Monsters in Calculus

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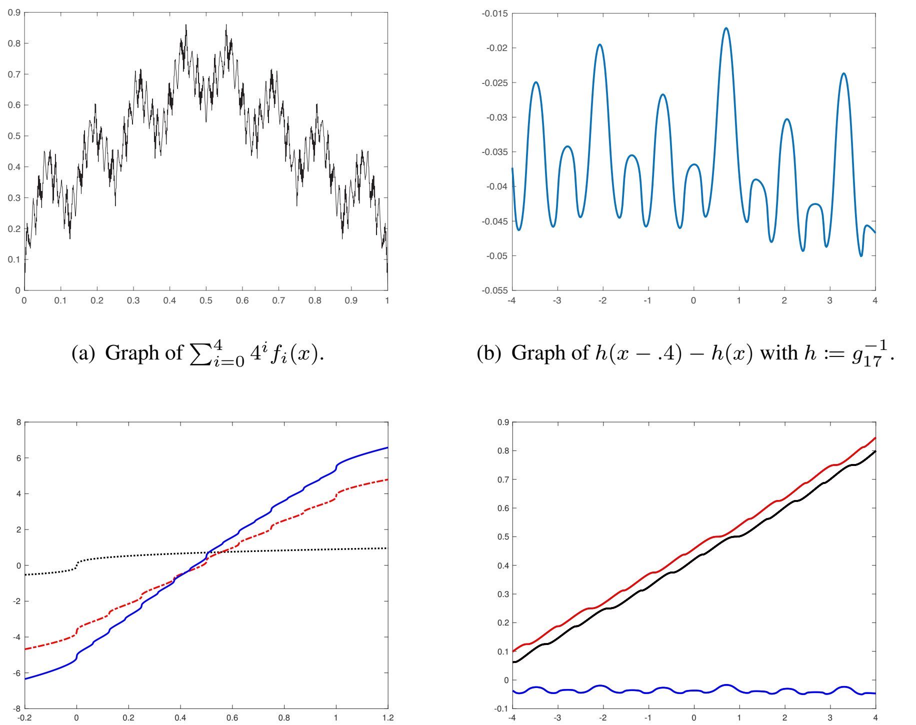
Abstract. One of the strangest, most mind-boggling examples in analysis is that of a function from $\mathbb{R}$ to $\mathbb{R}$ that is everywhere differentiable but monotone on no interval. The graph of such a "monstrous" function is simultaneously smooth and very rugged. Although such examples have been known for over 100 years, so far the existing constructions are quite involved. In this note we provide a simple example of such a map. It is presented in a broader context of other paradoxical examples related to differentiability of continuous maps from $\mathbb{R}$ to $\mathbb{R}$, including a differentiable function which maps a compact perfect subset $\mathfrak{X}$ of $\mathbb{R}$ onto itself even though its derivative vanishes on $\mathfrak{X}$.

1. INTRODUCTION. Most continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ one studies are C^1, that is, have continuous first derivative. This favorable property can fail for two related, but distinct reasons. The first is that a continuous function need not be differentiable, as typified by the absolute value map $f(x) := |x|$. The second reason is that an everywhere differentiable function can have a discontinuous derivative, as usually exemplified by the map $f(x) := x^2 \sin\left(x^{-1}\right)$ for $x \neq 0$ and $f(0) := 0$. Each of these anomalies is pushed to its limits in the following three classes of examples.

