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MINIMAL DEGREES OF GENOCCHI-PEANO FUNCTIONS: CALCULUS MOTIVATED NUMBER THEORETICAL ESTIMATES

Abstract

A rational function of the form $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ is a *Genocchi-Peano example, GPE*, provided it is discontinuous, but its restriction to any hyperplane is continuous. We show that the minimal degree $D(n)$ of a GPE of n-variables equals $2\left\lfloor \frac{e^2}{e^2-1} n \right\rfloor + 2i$ for some $i \in \{0,1,2\}$. We also investigate the minimal degree $D_b(n)$ of a bounded GPE of n-variables and note that $D(n) \leq D_b(n) \leq n(n+1)$. Finding better bounds for numbers $D_b(n)$ remains an open problem.

1 Historical background

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *separately continuous* provided it is continuous with respect to each variable separately, that is, for every $x, y \in \mathbb{R}$ the maps $t \mapsto f(t,y)$ and $t \mapsto f(x,t)$ are continuous. The 1821 mathematical analysis textbook *Cours d'analyse* [1] of Augustin-Louis Cauchy contains (on pages 38-39) the following

Theorem X: *A separately continuous map of real variables is continuous.*

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Since essentially every modern multivariable calculus textbook contains the following counterexample for Theorem X,

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{for } \langle x,y\rangle\neq\langle 0,0\rangle\\ 0 & \text{for } \langle x,y\rangle=\langle 0,0\rangle,\end{cases}$$

which first appeared in 1884 treatise on calculus by Genocchi and Peano [5], it is only natural to claim that Cauchy made a mistake. However, such claim would be unwarranted, since, actually, Theorem X is true in the setting of Cauchy's text, which is written for the set $\mathscr{R}$ of real numbers containing infinitesimals, rather than nowadays standard set $\mathbb{R}$ of reals. (See [3].)

Not surprisingly, since the mid 19th century, when analysts firmly chose the use of the standard set $\mathbb{R}$ of reals over one containing infinitesimals, there has been a great deal of research activity on the relationship between standard continuity and separate continuity and its generalizations. In particular, this subject was studied by E. Heine, H. Lebesgue, G. Peano, R. Baire, W. Sierpiński, N. Luzin, E. Marczewski, and A. Rosenthal, among others. For more on this history, see the recent survey [3]. For this paper, the crucial result in this direction is yet another example from the text of Genocchi and Peano, [5]:

$$g(x,y)=\begin{cases}\frac{xy^2}{x^2+y^4} & \text{when } (x,y)\neq(0,0)\\ 0 & \text{otherwise.}\end{cases}\tag{1}$$

Clearly, this function g is discontinuous along the parabola $x=y^2$, but its restriction to any line (i.e., hyperplane in $\mathbb{R}^2$) is continuous. Notice also that this g is bounded: $|g(x,y)|\leq 1$ for every $x,y\in\mathbb{R}$.

2 Preliminaries

In the recent article [4], the author and D. Miller investigated the problem of how to generalize the example (1) to higher dimensions, in the sense that it should be discontinuous, but have a continuous restriction to any hyperplane in $\mathbb{R}^n$. In particular, it was noticed there that such generalizations can be found among functions of the form

$$g(x_1,x_2,\ldots,x_n)=\begin{cases}\frac{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}} & \text{when } (x_1,x_2,\ldots,x_n)\neq(0,0,\ldots,0),\\ 0 & \text{otherwise,}\end{cases}\tag{2}$$

where $\frac{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ is a rational function, that is, $\alpha_i,\beta_i\in\mathbb{N}=\{1,2,3,\ldots\}$ for all i. Since g from (1) is clearly of this form, we say that $g\colon\mathbb{R}^n\to\mathbb{R}$ of the form of (2) is a *Genocchi-Peano example* (abbreviated *GPE*), if g is discontinuous but its restriction to any hyperplane in $\mathbb{R}^n$ is continuous.

Notice that if some β_i is odd, then g, in the form of (2), is not a GPE, since it is discontinuous on any hyperplane containing a non-origin point $y = (y_1, \dots, y_n)$ satisfying $\sum_{i=1}^n y_i^{\beta_i} = 0$ (e.g., with $y_j = 1$ for $j \neq i$ and $y_i = \sqrt[\beta_i]{1-n}$). Therefore, in the rest of the paper we will concentrate on the cases when all β_i's are even. With this, we will need to check the continuity of g only when restricted to the hyperplanes that contain the origin, that is, expressible as $\sum_{i=1}^n b_k x_k = 0$. For the rest of this paper we will assume that every rational function expression $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1} + x_2^{\beta_2} + \cdots + x_n^{\beta_n}}$ is of the form (2), that is, takes value 0 at the origin $(0, 0, \dots, 0)$. Also, because of symmetry, we will always assume that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$.

The following result from [4] gives a characterization of all GPEs.

Theorem 1. *Let g be given by (2) with $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ positive even numbers.*

(i) g is discontinuous if, and only if, $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \leq 1$.

