On fixed points of locally and pointwise contracting maps

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Abstract

We prove that the following self-mappings must have unique fixed points: *pointwise contractive*, (PC), maps on compact rectifiably path connected spaces; *uniformly locally contractive*, (ULC), maps on complete connected spaces.

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1. Introduction

Let $\langle X, d\rangle$ be a complete metric space. A mapping $f\colon X \to X$ is *contractive* (*with a contraction constant* λ), abbreviated (C), provided there exists a $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \le \lambda d(x, y)$ for every $x, y \in X$. The Banach Fixed Point Theorem also known as Contraction Mapping Principle, [1] states the following:

Theorem 1.1. (Banach 1922) *If X is a complete metric space and $f\colon X \to X$ is contractive, then f has a unique fixed point, that is, there exists a unique $\xi \in X$ such that $f(\xi) = \xi$.*

This Banach result inspired many generalizations, among which we are the most interested in those, where the assumption of contractiveness is relaxed to a local condition, see [5], [6], [17], [4], [10], [12], and the survey [16]. This work was the most influenced by the 1978 theorem of Hu and Kirk [10], see below, with a proof corrected, in 1982, by Jungck [12].

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Definition 1.2. A map $f: X \to X$ is *pointwise contractive*, denoted (PC), if for every point $x \in X$ there exists a $\lambda_x \in [0, 1)$ and an open neighborhood $U_x \subseteq X$ of x such that $d(f(x), f(y)) \leq \lambda_x d(x, y)$ for all $y \in U_x$.[1] We say that f is *uniformly pointwise contractive*, denoted (uPC),[2] provided the same $\lambda \in [0, 1)$ works for all $x \in X$.

Theorem 1.3. (Hu and Kirk 1978; Jungck 1982) *If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f: X \to X$ is* (uPC)*, then f has a unique fixed point.*

Recall, that a metric space $\langle X, d\rangle$ is *rectifiably path connected* provided any two points $x, y \in X$ can be connected in X by a path $p: [0, 1] \to X$ of finite length $\ell(p)$, that is, by a continuous map p satisfying $p(0) = x$ and $p(1) = y$, and having a finite length $\ell(p)$ defined as the supremum over all numbers: $\sum_{i=1}^{n} d(p(t_i), p(t_{i-1}))$, where $0 < n < \omega$ and $0 = t_0 < t_1 < \cdots < t_n = 1$.

It is worth noting that Munkres [14, p. 182] provides an example of a (PC) map $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\left(x + \sqrt{x^2 + 1}\right)$, without fixed or even periodic points. This shows that in Theorem 1.3 the assumption of (uPC) cannot be weakened to (PC). One of two principal results of this paper is to show, that the fixed point result with the weaker requirement of f being (PC) remains true, when we additionally assume that X is compact.

Theorem 1.4. *Assume that $\langle X, d\rangle$ is compact and rectifiably path connected metric space. If $f: \langle X, d\rangle \to \langle X, d\rangle$ is* (PC)*, then f has a unique fixed point.*

Theorem 1.4 stands in contrast with the main result of [3, Theorem 1], where we construct a (uPC)[3] self-map on a compact (zero-dimensional) subset X of $\mathbb{R}$ with every orbit being dense, hence having neither fixed nor periodic points.

Our second result on fixed points, without the assumption that the space is rectifiably path connected, requires the following natural definition, in which $B(x, \varepsilon)$ denotes an open ball centered at x and of radius ε.

Definition 1.5. A map $f: X \to X$ is *locally contractive*, denoted (LC), if for every $x \in X$ there exist numbers $\lambda_x \in [0, 1)$ and $\varepsilon_x > 0$ such that $f \restriction B(x, \varepsilon_x)$ is contractive with constant λ_x. Moreover, f is *uniformly locally contractive*, (ULC), if the same λ and ε work for all $x \in X$, which we also indicate by saying that f is (ε, λ)-(ULC).