Weierstrass's monsters. There exist continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that are nowhere differentiable. The first published example of such a map was given by K. Weierstrass and appeared in the 1872 paper [**19**]. At that time, mathematicians commonly believed that a continuous function must have a derivative at a "significant" set of points. Thus, the example was received with disbelief and such functions became known as *Weierstrass's monsters*. A large number of simple constructions of Weierstrass's monsters have appeared in the literature; see for example [**10**]. Our favorite is the following variant of an example of van der Waerden [**18**] (see also [**16**, Thm. 7.18]), since the proof of its properties requires only the standard tools of one-variable analysis. Set $f(x) := \sum_{n=0}^{\infty} 4^n f_n(x)$, where $f_n(x) := \min_{k\in\mathbb{Z}} \left|x - \frac{k}{8^n}\right|$ is the distance from $x \in \mathbb{R}$ to the set $\frac{1}{8^n}\mathbb{Z} = \left\{\frac{k}{8^n} \colon k \in \mathbb{Z}\right\}$; see Figure 1(a). It is continuous at each $x \in \mathbb{R}$, since $|f(y) - f(x)| \le \left|\sum_{i=0}^{n} 4^i f_i(y) - \sum_{i=0}^{n} 4^i f_i(x)\right| + \frac{1}{2^n}$ for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$. It is not differentiable at any $x \in \mathbb{R}$, since for every $n \in \mathbb{N}$ with $x \in \left[\frac{k}{8^n}, \frac{k+1}{8^n}\right]$, $k \in \mathbb{Z}$, there exists a $y_n \in \left\{\frac{k}{8^n}, \frac{k+1}{8^n}\right\} \setminus \{x\}$ such that $\left|\frac{f(x)-f(y_n)}{x-y_n}\right| \ge \left|\frac{f(\frac{k+1}{8^n})-f(\frac{k}{8^n})}{\frac{k+1}{8^n}-\frac{k}{8^n}}\right| = 8^n \left|\sum_{i=0}^{n} 4^i f_i(\frac{k+1}{8^n}) - \sum_{i=0}^{n} 4^i f_i(\frac{k}{8^n})\right| \ge \frac{2}{3}4^{n-1}$.

Differentiable monsters. There exist maps $f\colon \mathbb{R} \to \mathbb{R}$ that are everywhere differentiable and nowhere monotone—simultaneously smooth and very rugged. Does that sound like an oxymoron? We think so. In particular, since they seem to us even more monstrous than Weierstrass's monsters, in what follows we will refer to them as *differentiable monsters*. Of course, by the mean value theorem, f is a differentiable monster if, and only if, its derivative f' attains both positive and negative values on every interval. In particular, the derivative f' of a differentiable monster is discontinuous on a dense set.

The history of differentiable monsters is described in detail in the 1983 paper of A. M. Bruckner [**3**]. The first construction of such a function is given in A. Köpcke's



(a) Graph of $\sum_{i=0}^{4} 4^i f_i(x)$.

(b) Graph of $h(x - .4) - h(x)$ with $h := g_{17}^{-1}$.

(c) The graphs of g_1 (dotted), g_9 (dashed), and g_{17} (solid). The inverse of g_{17} is the map h from (d).

(d) Top to bottom: the graphs of $h := g_{17}^{-1}$, $h(x - .4)$, and $h(x - .4) - h(x)$, same as (b) but different scale.

Figure 1. Approximations of (a) Weierstrass's and (b) differentiable monsters. Note the different scales and shapes. We used dyadic numbers $\langle q_1 \ldots, q_{17}\rangle = \langle 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}\rangle$ and approximations $g_n(x) := \sum_{i=1}^{n} 0.9^i (x - q_i)^{1/3}$ of g.

1887 paper [**12**]; see also [**13, 14**]. The most influential study of this subject is a 1915 paper of A. Denjoy [**7**]. Until now, we find the simplest constructions of differentiable monsters to be from the 1974 paper [**11**] of Y. Katznelson and K. Stromberg and from the 1976 article [**20**] of C. Weil. However, the first of these, while elementary, still requires a delicate inductive construction. The second one uses a Baire category argument on the space D of bounded derivatives, where D is endowed with the supremum norm. Specifically, Weil shows that $D_0 := \{f \in D \colon f^{-1}(0) \text{ is dense in } \mathbb{R}\}$ is a closed linear subspace of D and, using Pompeiu's functions (which we discuss shortly), that $E := \{f \in D_0 \colon f \text{ is not a differentiable monster}\}$ is of first category in D_0.

