglement, giving us (ii).

(ii)$\Rightarrow$(iii): Let $\mathscr{E}$ be a strong entanglement for $\bar{X}$ from (ii). Then, by Theorem 5, $Y = \bar{X}_{\mathscr{E}}$ satisfies (iii).

The implications (iii)$\Rightarrow$(iv) and (iii)$\Rightarrow$(v) are obvious, since $Z = [0,1]$ is a Hausdorff space with a countable pseudo-character.

The implications (iv)$\Rightarrow$(i) and (v)$\Rightarrow$(i) follow from the fact, that any countable regular space is completely regular and disconnected. $\square$

## 2.4 Separable spaces with strong entanglements

The goal of this section is to prove the following theorem.

**Theorem 8.** *The following conditions are equivalent.*

*(i)* $\kappa$ *is an uncountable cardinal with* $\kappa \le 2^{\mathfrak{c}}$.

*(ii) There exists a separable regular space* $\bar{X}$ *with a strong entanglement such that* $|\bar{X}| = \kappa$.

*(iii) There exists a separable regular space* $Y^*$ *of cardinality* $\kappa$ *such that any continuous function from* $Y^*$ *into any Hausdorff space* $Z$ *with a countable pseudo-character is constant.*

*(iv) There exists a separable, regular, and not completely regular space* $Y^*$ *of cardinality* $\kappa$.

*(v) There exists a separable, regular, and connected space* $Y^*$ *of cardinality* $\kappa$.

To prove the theorem we will need the following lemma.

**Lemma 9.** *There exists a separable regular space* $\hat{X}$ *with a strong entanglement such that* $|\hat{X}| = \omega_1$.

*Proof.* Let $X$ be a zero-dimensional Hausdorff space of cardinality $\omega_1$ that contains a strong entanglement $\mathscr{E} = \{U_k : k \in \mathbb{Z}\}$ w.r.t. $\mathscr{I}$ such that $X = \bigcup \mathscr{E}$. For example, $X$ constructed in Section 2.1 has such property. Let $\{D_k : k \in \mathbb{Z}\}$ be a family of pairwise disjoint countable sets with each $D_k$ disjoint with $X$. For each $k \in \mathbb{Z}$, let $\{D_{k,x} : x \in U_k \setminus U_{k+1}\}$ be an almost disjoint family of infinite subsets of $D_k$, that is, such that any two distinct sets from the family have a finite intersection. Consider $\hat{X} = X \cup \bigcup_{k\in\mathbb{Z}} D_k$ with a topology generated by a basis $\bigcup_{x\in\hat{X}} \mathscr{B}_x$, where $\mathscr{B}_x$ is a singleton $\{\{x\}\}$ for each $x \in \bigcup_{k\in\mathbb{Z}} D_k$, while, for each $k \in \mathbb{Z}$ and $x \in U_k \setminus U_{k+1}$, the family $\mathscr{B}_x$ consists of all sets of the form $V \cup D_{k,x} \setminus F$, where $V$ is a neighborhood of $x$ in $X$ and $F$ is finite. Clearly, $\hat{X}$ is a zero-dimensional Hausdorff space of cardinality $\omega_1$. It is separable, since $\bigcup_{k\in\mathbb{Z}} D_k$ is dense in $\hat{X}$. Moreover, it is not difficult to check that $\mathscr{E}' = \{U_k \cup D_k : k \in \mathbb{Z}\}$ a strong entanglement for $\hat{X}$ with respect to the same ideal $\mathscr{I}$. □

*Proof of Theorem 8.* (i)⇒(ii): Fix an uncountable cardinal $\kappa \le 2^{\mathfrak{c}}$ and notice that there exists a regular separable space $T$ of cardinality $\kappa$. Indeed, the Hewitt-Marczewski-Pondiczery Theorem (see Engelking [En]) implies that if $Y$ is separable, then $Y^{\mathfrak{c}}$ is also separable. In particular, the Cantor cube $2^{\mathfrak{c}}$ has a countable dense subset $D$. Then a subspace $T$ of $2^{\mathfrak{c}}$ containing $D$ and of cardinality $\kappa$ is as required.

Now, let $\hat{X}$ be as in Lemma 9 and let $\bar{X}$ be the direct sum of $\hat{X}$ and $T$. Then $\bar{X}$ is a separable regular space with $|\bar{X}| = \kappa$, and, by Fact 6, $\bar{X}$ has a strong entanglement.