Theorem 1.6. *Assume that $\langle X, d\rangle$ is complete and that $f: X \to X$ is* (ULC).

(i) If X is connected, then f has a unique fixed point.

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[1] So, a differentiable function $f: \mathbb{R} \to \mathbb{R}$ is (PC) if, and only if, $|f'(x)| < 1$ for every $x \in \mathbb{R}$, see Fact 3.1.

[2] The notion of (uPC) maps was introduced by Holmes [8], where it was called *local radial contraction*. (See also [11, 10].) The term *radial* is often used elsewhere in mathematics and we find the adverb *pointwise* to be more adequate for this notion, see for example [9, p 104].

[3] In [3] the property (PC) is denoted (LRC), for *locally radially contractive*, and (uPC) is denoted as (uLRC).

(ii) *If X has a finite number of components, then f has a periodic point, that is, $f^{(n)} = f \circ \cdots \circ f$ has a fixed point for some $n > 0$.*

Notice, that for the case of compact X, the conclusion of Theorem 1.6 follows from the results of Edelstein [6] we discuss in Section 4.

The following examples show that the map f in Theorem 1.6 need to have neither periodic points, when X has an infinite number of components, nor fixed points, when X is disconnected with a finite number of components.

Example 1.7. *Consider $X = \bigcup_{n<\omega}[n, n+2^{-(n+1)}]$ with the standard distance and define $f\colon X \to X$ by putting $f(x) = n + 1 + \frac{x-n}{2}$ for every $n < \omega$ and $x \in [n, n+2^{-(n+1)}]$. Then X is complete, while f is a $(\frac{1}{2}, \frac{1}{2})$-(ULC) map with no periodic point.*

Example 1.8. *Consider $X = [-2, -1] \cup [1, 2]$ with the standard distance and define $f\colon X \to X$ by putting $f(x) = -\frac{x}{|x|}$ for every $x \in X$. Then X is compact, while f is a (ULC) map with no fixed point.*

2. Proof of Theorem 1.6

For $\varepsilon > 0$, we say that X is *ε-chainable*, provided for every $p, q \in X$ there exists a finite sequence $s = \langle x_0, x_1, \ldots, x_n\rangle$, referred to as an *$\varepsilon$-chain from p to q*, such that $x_0 = p$, $x_n = q$, and $d(x_i, x_{i+1}) \le \varepsilon$ for every $i < n$. The length of the ε-chain s is defined as $\mathfrak{l}(s) = \sum_{i<n} d(x_{i+1}, x_i)$.

Fact 2.1. *Connected spaces are ε-chainable for any $\varepsilon > 0$.*

Proof. (See Engelking [7, Exercise 6.1.D(a) p 359].) Fix $x, y \in X$ and $\varepsilon > 0$. Define, by induction on $n < \omega$, a sequence $\langle B_n \subset X : n < \omega\rangle$ as $B_0 = \{x\}$ and $B_{n+1} = \{z \in X : \exists b \in B_n(d(z, b) < \varepsilon)\}$. The union $\bigcup_{n<\omega} B_n \neq \emptyset$ is a clopen, so by, connectedness of the space X, we have $\bigcup_{n<\omega} B_n = X$. Thus, $y \in B_n$ for some $n < \omega$ and so, there exists an ε-chain, with $n + 1$ terms, from x to y. ■

The next lemma shows that in connected spaces a new metric may be defined such that functions locally contractive in original metric become globally contractive in the new one, see also Lemma 3.6. Some of these ideas can be found in Rosenholtz [17], Jungck [12], and Hu and Kirk [10].