A cellar full of monsters. Perhaps the most familiar monster is Cantor's. Often called *Cantor's ternary function* or simply *the Cantor function*, it also goes by *Cantor staircase function* and *the devil's staircase*; see for example [**17**]. The Cantor function is a continuous, nondecreasing map $c \colon [0, 1] \to [0, 1]$ that monstrously (or devilishly) manages to do all its increasing on a set $\mathfrak{C}$ of measure zero called the *Cantor ternary set*, while being constant on each of the intervals comprising $[0, 1] \setminus \mathfrak{C}$. The reader is probably familiar with the usual description of $\mathfrak{C}$ as the set of all numbers in $[0, 1]$ whose ternary representation can be expressed using only the digits 0 and 2. An equivalent description, which follows, provides us a means to open the cellar door. First note that $\mathfrak{C} = \left\{\sum_{n=0}^{\infty} \frac{2s(n)}{3^{n+1}} \colon s \in 2^\omega\right\}$, where 2^ω is the set of all functions from

$\omega := \{0, 1, 2, \dots\}$ into $2 := \{0, 1\}$. On $\mathfrak{C}$ we define $c\left(\sum_{n=0}^{\infty} \frac{2s(n)}{3^{n+1}}\right) := \sum_{n=0}^{\infty} \frac{s(n)}{2^{n+1}}$ and, for $x \in [0,1] \setminus \mathfrak{C}$, set $c(x) := c(\inf \mathfrak{C} \cap [x, 1])$. Then c is continuous, nondecreasing, and has zero derivative on the open dense set $[0,1] \setminus \mathfrak{C}$ of full Lebesgue measure. At the same time, it maps the nowhere dense measure zero set $\mathfrak{C}$ onto $[0,1]$, which, at first glance, seems counterintuitive.

It is well known that c is not differentiable on $\mathfrak{C}$. (For example, $c'(x)$ does not exist for $x = 2/3 \in \mathfrak{C}$, since $\frac{c(x+2\cdot 3^{-n})-c(x)}{2\cdot 3^{-n}} = \frac{2^{-n}}{2\cdot 3^{-n}} \to \infty$ as $n \to \infty$. But note that the set of points of nondifferentiability of c is considerably smaller than $\mathfrak{C}$, since, as proved by R. Darst [**6**], it has Hausdorff dimension $[\ln(2)/\ln(3)]^2$, while $\mathfrak{C}$ has Hausdorff dimension $\ln(2)/\ln(3)$.) For our purposes, a convenient way to see the nondifferentiability of c is by noting the following result; see for example [**9**, p. 355].

Fact. *If $F \colon \mathbb{R} \to \mathbb{R}$ is differentiable on a measurable set $X \subset \mathbb{R}$ and $|F'(x)| \le L$ for all $x \in X$, then the measure $m(F[X])$ of $F[X]$ is less than or equal to $L \cdot m(X)$.*

Indeed, the fact implies that the function c cannot be differentiable, as otherwise $[0,1] = c[\mathfrak{C}] = \bigcup_{L=1}^{\infty} c[\{x \in \mathfrak{C} \colon |c'(x)| \le L\}]$ would have measure zero.

The fact seems also to imply that if $L \in [0, 1)$, then $F[X]$ should be smaller than X, making it impossible for $F[X]$ to contain X. Although this claim is true for any X of finite positive measure, the next example shows that it can fail badly for compact perfect sets X of measure zero. More specifically, in [**5**], K. C. Ciesielski and J. Jasinski constructed a differentiable function $F \colon \mathbb{R} \to \mathbb{R}$ and a compact perfect set $\mathfrak{X} \subset \mathbb{R}$ such that $F[\mathfrak{X}] = \mathfrak{X}$ while $F' \equiv 0$ on $\mathfrak{X}$. Thus, the map $\mathfrak{f} := F \restriction \mathfrak{X}$, the restriction of F to $\mathfrak{X}$, is *pointwise contractive but globally stable* (in the sense that $\mathfrak{f}[\mathfrak{X}] = \mathfrak{X}$). The map $\mathfrak{f}$ is defined as $\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $\sigma \colon 2^\omega \to 2^\omega$ is the *add-one-and-carry adding machine* (i.e., $\sigma(\langle 1, 1, \dots, 1, 0, s_{k+1}, s_{k+2}, \dots\rangle) := \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots\rangle$ and $\sigma(\langle 1, 1, 1, \dots\rangle) := \langle 0, 0, 0, \dots\rangle$) and $h \colon 2^\omega \to \mathbb{R}$ is an appropriate embedding that ensures that $\mathfrak{f}' \equiv 0$. (For more on adding machines, see [**8**].) Such an $\mathfrak{f}$ can be extended to a differentiable $F \colon \mathbb{R} \to \mathbb{R}$ by Jarník's differentiable extension theorem, an elementary proof of which can be found in [**4**]. The embedding h can be defined by the following formula, based on its variants from [**5**] and [**2**]:

$$h(s) := \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)},$$

where $N(s \restriction n)$ is the natural number for which the following 0-1 sequence[1] $\nu(s, n) = \langle 1, 1 - s_{n-1}, s_{n-2}, \dots, s_0\rangle$ is its binary representation, that is, we have $N(s \restriction n) := \sum_{i<n-1} s_i 2^i + (1 - s_{n-1}) 2^{n-1} + 2^n$.

Clearly, $2^n \le N(s \restriction n) \le \sum_{i \le n} 2^i < 2^{n+1}$ for every $s \in 2^\omega$ and $n \in \omega$. Hence, the sequence $\langle N(s \restriction n) \colon n \in \omega\rangle$ is strictly increasing and h is an embedding into $\mathfrak{C}$. So, $\mathfrak{X} = h[2^\omega] \subset \mathfrak{C}$. Actually, $\mathfrak{f}$ is an auto-homeomorphism (of $\mathfrak{X}$) with every orbit dense, since so is σ. The proof that $\mathfrak{f}' \equiv 0$ is presented in Section 4.

2. THE CONSTRUCTION OF A DIFFERENTIABLE MONSTER. Our example is given as $f(x) := h(x - t) - h(x)$, where h is the inverse of a version of a Pompeiu's function defined below. The new part in our construction is a *shift trick*: a short and simple argument that a Baire-category typical $t \in \mathbb{R}$ gives a correct f.

Pompeiu's functions. Let $Q := \{q_i \colon i \in \mathbb{N}\}$ be an enumeration of any countable dense subset of $\mathbb{R}$ (e.g., the set $\mathbb{Q}$ of rational numbers) such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

[1] $\nu(s, n)$ is obtained from $s \restriction n = \langle s_0, \dots, s_{n-1}\rangle$ by changing its last digit s_{n-1} to $1 - s_{n-1}$, appending 1 at the end, and reversing the order. The "appending 1" step is to ensure that $2^n \le N(s \restriction n)$. The "changing" step is the key new trick, that comes from [**2**], needed to prove property (a) from Section 4.

Fix an $r \in (0,1)$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sum_{i=1}^{\infty} r^i (x-q_i)^{1/3}$. We will rely on the following result, the proof of which is given in Section 3. Intuitively, g has a non-horizontal tangent line at every point, vertical at each $(q_i, g(q_i))$, since the same is true for the map $(x-q_i)^{1/3}$; see Figure 1(c). The graph of its inverse, h, is the reflection of the graph of g with respect to the line $y = x$. So, h also has a tangent line at every point, the vertical tangent lines of g becoming horizontal tangents of h; see Figure 1(d).

Proposition. *The function g is continuous, strictly increasing, and onto $\mathbb{R}$. Its inverse h is everywhere differentiable with $h' \geq 0$ and such that $Z := \{x \in \mathbb{R}\colon h'(x) = 0\}$ is dense in $\mathbb{R}$. Moreover, $\mathbb{R} \setminus Z$ is also dense in $\mathbb{R}$ and Z is a G_δ-set (i.e., it is a countable intersection of open sets).*