(ii) g has a continuous restriction to every hyperplane if, and only if,

$$\left(\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \right) - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} > 1 \text{ for every } k \in \{2, \dots, n\}. \tag{3}$$

In particular, g is a GPE if, and only if, $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \leq 1$ and (3) holds. Moreover,

(iii) g is a bounded GPE if, and only if, $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} = 1$ and all β_is are distinct.

Notice, that the value of $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$ from (3) can be calculated by replacing β_k with β_{k-1} in the expression $\frac{\alpha_1}{\beta_1} + \cdots + \frac{\alpha_k}{\beta_k} + \cdots + \frac{\alpha_n}{\beta_n} = \sum_{i=1}^n \frac{\alpha_i}{\beta_i}$.

SKETCH OF PROOF OF THEOREM 1. The argument is based on the equation $g(x_1, \dots, x_n) = \frac{1}{d^{1-\gamma}} \prod_{i=1}^n \frac{(x_i)^{\alpha_i}}{d^{\alpha_i/\beta_i}}$, where $\gamma = \sum_{i=1}^n \frac{\alpha_i}{\beta_i}$ and $d = x_1^{\beta_1} + \cdots + x_n^{\beta_n}$. From this we get $|g(x_1, \dots, x_n)| \leq d^{\gamma - 1}$ and (i) follows from $g\left(t^{1/\beta_1}, \dots, t^{1/\beta_n}\right) = \frac{t^{\gamma-1}}{n}$. To see the necessity of (3) notice that, for $\delta_k = \sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}}$ and $f_i(t)$ defined as t^{1/β_i} for $i \neq k$ and as $t^{1/\beta_{k-1}}$ for $i = k$, we have the following equality $g(f_1(t), \dots, f_n(t)) = \frac{1}{(n-1) + t^{(\beta_k/\beta_{k-1}) - 1}} t^{\delta_k - 1}$. The condition (3) is sufficient since, for every hyperplane given by an equation $x_k = \sum_{i=1}^{k-1} a_i x_i$, we have $|g(x_1, \dots, x_n)| \leq A^{\alpha_k} d^{\delta_k - 1}$, where $A = \sum_{i=1}^{k-1} |a_i|$. The boundedness claim is justified by $g(x_1, \dots, x_n) = \frac{1}{d^{1-\gamma}} \prod_{i=1}^n \frac{(x_i)^{\alpha_i}}{d^{\alpha_i/\beta_i}}$. For more details, see [4]. ∎

Theorem 1 immediately implies that, for any $n \geq 2$, the following functions are bounded GPEs, compare [4] and, for the first example, also [2]:

$$\frac{x_1 x_2 \cdots x_{n-1} x_n^2}{x_1^2 + x_2^4 + \cdots + x_{n-1}^{2^{n-1}} + x_n^{2^n}} \quad \text{and} \quad \frac{x_1^2 \cdots x_i^{2i} \cdots x_n^{2n}}{x_1^{2n} + \cdots + x_i^{2in} + \cdots + x_n^{2n^2}}. \tag{4}$$

3 The simplest GPEs of n-variables

If g given by (2) is a GPE then, by Theorem 1, $1 \geq \sum_{i=1}^n \frac{\alpha_i}{\beta_i} > \frac{\sum_{i=1}^n \alpha_i}{\beta_n}$. In particular, the degree $\sum_{i=1}^n \alpha_i$ of the numerator of such g is always smaller that the degree of its denominator, β_n. Thus, for a GPE g we define its degree, $\deg(g)$, as the degree of its denominator, that is, $\deg(g) = \beta_n$. In particular, the numbers

$$D(n) = \min\{\deg(g) \colon g \text{ is a GPE of } n \text{ variables}\},$$

defined for $n \geq 2$, represent a measure of how simple the GPEs of n variables can be. It has been noticed in [4] that $2n \leq D(n) \leq \min\{2^n, 2n^2\}$, where the second inequality is justified by (4). The main goal of this paper is to prove the following theorem, which provides a very tight estimate of the value of $D(n)$. The examples of minimal degree GPEs can be seen in Table 2.

In what follows, for every $n \geq 2$, the symbol k_n indicates the smallest term in the harmonic series after which the sum of n consecutive terms is ≤ 2:

$$k_n = \min\left\{k \in \{0, 1, 2, \ldots\} \colon \sum_{i=1}^n \frac{1}{k+i} \leq 2\right\}.$$

Some of the numbers k_n can be seen in Tables 1, 2, and 3. The limiting value of k_n will be established in Lemma 5.

The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ denote, respectively, the floor and ceiling functions of x, that is, the largest integer $\leq x$ and the smallest integer $\geq x$.