Lemma 2.2. *Let $\varepsilon > 0$ and assume that $\langle X, d\rangle$ is connected or, more generally, ε-chainable. Then the map $D_\varepsilon\colon X^2 \to [0, \infty)$ given as*

$$D_\varepsilon(x, y) = \inf\{\mathfrak{l}(s) : s \text{ is an } \varepsilon\text{-chain from } x \text{ to } y\}$$

is a metric on X topologically equivalent to d. Moreover,

(i) *If $\langle X, d\rangle$ is complete, then so is $\langle X, D_\varepsilon\rangle$.*

(ii) *If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (η, λ)-(ULC) for some $\eta > \varepsilon$ and $\lambda \in [0, 1)$, then $f\colon \langle X, D_\varepsilon\rangle \to \langle X, D_\varepsilon\rangle$ is (C) with constant λ.*

Proof. To see that D_ε is a metric on X it is enough to show that D_ε satisfies the triangle inequality. So, fix $x, y, z \in X$ and $\delta > 0$. Then, there exist the ε-chains $s = \langle x_0, \ldots, x_n \rangle$ from x to y and $t = \langle y_0, \ldots, y_m \rangle$ from y to z with $D_\varepsilon(x, y) \geq \mathfrak{l}(s) - \delta$ and $D_\varepsilon(y, z) \geq \mathfrak{l}(t) - \delta$. Since $u = \langle x_0, \ldots, x_n, y_0, \ldots, y_m \rangle$ is an ε-chain from x to z with $\mathfrak{l}(u) = \mathfrak{l}(s) + \mathfrak{l}(t)$, we have

$$D_\varepsilon(x, y) + D_\varepsilon(y, z) \geq \mathfrak{l}(s) - \delta + \mathfrak{l}(t) - \delta = \mathfrak{l}(u) - 2\delta \geq D_\varepsilon(x, z) - 2\delta.$$

Since the constant $\delta > 0$ was arbitrary, we obtain the desired triangle inequality $D_\varepsilon(x, y) + D_\varepsilon(y, z) \geq D_\varepsilon(x, z)$.

Notice also that if $d(x, y) \leq \varepsilon$, then we have $D_\varepsilon(x, y) = d(x, y)$ (since then $d(x, y) \leq D_\varepsilon(x, y) \leq \mathfrak{l}(\langle x, y \rangle) = d(x, y)$). This implies topological equivalence.

To see (i) notice that for any $x, y \in X$, $D_\varepsilon(x, y) \geq d(x, y)$ so any D_ε-Cauchy sequence is also d-Cauchy. Since the metrics are topologically equivalent, if $\langle X, d \rangle$ is complete, then so is $\langle X, D_\varepsilon \rangle$.

Suppose that $f \colon \langle X, d \rangle \to \langle X, d \rangle$ is (η, λ)-(ULC) for some $\eta > \varepsilon$. To prove (ii), fix $x, y \in X$. We need to show that $D_\varepsilon(f(x), f(y)) \leq \lambda D_\varepsilon(x, y)$. For this, fix a $\delta > 0$ and let $s = \langle x_0, \ldots, x_n \rangle$ be an ε-chain from x to y with $D_\varepsilon(x, y) \geq \mathfrak{l}(s) - \delta$. Notice that, by (η, λ)-(ULC), for every $i < n$ we have $d(f(x_{i+1}), f(x_i)) \leq \lambda d(x_{i+1}, x_i)$. In particular, $t = \langle f(x_0), \ldots, f(x_n) \rangle$ is an ε-chain and $\mathfrak{l}(t) = \sum_{i<n} d(f(x_{i+1}), f(x_i)) \leq \sum_{i<n} \lambda d(x_{i+1}, x_i) = \lambda \mathfrak{l}(s)$. Hence, $D_\varepsilon(f(x), f(y)) \leq \mathfrak{l}(t) \leq \lambda \mathfrak{l}(s) \leq \lambda(D_\varepsilon(x, y) + \delta)$. Since $\delta > 0$ was arbitrary, we obtain the desired inequality $D_\varepsilon(f(x), f(y)) \leq \lambda D_\varepsilon(x, y)$. ■

Proof of Theorem 1.6. The first part follows immediately from the Banach Contraction Principle, Theorem 1.1, Lemma 2.2, and the fact that every connected space is ε-chainable for every $\varepsilon > 0$, see Fact 2.1.