The maps g are closely related to the well-studied functions (see, e.g., [**?**], [**3**], or [**17**, Sec. 9.7]) constructed in the 1907 paper [**15**] of D. Pompeiu, where they are defined by $\gamma(x) := \sum_{i=1}^{\infty} A_i (x-q_i)^{1/3}$, with $\{q_i\colon i \in \mathbb{N}\}$ being dense in some bounded interval (a,b), while the numbers A_i are positive with $\sum_{i=1}^{\infty} A_i < \infty$. These definitions ensure that the functions $\gamma(x)$, as well as our $g(x)$, are continuous, being uniform limits of continuous functions. Notice, the set Q is not restricted to be contained in a bounded interval as in Pompeiu's definition. We can do this since we replace the more general absolutely convergent series $\sum A_i$ with the geometric series $\sum r^i$, so we have more leeway with the choice of q_i. (For instance, bounding $|q_i|$ by any fixed power of i still leads to a continuous g.) The proof that the inverses of g and γ have derivative 0 on the image of $\{q_i\colon i \in \mathbb{N}\}$ is, in both cases, the same.

A differentiable monster. Using the properties of our variant of Pompeiu's functions given in the proposition, the construction is just a few lines, as presented in the box below. Note, that the construction works for any increasing function g for which $g'(x) \in (0,\infty]$ exists for all $x \in \mathbb{R}$ and equals ∞ on a dense subset of $\mathbb{R}$. The density of G from the box follows from the Baire category theorem on $\mathbb{R}$.

Smooth and rugged $f(x) := h(x-t) - h(x)$: an oxymoron in calculus?

Let h and Z be as in the proposition and $D \subset \mathbb{R} \setminus Z$ be countable and dense. Since Z is a dense G_δ-set, so is $G := \bigcap_{d \in D} \big((-d+Z) \cap (d-Z)\big)$. Any $t \in G$ makes f, shown in Figure 1(b), a differentiable monster.

Indeed, f is clearly differentiable with $f'(x) = h'(x-t) - h'(x)$. Also, $f' > 0$ on $t+D$, since for every $d \in D$ we have $t+d \in Z$, so that $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$. Similarly, $f' < 0$ on D, since for every $d \in D$ we have $d - t \in Z$, so that $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$.

It is not clear why this argument was not previously discovered, as the idea of the construction of a differentiable monster as a difference of two Pompeiu-like functions was considered earlier, as in [**3**, pp. 66–67]. Our argument might have been easily included in the 1907 paper [**15**] of D. Pompeiu, as all the tools we use were already present there.

3. PROOF OF THE PROPOSITION. The series $g(x) = \sum_{i=1}^{\infty} r^i (x-q_i)^{1/3}$ converges uniformly on every bounded set: $|g(x)| \leq \sum_{i=1}^{\infty} r^i (|x| + i + 1)$, as $\big|(x-q_i)^{1/3}\big| \leq (|x| + |q_i| + 1)^{1/3} \leq |x| + |q_i| + 1 \leq |x| + i + 1$. Thus, g is continuous. It is strictly increasing and onto $\mathbb{R}$, since that is true of every term $\psi_i(x) := r^i (x-q_i)^{1/3}$.

The trickiest part is to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x) \left(= \sum_{i=1}^{\infty} r^i \tfrac{1}{3} \tfrac{1}{(x-q_i)^{2/3}}\right). \quad (1)$$

However, this holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$, we have $\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$, and the last expression is arbitrarily large for large enough n and y close enough to x. On the other hand, when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, then (1) follows from the fact that $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$ for every $y \neq x$.[2] Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\begin{aligned}\left|\tfrac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x)\right| &\leq \sum_{i=1}^{n} \left|\tfrac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + 7\left|\sum_{i=n+1}^{\infty} \psi_i'(x)\right| \\ &\leq \sum_{i=1}^{n} \left|\tfrac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + \tfrac{\varepsilon}{2},\end{aligned}$$

which is less than ε for y close enough to x.