(ii)⇒(iii) is ensured by Theorem 5, while (iii)⇒(iv) and (iii)⇒(v) are obvious.

(iv)⇒(i) and (v)⇒(i): In both cases the uncountability of $\kappa = |Y^*|$ follows from Theorem 7. The inequality $\kappa \le 2^{\mathfrak{c}}$ follows the fact that a separable Hausdorff space cannot have cardinality larger than $2^{\mathfrak{c}}$, see Engelking [En]. □

## 2.5 $\mathbb{R}$-entanglements from Jones' machine

In [Jo] (compare also Rudin [Ru, pp. 27-28]) Jones presented a general construction that transforms a regular topological space that is not normal into a regular space that is not completely regular. We show here that this construction can be modified slightly to give a space with an $\mathbb{R}$-entanglement. This modified construction possess all the required properties of the original construction.

Let $Z$ be a topological space that is not normal and let $A$ and $B$ be fixed disjoint closed sets in $Z$ that cannot be separated by open sets. Note that

the family $\mathscr{J}_A$ of subsets of $A$ that can be separated from $B$ by open sets

is a proper ideal on $A$. Indeed, $\varnothing \in \mathscr{J}_A$, $A \notin \mathscr{J}_A$, and $\mathscr{J}_A$ is closed under taking subsets. Moreover, if $S_0, S_1 \in \mathscr{J}_A$ and for each $i = 1,2$ the sets $U_i \supseteq S_i$ and $V_i \supseteq B$ are open and disjoint, then $U_0 \cup U_1$ and $V_0 \cap V_1$ constitute a separation of $S_0 \cup S_1$ and $B$, implying that $S_0 \cup S_1 \in \mathscr{J}_A$. Similarly, the family $\mathscr{J}_B$ of subsets of $B$ that can be separated from $A$ by open sets is a proper ideal on $B$. It follows that

$$\mathscr{J} = \{S \subseteq Z \colon S \cap A \in \mathscr{J}_A \text{ and } S \cap B \in \mathscr{J}_B\}$$

is an ideal on $Z$. Given $D, E \subseteq Z$, we say that $D$ *is $\mathscr{J}$-almost contained in $E$* provided that $D \setminus E \in \mathscr{J}$.

Note that, for every open subset $W \subseteq Z$, if $A \setminus W \in \mathscr{J}_A$, then $B \setminus \mathrm{cl}(W) \in \mathscr{J}_B$. Indeed, if $U, V$ are disjoint open sets with $A \setminus W \subseteq U$ and $B \subseteq V$, then $U \cup W$ and $V \setminus \mathrm{cl}(W)$ are disjoint open sets containing $A$ and $B \setminus \mathrm{cl}(W)$, respectively. Thus,

(∗) For every open $W \subseteq Z$, if $A$ is $\mathscr{J}$-almost contained in $W$, then $B$ is $\mathscr{J}$-almost contained in cl($W$). Similarly, if $B$ is $\mathscr{J}$-almost contained in $W$, then $A$ is $\mathscr{J}$-almost contained in cl($W$).

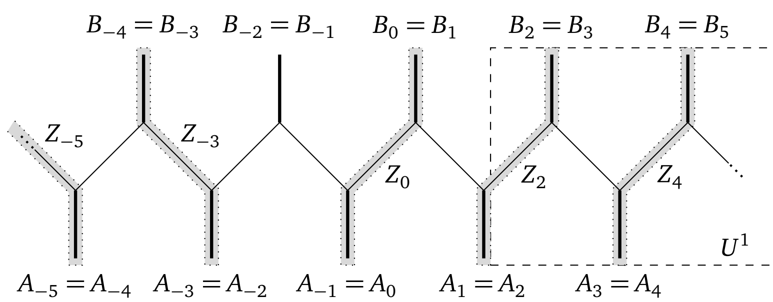


Figure 2: The space $X$ and the set $U^1$ encompassed by a dashed line.

Now, following Jones' construction, let $X$ be the quotient space of $Z \times \mathbb{Z}$ (the topology on $\mathbb{Z}$ is discrete), where for every $k \in \mathbb{Z}$, $a \in A$, and $b \in B$ we identify $\langle a, 2k-1\rangle$ with $\langle a, 2k\rangle$ and $\langle b, 2k\rangle$ with $\langle b, 2k+1\rangle$. If $k \in \mathbb{Z}$, then let

$$Y_k = \{[\langle z, k\rangle] \in X : z \in Y\}$$

whenever $Y \subseteq Z$, and let $U^k \subseteq X$ be defined by

$$U^k = \left( \bigcup_{i \geq 2k} Z_i \right) \setminus A_{2k},$$

see Figure 2.