To see the second part, let $C_1, \ldots, C_m$ be the connected components of X. Since $f^{(n)}[C_1]$ is connected, there must exist $i < i+k$ with $f^{(i)}[C_1]$ and $f^{(i+k)}[C_1]$ intersecting the same component of X, call it C. Then $f^{(k)}[C] \subset C$. Applying the first part of the theorem to $f^{(k)} \restriction C \colon C \to C$, we can find an $x \in C$ with $f^{(k)}(x) = x$. Thus, x is a periodic point of f. ■

3. Proof of Theorem 1.4

Now let us highlight the connection between the (PC) property and the derivative. If $P \subset X$ and $f \colon P \to X$, then, for every $x \in P$,

$$D^* f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}$$

whenever x is a limit point of P and $D^* f(x) = 0$ otherwise. We have the following relationship, which is related to a discussion at [11, p. 569].

Fact 3.1. *For any $f \colon X \to X$,* (PC) *is equivalent to having $D^* f(x) < 1$ for all $x \in X$. Also,* (uPC) *property simply says that* $\sup\{D^* f(x) \colon x \in X\} < 1$.

Proof. If $D^*f(x) < 1$ and $\lambda_x \in (D^*f(x), 1)$, then $\frac{d(f(x),f(y))}{d(x,y)} \leq \lambda_x$ for all $y \in X$ sufficiently close to x. Conversely, if f is (PC) at $x \in X$ with $\lambda_x \in [0,1)$, then $\frac{d(f(x),f(y))}{d(x,y)} \leq \lambda_x$ for all $y \in X$ sufficiently close to x, so that $D^*f(x) \leq \lambda_x$.

An argument for the other equivalence is similar. ∎

We will also need the following 1945 result of Myers [15, page 219]. For reader convenience, we include a sketch of its proof.

Lemma 3.2. *Let $\langle X, d\rangle$ be a compact metric space and assume that, for any $n < \omega$, $p_n : [0,1] \to X$ is a rectifiable path such that $\ell(p_n \restriction [0,t]) = t\ell(p_n)$ for any $t \in [0,1]$. If $L = \liminf_{n\to\infty} \ell(p_n) < \infty$, then there exists a subsequence $\langle p_{n_k} : k < \omega\rangle$ converging uniformly to a rectifiable path $p : [0,1] \to X$ with $\ell(p) \leq L$.*

Sketch of proof. Select a countable dense subset $U = \{u_m : m < \omega\}$ of $[0,1]$. By compactness of X, using a diagonal argument, it is possible to find a subsequence $\langle p_{n_k} : k < \omega\rangle$ which is pointwise convergent on U, that is, such that $\lim_{k\to\infty} p_{n_k}(u_m) = p(u_m)$ for all $m < \omega$. Then, functions $\{p_{n_k} : k < \omega\}$ are equicontinuous and converge uniformly to a continuous $p : [0,1] \to X$ with $\ell(p) \leq L$. ∎

Notice that Lemma 3.2 implies immediately the following 1930 theorem of Menger [13] that we will need as well.

Fact 3.3. *In any compact metric space $\langle X, d\rangle$, any two points that can be joined by a rectifiable curve, can be joined by a length minimizer.*

Proof. Assume that $x, y \in X$ can be joint by a rectifiable curve. Let L be the infimum of the lengths of all such curves and choose rectifiable curves $q_n : [0,1] \to X$ from x to y such that $\lim_{n\to\infty} \ell(q_n) = L$. For every $n < \omega$ define

$$p_n = \left\{ \left\langle \frac{\ell(q_n \restriction [0,t])}{\ell(q_n)}, q_n(t) \right\rangle : t \in [0,1] \right\} \tag{1}$$

that is, p_n is a reparametrization of q_n via (rescaled) path length of (the initial segments of) q_n. Observe that if we let $v = \frac{\ell(q_n \restriction [0,t])}{\ell(q_n)}$, then we have $\ell(p_n \restriction [0,v]) = \ell(q_n \restriction [0,t]) = v\ell(q_n) = v\ell(p_n)$. Application of Lemma 3.2 to the sequence $\langle p_n : n < \omega\rangle$ gives a path $p : [0,1] \to X$ from x to y with $\ell(p) = L$. ∎