Now, by (1), $g'(x) = \infty$ on the dense set Q. So, $h = g^{-1}$ is strictly increasing and differentiable, with $h' \geq 0$. The set $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ is dense in $\mathbb{R}$, since it contains the dense set $g[Q]$. It is a G_δ-set, since h' is Baire class one (being a pointwise limit of continuous functions $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$) and a preimage of any closed set for such a map is a G_δ-set. (In our case, $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\geq N} h_n^{-1}(-1/i, 1/i)$.) The complement of Z must be dense, since otherwise h would be constant on some interval.

4. PROOF THAT $\mathfrak{f}' \equiv 0$. This follows from the following two observations:

(a) For every $s \in 2^\omega$ there is a $k \in \omega$ such that $N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for every $n > k$.

(b) If $n = \min\{i \in \omega \colon s_i \neq t_i\}$ for some distinct $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ from 2^ω, then $3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}$.

Assuming these two observations to be true, to see that $\mathfrak{f}'(h(s)) = 0$ for an $s \in 2^\omega$, choose a $k \in \omega$ satisfying (a) and let $\delta > 0$ be such that $0 < |h(s) - h(t)| < \delta$ implies that $n = \min\{i \in \omega \colon s_i \neq t_i\}$ is greater than k. Fix a $t \in 2^\omega$ for which $0 < |h(s) - h(t)| < \delta$. Then $n = \min\{i \in \omega \colon s_i \neq t_i\} = \min\{i \in \omega \colon \sigma(s)_i \neq \sigma(t)_i\}$ and, using (a) and (b) for the pairs $\langle s, t\rangle$ and $\langle \sigma(s), \sigma(t)\rangle$,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

Hence $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for δ small enough.

Property (a) holds since $N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for every $s \in 2^\omega$ and $n > 0$ for which $\nu(s, n) \neq \langle 1, \ldots, 1\rangle$, that is, when $s \restriction n \neq \langle 1, \ldots, 1, 0\rangle$.

To see (b), notice that, for every $s \in 2^\omega$ and $n > 0$,

$$2s_n 3^{-(n+1)N(s\restriction n)} \leq \sum_{k\geq n} 2s_k 3^{-(k+1)N(s\restriction k)} \leq (2s_n + 1)3^{-(n+1)N(s\restriction n)}, \quad (*)$$

[2] It is enough to prove this for $\psi(x) := x^{1/3}$. It holds for $x = 0$, as $\psi'(0) = \infty$. Also, since $\psi(x)$ is odd and concave on $(0, \infty)$, we can assume that $x > 0$ and $y < x$. Then $L(y) < \psi(y)$, where L is the line passing through $(x, \psi(x))$ and $(0, -\psi(x))$. So, $0 < \frac{\psi(x)-\psi(y)}{x-y} < \frac{L(x)-L(y)}{x-y} = \frac{2x^{1/3}}{x} = 6\psi'(x)$, as needed.

where the second inequality holds, since series reminder $\sum_{k>n} 2s_k 3^{-(k+1)N(s\upharpoonright k)}$ is bounded above by the quantity $2\sum_{i=1}^{\infty} 3^{-((n+1)N(s\upharpoonright n)+i)} = 3^{-(n+1)N(s\upharpoonright n)}$. We can assume that $s_n = 0$ and $t_n = 1$, as $s \upharpoonright n = t \upharpoonright n$. Then, by property $(*)$, $h(t) - h(s) = \sum_{k\geq n} 2t_k 3^{-(k+1)N(t\upharpoonright k)} - \sum_{k\geq n} 2s_k 3^{-(k+1)N(s\upharpoonright k)}$ is bounded below by $2t_n 3^{-(n+1)N(t\upharpoonright n)} - (2s_n+1)3^{-(n+1)N(s\upharpoonright n)} = 2\cdot 3^{-(n+1)N(s\upharpoonright n)} - 3^{-(n+1)N(s\upharpoonright n)}$, as needed. Similarly, again by $(*)$, $|h(t) - h(s)| = h(t) - h(s)$ is bounded above by $\sum_{k\geq n} 2t_k 3^{-(k+1)N(t\upharpoonright k)} \leq (2t_n + 1)3^{-(n+1)N(t\upharpoonright n)} = 3 \cdot 3^{-(n+1)N(s\upharpoonright n)}$.

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