For each $k \in \mathbb{Z}$ we have the ideal $\mathscr{J}_k = \{J_k : J \in \mathscr{J}\}$ on $Z_k$. Let $\mathscr{I}$ be the ideal of subsets of $X$ defined by

$$\mathscr{I} = \{S \subseteq X : S \cap Z_k \in \mathscr{J}_k \text{ for every } k \in \mathbb{Z}\}.$$

It follows from $(*)$ that

$(**)$ For every open $W \subseteq X$ and $k \in \mathbb{Z}$, if $A_k$ is $\mathscr{I}$-almost contained in $W$, then $B_k$ is $\mathscr{I}$-almost contained in $\text{cl}(W)$ and if $B_k$ is $\mathscr{I}$-almost contained in $W$, then $A_k$ is $\mathscr{I}$-almost contained in $\text{cl}(W)$.

**Theorem 10.** *If $Z$ is regular, then so is $X$. Moreover, $\mathscr{E} = \{U^k : k \in \mathbb{Z}\}$ is an $\mathbb{R}$-entanglement in $X$ w.r.t. $\mathscr{I}$.*

*Proof.* The regularity of $X$ is easy to see. The property $(E)$ is satisfied, since $\bigcap_{k \in \mathbb{Z}} U^k = \varnothing$, $\bigcup_{k \in \mathbb{Z}} U^k = X$, and

$$\text{cl}\left(U^{k+1}\right) \subseteq U^{k+1} \cup A_{2(k+1)} \subseteq U^k \text{ and } \text{cl}\left(U^k\right) \subseteq U^k \cup A_{2k} \subseteq U^{k-1}$$

for every $k \in \mathbb{Z}$. Since $Z_{2k+1} \subseteq U^k$ and $Z_{2k-2} \subseteq X \setminus \text{cl}\left(U^k\right)$ for every $k \in \mathbb{Z}$, the property $(J)$ holds.

It remains to verify that $(C_{\mathbb{R}})$ holds. Let $f : X \to \mathbb{R}$ be continuous and let $y \in \mathbb{R}$ be such that $f^{-1}(W)$ contains some $U^k \in \mathscr{E}$ for every open $W \ni y$. Let $W$ be any open neighborhood of $y$. We want to show that $X \setminus f^{-1}(W) \in \mathscr{I}$, that is, that both $A_k$ and $B_k$ are $\mathscr{I}$-almost contained in $f^{-1}(W)$ for each $k \in \mathbb{Z}$. Let $W_0, W_1, \ldots$ be an infinite sequence of open subsets of $W$ such that

$$y \in W_0 \subseteq \text{cl}(W_0) \subseteq W_1 \subseteq \text{cl}(W_1) \subseteq \cdots.$$

Thus

$$H_0 \subseteq \text{cl}(H_0) \subseteq H_1 \subseteq \text{cl}(H_1) \subseteq \cdots,$$

where $H_i = f^{-1}(W_i)$ for each $i \in \mathbb{N}$. So, there exists a $t \in \mathbb{Z}$ such that $U^t \subseteq H_0$. Then $A_k, B_k \subseteq H_0$ for every $k \geq 2t+1$. In particular, $A_k$ and $B_k$ are both $\mathscr{I}$-almost contained in $H_0$ for each $k \geq 2t+1$. Assume now that $s \in \mathbb{Z}$ and $A_s$ and $B_s$ are both $\mathscr{I}$-almost contained in $H_0$. If $s$ is even, then $A_{s-1} = A_s$ so it follows from $(**)$ that both $A_{s-1}$ and $B_{s-1}$ are $\mathscr{I}$-almost contained in $H_1$. If $s$ is odd, then $B_{s-1} = B_s$ and again $(**)$ implies that both $A_{s-1}$ and $B_{s-1}$ are $\mathscr{I}$-almost contained in $H_1$. Using induction, we conclude that $A_k$ and $B_k$ are $\mathscr{I}$-almost contained in $f^{-1}(W)$ for all $k \in \mathbb{Z}$. □