Lemma 3.4. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and $f : X \to Y$ be continuous. For every $\varepsilon, L > 0$ the set*

In the version of the paper that appeared in print, Lemma 3.4 had incorrect direction of inequality. This version corrects the "typo."

$$K^L_\varepsilon = \{x \in X : \rho(f(x), f(x')) \leq Ld(x,x') \text{ for all } x' \in X \text{ with } d(x,x') < \varepsilon\}$$

is closed in X.

Proof. It is enough to show that the complement of K^L_ε is open in X. Indeed, if $x \in X \setminus K^L_\varepsilon$ then there is an $x' \in B(x,\varepsilon)$ such that $\rho(f(x), f(x')) > Ld(x,x')$. In particular, we have $x' \neq x$. Also, by the continuity of the map $X \ni x \mapsto$

$\rho(f(x),f(x'))-Ld(x,x')$, there exists a $\delta>0$ such that $\rho(f(x''),f(x'))>Ld(x'',x')$ for every $x''\in B(x,\delta)$. Decreasing δ, if necessary, we can also assume that $d(x'',x')<\varepsilon$ for every $x''\in B(x,\delta)$. So, $B(x,\delta)\subset X\setminus K^L_\varepsilon$ (as, for every $x''\in B(x,\delta)$ there exists an $x'\in B(x'',\varepsilon)$ with $\rho(f(x''),f(x'))>Ld(x'',x')$). ■

Corollary 3.5. *Assume that f is a* (PC) *self-map on a complete metric space $\langle X,d\rangle$. If $Y\subset X$ is closed in X (or, more generally, a Baire space), then there exists a dense open subset U of Y such that $f\restriction U$ is* (LC).

Proof. For $0<n<\omega$ let $F_n=K^{(n-1)/n}_{1/n}$ be defined as in Lemma 3.4. Since f is (PC), we have $X=\bigcup_{n=1}^\infty F_n$. Then $U=\bigcup_{n=1}^\infty \operatorname{int}_Y(Y\cap F_n)$ is as needed. Indeed, since F_ns are closed, U is dense in Y by Baire category theorem. Also, if $y\in\operatorname{int}_Y(Y\cap F_n)$, then there is a $\delta_y\in(0,1/(2n))$ such that $Y\cap B(y,\delta_y)\subseteq\operatorname{int}_Y(Y\cap F_n)$ and then $f\restriction Y\cap B(y,\delta_y)$ is (C) with constant $(n-1)/n$. ■

Lemma 3.6. *If $\langle X,d\rangle$ is a rectifiably path connected metric space, then the map $D_0\colon X^2\to[0,\infty)$ given as*

$$D_0(x,y)=\inf\{\ell(p)\colon p \text{ is a rectifiable path from } x \text{ to } y\}$$

is a metric on X with the following properties.

(i) If $\langle X,d\rangle$ is complete, then so is $\langle X,D_0\rangle$.

(ii) If P is the range of a rectifiable path p in X, $\lambda\ge 0$, and $f\colon X\to X$ is such that, for every $x\in P$, $D^*(f\restriction P)(x)\le\lambda$ with respect to the metric d, then $\ell(f\circ p)\le\lambda\ell(p)$.[4]

Proof. For (i), see Hu and Kirk [10, Proof of Theorem 1, page 122].

