Real Analysis Exchange

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ON THE COMPOSITION OF DERIVATIVES

Abstract

We show that there exists a derivative $f\colon [0,1] \to [0,1]$ such that the graph of $f \circ f$ is dense in $[0,1]^2$, so not a G_δ-set. In particular, $f \circ f$ is everywhere discontinuous, so not of Baire class 1, and hence it is not a derivative.

1 Introduction

In [7] K. Kuratowski and W. Sierpiński proved that any function $f\colon \mathbb{R} \to \mathbb{R}$ possessing the Darboux property and belonging to the first Baire class has a connected graph. It follows that if such an f maps $I = [0,1]$ into itself, it must have a fixed point in I; that is, there is a $p \in I$ such that $f(p) = p$. More recently, it has been shown by P. Szuca [10], that the same is true of a finite composition of such functions.

If $f\colon I \to I$ is a composition of finite number of maps, each having a connected G_δ graph, then f has a fixed point.

(For a composition of two Darboux Baire 1 functions this was shown also in [4] and [5].) Since any derivative has a connected G_δ graph, Szuca's result immediately implies that a finite composition of derivatives from I to I has a fixed point.

Since the Darboux property is clearly preserved under the composition, this last result would have followed immediately from the theorem of Kuratowski

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and Sierpiński if the composition of two derivatives were always in the first Baire class. The main aim of this note is to show that such an argument cannot be made, by giving an example of a derivative $f\colon [0,1] \to [0,1]$ for which $f \circ f$ does not have a G_δ graph, so that it is not of Baire class 1.

Notice, that in [9, example 4.1] Szuca gives an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ with a connected G_δ graph such that the graph of $f \circ f$ is not a G_δ. However, this map f is not of Baire class 1.

2 The example

In 1907 Pompeiu [8] gave an example of a differentiable, strictly increasing function $F\colon I \to \mathbb{R}$ whose derivative $f = F'$, known as a *Pompeiu derivative*, is bounded and equal 0 on a dense subset Z of its domain. The constructions of such functions can be found in many places including [2], [12, sec. 9.7], and [3]. In fact, see [3], F can be defined on the entire real line, as the inverse of the map $\sum_{i=1}^{\infty} 2^{-i}(x - q_i)^{1/3}$, where $\{q_i\colon i \in \mathbb{N}\}$ is an enumeration of $\mathbb{Q}$ with $|q_i| \le i$. Notice that as f is of Baire class 1, Z must be a G_δ-set. Also, since F is nowhere constant, the complement Z^c of Z must be dense.

Example 2.1. There exists a rescaling f of a Pompeiu derivative such that $f\colon I \to I$ and $(f \circ f)^{-1}(b)$ is dense in I for every $b \in [0,1)$. In particular, the graph of $f \circ f$ is dense in $[0,1]^2$, hence not a G_δ-set. In addition, $f \circ f$ is everywhere discontinuous, so not of Baire class 1, and hence is not a derivative.

Proof. Let $f\colon I \to [0,\infty)$ be a Pompeiu derivative. Since f is bounded, the real number $s = \sup\{f(x)\colon x \in [0,1]\}$ is well defined. Using the Darboux property of f, we can choose a strictly monotone sequence $\{x_n\}_{n=1}^{\infty}$ converging to a point $x_0 \in [0,1]$ such that $f(x_n) \to_n s$. Since $f = 0$ on a dense set, the oscillation $\omega_f(x_0)$ of f at x_0 equals s. By replacing f, if necessary, with $s^{-1} f \circ L$, where $L\colon I \to \mathbb{R}$ is linear with $L(0) = x_0$ and $L(1) = x_1$, we may assume that f maps I into I, $s = 1$, and $x_0 = 0$. (It is easy to show that a derivative composed with a linear function is a derivative.) Now let J be an open interval in $[0,1]$. Since J intersects Z and Z^c, both sets being dense, $f[J]$ is a non-trivial interval K containing 0. Since the oscillation of f at 0 is 1 and f has the Darboux property, $(f \circ f)[J] = f[K]$ contains $[0,1)$. It follows that for every $b \in [0,1)$, the set $\{x\colon (f \circ f)(x) = b\}$ is dense in $[0,1]$. This implies that the graph of $f \circ f$ cannot be a dense G_δ-set, since, otherwise, all level sets $\{x\colon (f \circ f)(x) = b\}$, with $b \in [0,1)$, would be dense G_δ sets in I, so, they could not be disjoint. ■

Note that a function from [11, thm 4.1] has similar properties to those of Example 2.1 and has also been examined in the context of its self composition.

However, it was constructed with more advanced tools and does not explicitly addresses the composition of derivatives.

3 Remarks

While the composition of two Baire 1 functions might not itself be in Baire class 1, it will always be in Baire class 2, see e.g. [6, thm 24.3]. Thus, our function $f \circ f$ is in Baire class 2, but not Baire class 1. Because of the works [4, 5, 10], we know $f \circ f$ does have a fixed point. This means that the graph of $h = f \circ f$ cannot be separated by the line $y = x$. It does not in itself say the graph of h is connected — there could possibly be some other continuum separating the graph.[1] We do not know whether the composition of two derivatives must have a connected graph. (Just because a function takes every value in every interval and has fixed points does not suffice for connectedness of the graph. Simply change the values of h, say whenever $h(x) = x^2$, and the resulting function will still take every value in every interval and will have fixed points but its graph will not be connected. Of course, we are not claiming this function is a composition of two derivatives.)

The n-fold composition of Baire 1 functions is always in the Baire class n, but not always in the class $n-1$, see e.g. [6, thm 24.3]. What can one say about the class of an n-fold composition of derivatives? Can one do better than saying it is in the class n? Of course, our example show that the answer is *no* for $n = 2$.

The Pompeiu's derivatives provide a quick construction of a nowhere monotone differentiable function $\varphi\colon \mathbb{R} \to \mathbb{R}$, as recently shown in [3]. Although, there are many known proofs of the existence of such maps, most are quite complicated or use modern techniques, see [2]. A construction from [3] can be outlined in a single sentence. If $G\colon \mathbb{R} \to \mathbb{R}$ is such that $g = G'$ is Pompeiu's derivative and $Z = \{x\colon g(x) = 0\}$, then we can define φ as

$$\varphi(x) = G(x - t) - G(x),$$

where $t \in \bigcap_{d \in D} \big((-d + Z) \cap (d - Z)\big)$ and $D \subset Z^c$ is countable and dense.

[1] In [9, p. 28] the question is explicitly asked as: "it seems interesting to find out if the composition of derivatives is connected." A partial answer to this inquiry, in the positive direction, can be found in [1].

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