## 2.6 Separable and first countable spaces with $\mathbb{R}$-entanglements

First notice that

**Fact 11.** *If $X$ is Hausdorff and first countable, then $X$ cannot have a strong entanglement.*

*Proof.* Indeed, if such space $X$ had a strong entanglement $\mathscr{E}$, then the associated space $X_{\mathscr{E}}$ would have satisfied $(H_Z)$ for any Hausdorff space $Z$ of countable pseudo-character, including $Z = X_{\mathscr{E}}$. However, the identity function contradicts $(H_{X_{\mathscr{E}}})$. □

Fact 11 shows that we cannot hope for strong entanglements in first countable Hausdorff spaces. Thus, we will restrict our attention in this section to $\mathbb{R}$-entanglements. For the spaces that are both first countable and separable we have:

**Theorem 12.** *The following conditions are equivalent.*

*(i) $\kappa$ is an uncountable cardinal with $\kappa \le \mathfrak{c}$.*

*(ii) There exists a first countable, separable, and regular space $\bar{X}$ with an $\mathbb{R}$-entanglement such that $|\bar{X}| = \kappa$.*

*(iii) There exists a first countable, separable, and regular space $Y^*$ of cardinality $\kappa$ such that any continuous function from $Y^*$ into $\mathbb{R}$ is constant.*

*(iv) There exists a first countable, separable, regular, and not completely regular space $Y^*$ of cardinality $\kappa$.*

*(v) There exists a first countable, separable, regular, and connected space $Y^*$ of cardinality $\kappa$.*

*Proof.* (i)⇒(ii): Fix an uncountable cardinal $\kappa \le \mathfrak{c}$. First notice that there exists a first countable, separable, and regular space $X$ with an $\mathbb{R}$-entanglement such that $|X| = \omega_1$. Indeed, there exists a non-normal, separable, and first countable topological space $Z$ of cardinality $\omega_1$, see for example [NV, pp. 466] or [vD]. Applying the construction described in Section 2.5 to such space $Z$, we obtain the desired space $X$.

Let $T$ be a separable subspace of the Sorgenfrey plane (see e.g. [Mu]) of cardinality $\kappa$ and notice that it is first countable. Let $\bar{X}$ be the direct sum of $X$ and $T$. Then $\bar{X}$ is first countable, separable, and regular with $|\bar{X}| = \kappa$. In addition, by Fact 6, $\bar{X}$ has an $\mathbb{R}$-entanglement.

(ii)⇒(iii) is ensured by Theorem 5, while (iii)⇒(iv) and (iii)⇒(v) are obvious.

(iv)⇒(i) and (v)⇒(i): In both cases the uncountability of $\kappa = |Y^*|$ follows from Theorem 7. The inequality $\kappa \le \mathfrak{c}$ is justified by Hajnal-Juhász cardinal inequality $|X| \le 2^{c(X)\chi(X)}$ satisfied by any Hausdorff space (see e.g. [En, problem 3.12.10]), since this result implies that any Hausdorff, separable, and first countable space has cardinality at most $2^\omega = \mathfrak{c}$ (as $c(X) \le d(X)$ and, in our case, $d(X) = \chi(X) = \omega$). □

For the spaces that are just first countable, the cardinal restrictions are different:

**Theorem 13.** *The following conditions are equivalent.*

*(i) $\kappa$ is an uncountable cardinal.*

*(ii) There exists a first countable regular space $\bar{X}$ with an $\mathbb{R}$-entanglement such that $|\bar{X}| = \kappa$.*

*(iii) There exists a first countable regular space $Y^*$ of cardinality $\kappa$ such that any continuous function from $Y^*$ into $\mathbb{R}$ is constant.*

*(iv) There exists a first countable, regular, and not completely regular space $Y^*$ of cardinality $\kappa$.*

*(v) There exists a first countable, regular, and connected space $Y^*$ of cardinality $\kappa$.*

*Proof.* (i)⇒(ii): Fix an uncountable cardinal $\kappa \leq \mathfrak{c}$. Let $X$ be as in the proof of Theorem 12, that is, regular, separable, and first countable with an $\mathbb{R}$-entanglement such that $|X| = \omega_1$. Let $\bar{X}$ be the direct sum of $X$ and a discrete space of cardinality $\kappa$. Then $\bar{X}$ is a separable regular space with $|\bar{X}| = \kappa$, and, by Fact 6, $\bar{X}$ has an $\mathbb{R}$-entanglement.