To show (ii), fix an $\varepsilon>0$. First notice that

$$d(f(p(t)),f(p(s)))\le(\lambda+\varepsilon)\ell(p\restriction[s,t]) \text{ for every } 0\le s<t\le 1. \tag{2}$$

Indeed, for every $x\in[s,t]$ we have $D^*(f\restriction P)(x)\le\lambda$, so there exists a proper open interval $U_x=(x-\delta_x,x+\delta_x)$ such that

$$d(f(p(x)),f(p(x')))\le(\lambda+\varepsilon)d(p(x),p(x')) \text{ for every } x'\in U_x\cap[s,t]. \tag{3}$$

Let J be a finite subset of $[s,t]$ such that $\mathcal{U}=\{U_x\colon x\in J\}$ is a cover of $[s,t]$ containing no proper subcover. Let $\langle x_1,x_3,\dots,x_{2n-1}\rangle$ be a list of elements of J in the increasing order. Then, by minimality of $\mathcal{U}$, for every $0<i<n$ there exists an $x_{2i}\in U_{x_{2i-1}}\cap U_{x_{2i+1}}\cap(x_{2i-1},x_{2i+1})$. Moreover, $x_0=s\in U_{x_1}$ and

[4] In [12, Lemma, page 505] Jungck proves the same estimate under the assumption that f is (uPC) with a constant $\lambda\in[0,1)$. Our proof is similar to that of [12, Lemma, page 505].

$x_{2n} = t \in U_{x_{2n-1}}$. In particular, $s = x_0 \le x_1 < x_2 < \cdots < x_{2n-1} \le x_{2n} = t$ and $x_{2i}, x_{2i+2} \in U_{x_{2i+1}}$ for every $i < n$. Therefore, by (3),

$$\begin{aligned} d(f(p(t)), f(p(s))) &\le \sum_{k<2n} d(f(p(x_k)), f(p(x_{k+1}))) \\ &\le \sum_{k<2n} (\lambda+\varepsilon) d(p(x_k), p(x_{k+1})) \le (\lambda+\varepsilon)\ell(p \restriction [s,t]), \end{aligned}$$

justifying (2).

To finish the argument for (ii) choose the numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\ell(f \circ p) \le \sum_{i<n} d(f(p(t_{i+1})), f(p(t_i))) + \varepsilon$. Then, by (2),

$$\begin{aligned} \ell(f \circ p) &\le \sum_{i<n} d(f(p(t_{i+1})), f(p(t_i))) + \varepsilon \\ &\le \sum_{i<n} (\lambda+\varepsilon)\ell(p \restriction [t_{i-1}, t_i]) + \varepsilon = (\lambda+\varepsilon)\ell(p) + \varepsilon. \end{aligned}$$

As this holds with any $\varepsilon > 0$, the desired inequality, $\ell(f \circ p) \le \lambda \ell(p)$, follows. ■

Remark 3.7. Notice that, unlike the metrics d and D_ε from Lemma 2.2, the metrics d and D_0 from Lemma 3.6 do not need to be topologically equivalent. For example, let X be union of the "topologist's sine curve" (see Munkres [14, p. 156]) and a semi-circular curve connecting one end of the vertical segment with the "end" of the sine curve. If d is the standard metric on $\mathbb{R}^2$, then $\langle X, d \rangle$ is compact rectifiably path connected, while $\langle X, D_0 \rangle$ is not compact—it is homeomorphic to $[0, \infty)$.

For our next lemma let us recall a notion similar to contractiveness but without the λ.

Definition 3.8. A map $f\colon X \to X$ is *shrinking*, denoted (S), provided that $d(f(x), f(y)) < d(x, y)$ for any two different $x, y \in X$.[5]

The following lemma is the key step needed in our proof of Theorem 1.6.

Lemma 3.9. *Assume that $\langle X, d \rangle$ is compact rectifiably path connected metric space. If $f\colon \langle X, d \rangle \to \langle X, d \rangle$ is* (PC)*, then $f\colon \langle X, D_0 \rangle \to \langle X, D_0 \rangle$ is* (S).

Proof. Fix distinct $x, y \in X$. We need to show that $D_0(f(x), f(y)) < D_0(x, y)$.