(ii)⇒(iii) is ensured by Theorem 5, while (iii)⇒(iv) and (iii)⇒(v) are obvious.

(iv)⇒(i) and (v)⇒(i) follow from Theorem 7. □

# 3 The remaining proofs

## 3.1 Proof of Theorem 1

Let $Y$ be an infinite $T_1$ topological space with distinct $-\infty, \infty \in Y$. Consider a topology $\tau$ on $Y^\omega$, coarser than the box topology, generated by a basis $\mathscr{B}$ of sets of the form $\prod_{i<\omega} U_i$, where each $U_i$ is open in $Y$, such that

(a) there exists an $n < \omega$ such that $-\infty \in U_n = U_i \subset Y \setminus \{\infty\}$ for all $i \geq n$,[2] and

(b) if $\infty \in U_m$ for some $m < \omega$, then $U_i = Y$ for all $i < m$.[3]

Notice that $\mathscr{B}$ is closed under finite intersections, so it is a basis for the generated topology.

Let $Y^*$ be the family of all $p \in Y^\omega$ eventually equal to $-\infty$ and such that if $p(n) = \infty$ for some $n < \omega$, then also $p(i) = \infty$ for all $i < n$. This $Y^*$, considered as a subspace of $\langle Y^\omega, \tau \rangle$, is the space from Theorem 1.

Indeed, clearly $Y^*$ has the same cardinality than $Y$. The other conditions are justified as follows.

[2]The requirement that sets $U_i$ are eventually equal is needed only for preservation of separability and first countability. If we are not interested in these two properties, than the condition $-\infty \in U_n = U_i \subset Y \setminus \{\infty\}$ can be replaced with $-\infty \in U_n \subset Y \setminus \{\infty\}$.

[3]This condition means that if $p, q \in Y^\omega$ are such that, for some $m < \omega$, $p(m) = q(m) = \infty$ and $p(i) = q(i)$ for all $i > m$, then $p$ and $q$ are non-distinguishable with respect to the topology generated by $\mathscr{B}$.

**(i):** If $Y$ is first countable, fix a $p \in Y^*$ and for every $j < \omega$ let $\mathscr{B}_j$ be a countable basis of $Y$ at $p(j)$. Then the family of all sets $\prod_{i<\omega} U_i \in \mathscr{B}$ with the property that $U_i \in \{Y\} \cup \bigcup_{j<\omega} \mathscr{B}_j$ for every $i < \omega$ is a countable basis of $Y^*$ at $p$. So, $Y^*$ is first countable.

Next assume that $Y$ is Hausdorff and choose distinct $p_0, p_1 \in Y^*$. Let $k < \omega$ be the largest such that $p_0(k) \neq p_1(k)$. Note that $p_0(i) \neq \infty \neq p_1(i)$ for any $i > k$. Let $V_0 \ni p_0(k)$ and $V_1 \ni p_1(k)$ be disjoint open set. Fix an $\ell \in \{0,1\}$. We can assume that $\infty \notin V_\ell$, unless $p_\ell(k) = \infty$. Choose $W_\ell = \prod_{i<\omega} U_i^\ell \in \mathscr{B}$ containing $p_\ell$. We can assume that $U_i^\ell \subset Y \setminus \{\infty\}$ whenever $p_\ell(i) \neq \infty$. (If we replace each $U_i^\ell$ for which $p_\ell(i) \neq \infty$ with $U_i^\ell \setminus \{\infty\}$, then the resulting set $\prod_{i<\omega} U_i^\ell \ni p_\ell$ still belongs to $\mathscr{B}$.) Let $B_\ell = \{p \in W_\ell : p(k) \in V_\ell\}$. Then $B_\ell \in \mathscr{B}$ and $B_0$ and $B_1$ constitute the required open sets separating $p_0$ and $p_1$.

Finally, assume that $Y$ is regular. To see that $Y^*$ is regular, fix a $p \in Y^*$ and a $U = \prod_{i<\omega} U_i \in \mathscr{B}$ containing $p$. We will find a $V = \prod_{i<\omega} V_i \in \mathscr{B}$ containing $p$ whose closure is contained in $U$.

So, for every $i < \omega$ choose an open $W_i \subset Y$ containing $p(i)$ whose closure is contained in $U_i$. Let $m < \omega$ be the largest $i < \omega$ with $p(i) = \infty$, if such an index exists and let $m = 0$ otherwise. Moreover, choose $n < \omega$ satisfying (a) for which $p(i) = -\infty$ for every $i \geq n$. Let $V_i = Y$ for any $i < m$, $V_i = W_i$ for any $m \leq i \leq n$, and $V_i = W_n$ for all $i > n$. It is easy to see that such definition implies that $p \in W = \prod_{i<\omega} V_i \in \mathscr{B}$ and that $\prod_{i<\omega} \mathrm{cl}(V_i) \subset U$. To complete the argument, it is enough to notice that $Y^* \cap \prod_{i<\omega} \mathrm{cl}(V_i)$ is closed in $Y^*$.

Indeed, let $p \in Y^* \setminus \prod_{i<\omega} \mathrm{cl}(V_i)$ and $k < \omega$ be the largest such that $p(k) \notin \mathrm{cl}(V_k)$. Note that $p(i) \neq \infty$ for any $j > k$. Let $Z \subset Y \setminus \mathrm{cl}(V_k)$ be an open neighborhood of $p(k)$ which does not contain $\infty$, unless $p(k) = \infty$. If $B \in \mathscr{B}$ is such that $p \in B$, then $W = \{q \in B : q(k) \in Z\}$ belongs to $\mathscr{B}$, contains $p$, and is disjoint with $\prod_{i<\omega} \mathrm{cl}(V_i)$, completing the proof of (i).

**(ii):** If $Y \setminus \{\infty\}$ is separable with a countable dense set $D_0$, let $D = D_0 \cup \{-\infty, \infty\}$ and notice that $D^\omega \cap Y^*$ is a countable dense subset of $Y^*$. So, $Y^*$ is separable.

**(iii):** Let $Z$ be a space for which $(H_Z)$ holds and let $g : Y^* \to Z$ be continuous. It is enough to show that

(∗) if $n < \omega$ and $p, r \in Y^*$ are such that $p(n) = -\infty$, $r(n) = \infty$, and $p(i) = r(i) = -\infty$ for all $i > n$, then $g(p) = g(r)$.

Indeed, if (∗) holds and $p, q \in Y^*$, then $g(p) = g(q)$. To see this, let $n < \omega$ be such that $p(i) = q(i) = -\infty$ for all $i \geq n$ and let $r \in Y^*$ be such that $r(i) = \infty$ for every $i \leq n$ and $r(i) = -\infty$ for every $i > n$. Then, by (∗), $g(p) = g(r) = g(q)$.

To see (∗), let $p, r \in Y^*$ be as in its assumption and consider a subspace

$$Y' = \{r\} \cup \{q \in Y^* : q(j) = p(j) \text{ for all } j \neq n\}$$

of $Y^*$. Let $\pi_n : Y^\omega \to Y$ be the projection onto the $n$-th coordinate (i.e., given by $\pi_n(q) = q(n)$) and let $h : Y' \to Y$ be the restriction of $\pi_n$ to $Y'$. Then $h$ is a bijection, since $r$ is the only element of $Y'$ with $r(n)=\infty$, while for any $y\in Y\setminus\{\infty\}$ there is a unique $q\in Y'$ with $q(n)=y$.

Notice also that $h$ is an open map, that is, $h^{-1}\colon Y\to Y'$ is continuous. This follows from the fact that

$$h\left[Y'\cap\prod_{j<\omega}U_j\right]\in\{\emptyset,U_n\}\ \text{ for every } \textstyle\prod_{j<\omega}U_j\in\mathscr{B}. \tag{2}$$

Indeed, this is obvious, when $\infty\notin U_n$, since then $r\notin\prod_{j<\omega}U_j$. On the other hand, if $\infty\in U_n$, then $U_i=Y$ for all $i<n$ and, once again, (2) holds.

To finish the proof, of $(*)$ and the theorem, notice that $f=g\circ h^{-1}\colon Y\to Z$ is continuous. So, since $h^{-1}(-\infty)=p$ and $h^{-1}(\infty)=r$, using $(H_Z)$ to $f=g\circ h^{-1}$ we obtain

$$g(p)=g(h^{-1}(-\infty))=f(-\infty)=f(\infty)=g(h^{-1}(\infty))=g(r),$$

as required.