By Fact 3.3 there exists a path $p\colon [0,1] \to X$ from x to y with $D_0(x, y) = \ell(p)$. Since $\ell(f \circ p) \ge D_0(f(x), f(y))$, it is enough to prove that $\ell(p) > \ell(f \circ p)$.

To see this, let Y be the rage of p. By Corollary 3.5, there exists an open dense subset U of Y such that $f \restriction U$ is (LC). Thus, there exists also a non-empty open $W \subset U$ such that $f \restriction W$ is (C) with some constant $L \in (0, 1)$.

[5]The mappings with the property (S) are in the spotlight of Edelstein [6] where they are called *contractive* maps. In most of the literature contractive maps are the same as contractions, that is, maps with the property (C). Term *shrinking* for (S) is used in [14].

Choose $a < b$ in $[0,1]$ such that $P = p([a,b]) \subset W$. Then, $f \restriction P$ is Lipschitz with constant L. In particular, $D^*(f \restriction P)(x) \le L$ for every $x \in P$ and so, by Lemma 3.6(ii), $\ell(p \restriction [a,b]) \le L\ell(p \restriction [a,b])$. Moreover, by Fact 3.1, $D^* f(x) < 1$ for every $x \in X$. Thus, using Lemma 3.6(ii) also with $\lambda = 1$,

$$\begin{array}{rcl} \ell(p) & = & \ell(p \restriction [0,a]) + \ell(p \restriction [a,b]) + \ell(p \restriction [b,1]) \\ & > & \ell(p \restriction [0,a]) + L\ell(p \restriction [a,b]) + \ell(p \restriction [b,1]) \\ & \ge & \ell(f \circ p \restriction [0,a]) + \ell(f \circ p \restriction [a,b]) + \ell(f \circ p \restriction [b,1]) \\ & = & \ell(f \circ p), \end{array}$$

as needed. ■

Proof of Theorem 1.4. Let D_0 is the distance from Lemma 3.6. By Lemma 3.9, $f\colon \langle X, D_0\rangle \to \langle X, D_0\rangle$ is (S). Let $L = \inf\{D_0(x, f(x))\colon x \in X\}$. We will show that

($\bullet$) there exists an $\bar{x} \in X$ such that $D_0(\bar{x}, f(\bar{x})) = L$,

which is not completely obvious, since $\langle X, D_0\rangle$ need not be compact, see Remark 3.7.

Let $\langle x_n \in X\colon n < \omega\rangle$ be a sequence with $\lim_{n\to\infty} D_0(x_n, f(x_n)) = L$. By Fact 3.3, for every $n < \omega$ there exists a path $p_n\colon [0,1] \to X$ from x_n to $f(x_n)$ onto $P_n \subset X$ of length $D_0(x_n, f(x_n))$. Reparametrizing p_n as in (1), if necessary, we can assume that $\ell(p_n \restriction [0,t]) = t\ell(p_n)$ for any $t \in [0,1]$. Then, by Lemma 3.2, there exists a subsequence $\langle p_{n_k}\colon k < \omega\rangle$ converging uniformly to a rectifiable path $p\colon [0,1] \to X$ with $\ell(p) \le L$. If $\bar{x} = \lim_{k\to\infty} p_{n_k}(0) = \lim_{k\to\infty} x_{n_k}$, then p is from $\bar{x}$ to $p(1) = \lim_{k\to\infty} p_{n_k}(1) = \lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$. So, $D_0(\bar{x}, f(\bar{x})) \le \ell(p) \le L$, that is, $\bar{x}$ satisfies ($\bullet$).

To finish the proof it is enough to notice that L must be equal 0, since otherwise $D_0(f(\bar{x}), f(f(\bar{x}))) < D_0(\bar{x}, f(\bar{x}))$, contradicting minimality of L. Thus, $D(\bar{x}, f(\bar{x})) = 0$ and $f(\bar{x}) = \bar{x}$, as required.

The uniqueness of the fixed point is ensured by the fact that, according to Lemma 3.9, $f\colon \langle X, D_0\rangle \to \langle X, D_0\rangle$ is (S). ■

4. Concluding remarks and open questions

As we saw in Examples 1.7 and 1.8, when the assumption of connectivity of X is dropped, the fixed points of (ULC) map $f\colon X \to X$ may disappear. However, for compact spaces, the periodic points must exist, even when (ULC) is weakened to the following notions.

Definition 4.1. A map $f\colon X \to X$ is *locally shrinking*, denoted (LS), provided for every $x \in X$ there exists an open $\varepsilon_x > 0$ such that $f \restriction B(x, \varepsilon_x)$ is shrinking, (S). Also, f is *uniformly locally shrinking*, (ULS), if the same $\varepsilon > 0$ works for all $x \in X$.

Edelstein's [6, Theorem 2] implies the following result for compact spaces.

Theorem 4.2. (Edelstein 1962) *If $\langle X, d\rangle$ is compact and $f\colon X \to X$ is* (ULS)*, then f has a periodic point.*

Edelstein also gives an example [6, Example 2, p 79] that his theorem does not generalize to the (LS) functions on non-compact spaces. Nevertheless, Theorem 4.2 remains true when f is assumed only to be (LS). This follows from the following variation of [2, theorem 4.2].

Proposition 4.3. (LS) *implies* (ULS) *for maps $f\colon X \to X$ with compact X.*

PROOF. For each $y \in X$ find an open set $U_y \ni y$ such that $f \restriction U_y$ is shrinking. By compactness of X, there is a finite $X_0 \subset X$ such that $\mathcal{U}_0 = \{U_y \colon y \in X_0\}$ covers X. Let $\delta > 0$ be a Lebesgue number for the cover $\mathcal{U}_0$ of X. (See e.g. [14, lemma 27.5].) Then $\varepsilon = \delta/2$ satisfies (ULS). ■

Corollary 4.4. *If X is compact and $f\colon X \to X$ is* (LS)*, then f has a periodic point.*

Since the property (S) is the key in our proof of Theorem 1.4, it might be tempting to try proving it by showing, generalizing Proposition 4.3, that for compact spaces X every (PC) map $f\colon X \to X$ is (LS). However, there is no such implication, even if f is assumed to be (uPC), as justified by our example [3, Theorem 1]. (The function constructed there is (uPC). It cannot be (LS) by Corollary 4.4, since it does not have periodic points.)

Two interesting questions arise from Theorem 1.4 and the above discussion.

Problem 1. Can Theorem 1.4 be proved when we assume only that the compact space $\langle X, d\rangle$ is just connected? Or just path connected?

Clearly connectedness assumption on X is crucial, as shown by Example 1.8 and one from [3, Theorem 1].

A map $f\colon X \to X$ is *pointwise shrinking*, denoted (PS), if for any $x \in X$ there is an open neighborhood $U_x \ni x$ such that $d(f(x), f(y)) < d(x, y)$ for all $y \in U_x \setminus \{x\}$.

Problem 2. Can Theorem 1.4 be proved when we assume that the map f is just (PS), rather than (PC)?

A possible way of attacking Problem 2 would be to show that the conclusion of Lemma 3.9 when function f is assume to be just (PS). We do not know, if such generalization is true. However, our proof of Lemma 3.9 does not generalizes to such scenario. Indeed, in our proof of the lemma we use the fact, that

- if $p\colon [0,1] \to X$ is a rectifiable path and the restriction $f \restriction p[[0,1]]$ is (PC), then $\ell(f \circ p) < \ell(p)$.

However, such implication does not hold if $f \restriction p[[0,1]]$ is just (PS). Indeed, if p is a natural parameterization of $P = [0,1] \times \{2\} \subset \mathbb{R}^2$ and f maps each $\langle t, 2\rangle \in P$ onto $\langle \cos t, \sin t\rangle \in S^1 \subset \mathbb{R}^2$, then $f \restriction P$ is (PS) (in the standard metric of $\mathbb{R}^2$), while $\ell(f \circ p) = \ell(p)$.

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