## 3.2 Proof of Proposition 4

For disjoint sets $B$ and $C$, let $Y_{B,C}$ be the topological space on the set

$$Y_{B,C}=(B\times C)\cup B\cup C$$

with each point in $B\times C$ being isolated and the basic neighborhoods of $s\in B\cup C$ containing $s$ together with a cofinite subset of: $\{b\}\times C$ when $s=b\in B$, and of $B\times\{c\}$ when $s=c\in C$ (see Figure 3).

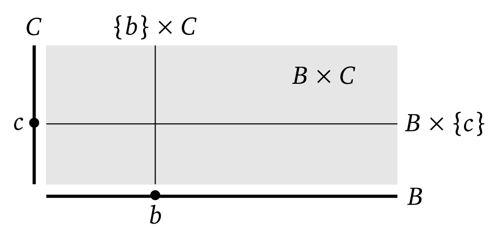


Figure 3: The space $Y_{B,C}$.

In particular, the space $Y_{B,C}$ is homeomorphic to $B^*\times C^*\setminus\{\langle\infty,\infty\rangle\}$, where $B^*=B\cup\{\infty\}$ denotes the one-point compactification of the discrete space $B$.

**Lemma 14.** *If $B$ is uncountable and $C$ is countable, then every continuous function from $Y_{B,C}$ into a Hausdorff space $Z$ with a countable pseudo-character is constant on some co-countable subset of $B$.*

*Proof.* Let $Z$ be a Hausdorff space of countable pseudo-character and $f : Y_{B,C} \to Z$ be continuous. Then, for each $c \in C$, the set

$$G_c = f^{-1}\big(f(c)\big) \cap (B \times \{c\})$$

is a closed $G_\delta$-set in $Y_{B,C}$, hence a co-countable subset of $B \times \{c\}$. Thus, the set

$$B' = \big\{b \in B : \langle b, c\rangle \in G_c \text{ for each } c \in C\big\}$$

is also co-countable. It suffices to show that $f$ is constant on $B'$.

Suppose $f(a) \neq f(b)$ for some $a, b \in B'$. Since $Y$ is Hausdorff, there are disjoint open neighborhoods $V_a$ and $V_b$ in $Y$ of $a$ and $b$, respectively. Since the sets

$$f^{-1}\big(V_a\big) \cap (\{a\} \times C) \qquad \text{and} \qquad f^{-1}\big(V_b\big) \cap (\{b\} \times C)$$

are cofinite subsets of $\{a\} \times C$ and $\{b\} \times C$, respectively, there exists a $c \in C$ with

$$\langle a, c\rangle \in f^{-1}\big(V_a\big) \qquad \text{and} \qquad \langle b, c\rangle \in f^{-1}\big(V_b\big).$$

Since $\langle a, c\rangle, \langle b, c\rangle \in G_c$, we have

$$f(a, c) = f(b, c) = f(c).$$

Since $V_a$ and $V_b$ are disjoint, that is a contradiction. $\square$

*Proof of Proposition 4.* Fix a continuous $f : X \to Z$. It is enough to show that, for every $k \in \mathbb{Z}$, the function $f$ is constant on a co-countable subset of $\Delta \cap (k, k+2)^2$. To see this, fix $k \in \mathbb{Z}$, put $B = A \cap (k, k+2)$, let $C$ be a countable subset of $A \cap (k-1, k)$, and consider the subspace

$$Y = (B \times C) \cup B' \cup C'$$

of $X$, where $B' = \Delta \cap B^2$ and $C' = \Delta \cap C^2$, see Figure 4.

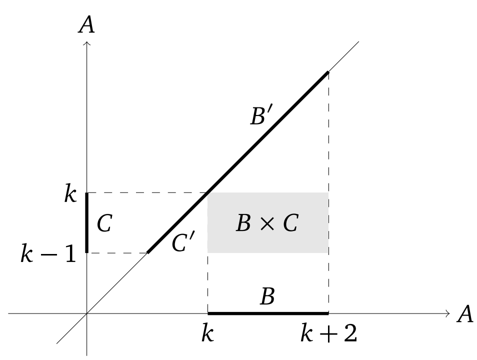


Figure 4: The space $Y$ homeomorphic to $Y_{B,C}$.

Note that $Y$ is homeomorphic to $Y_{B,C}$ upon the natural identification of $B$ with $B'$ and $C$ with $C'$. Therefore, Lemma 14 implies that $f$ is indeed constant on a co-countable subset of $B' = \Delta \cap (k, k+2)^2$. $\square$

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