Theorem 2. *For every* $n = 2, 3, 4, \ldots$ *we have*

$$k_n \in \left\{\left\lfloor \frac{1}{e^2 - 1} n \right\rfloor, \left\lceil \frac{1}{e^2 - 1} n \right\rceil\right\} \tag{5}$$

and

$$D(n) \in \{2(k_n + n), 2(k_n + n) + 2\}. \tag{6}$$

In particular,

$$D(n) = 2\left\lfloor \frac{e^2}{e^2 - 1} n \right\rfloor + i \in \left(\frac{2e^2}{e^2 - 1} n - 2, \frac{2e^2}{e^2 - 1} n + 4\right) \subset (2.31n - 2, 2.32n + 4)$$

for some $i \in \{0,2,4\}$. Moreover, for every $n \geq 2$ there is a GPE $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ of minimal degree such that $\alpha_i = 1$ for all but at most three indices i. In addition, if $D(n) = 2(k_n + n) + 2$, then β_js can be chosen as consecutive even numbers.

The proof of Theorem 2 will be based on two propositions and one lemma, each being of independent interest. In particular, the propositions give the conditions for $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ to be a GPE that are much easier to check than (3) in Theorem 1.

We start with the following simple corollary to Theorem 1.

Proposition 3. *Let $n > 1$, g be given by (2), and numbers $\beta_1 < \beta_2 < \cdots < \beta_n$ be positive and even. If $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \leq 1 < \sum_{i=1}^n \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n-2)}$, then g is a GPE.*

Proof. Indeed such g clearly satisfies (i) of Theorem 1. It also satisfies (ii) since $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} - \frac{\alpha_k}{\beta_k} + \frac{\alpha_k}{\beta_{k-1}} = \sum_{i=1}^n \frac{\alpha_i}{\beta_i} + \alpha_k \frac{\beta_k-\beta_{k-1}}{\beta_k\beta_{k-1}} \geq \sum_{i=1}^n \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n-2)} > 1$. ■

The next result is the key step in the proof of Theorem 2, used for finding the upper bound for $D(n)$.

Proposition 4. *Let $k, n \in \{0,1,2,\ldots\}$ be such that $n \geq k+2$ and for every $i = 1,\ldots,n$ let $\beta_i = 2(k+i)$. If $\sum_{i=1}^n \frac{1}{\beta_i} + \frac{4}{\beta_n} \leq 1$, then there exist α_i's such that $\frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ is a GPE. Moreover, $\alpha_i = 1$ for all but at most three indices i.*

Proof. It is enough to find α_i's, at most three of which are greater than 1, for which the assumptions of Proposition 3 are satisfied. Our first approximation of α_i's will be by putting $\alpha_i = 1$ for all $i < n$ and defining α_n as the largest integer for which $S_0 = \sum_{i=1}^n \frac{\alpha_i}{\beta_i} = \sum_{i=1}^{n-1} \frac{1}{\beta_i} + \frac{\alpha_n}{\beta_n} \leq 1$. By our assumption that $\sum_{i=1}^n \frac{1}{\beta_i} + \frac{4}{\beta_n} \leq 1$, we have $\alpha_n \geq 5$.

Now, if $S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} > 1$, then $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n-2)} = S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} > 1$ and the assumptions of Proposition 3 are already satisfied, so the conclusion of Proposition 4 holds. Thus, assume that $S_0 + \frac{1}{\beta_{n-1}} - \frac{1}{\beta_n} \leq 1$. Pick the smallest $j \in \{1,2,\ldots,n-1\}$ for which $S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \leq 1$. Such a j exists, as $j = n-1$ satisfies this inequality.

Since, by the maximality of α_n, we have the inequality $S_0 + \frac{1}{\beta_n} > 1$, we conclude that $S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \leq 1 < S_0 + \frac{1}{\beta_n}$. Therefore, $\beta_n/2 < \beta_j$. In particular, $j > 1$, since otherwise $k + n = \beta_n/2 < \beta_1 = 2(k+1)$, what contradicts our assumption that $n \geq k+2$.

As the next approximation, modify α_i's by decreasing the previous value of α_n by 1 and by putting $\alpha_j = 2$. Then, for the new α_i's, we have $\alpha_n \geq 4$ and the relation $S_1 = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = S_0 + \frac{1}{\beta_j} - \frac{1}{\beta_n} \leq 1 < S_0 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_n} = S_1 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_j}$, where the strict inequality follows from the minimality of j. Since, by $\beta_n/2 < \beta_j$, we also have $\beta_{n-1}/2 = (\beta_n/2) - 1 \leq \beta_j - 2 = \beta_{j-1}$, we get

$$1 < S_1 + \frac{1}{\beta_{j-1}} - \frac{1}{\beta_j} = S_1 + \frac{2}{\beta_{j-1}\beta_j} < S_1 + \frac{2}{(\beta_{n-1}/2)(\beta_n/2)} = S_1 + 4\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right).$$

Let $m \in \{0,1,2,3\}$ be such that $S_1+m\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) \leq 1 < S_1+(m+1)\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right)$ and modify α_i's by decreasing the previous value of α_n by m (afterwards we will still have $\alpha_n \geq 1$) and increasing the previous value of α_{n-1} by m. We claim that with these new α_i's the assumptions of Proposition 3 are satisfied. Indeed, for these new coefficients α_i's we have

$$\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = S_1+m\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) \leq 1 < S_1+(m+1)\left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_n}\right) = \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} + \frac{2}{\beta_n(\beta_n - 2)},$$

as required. ■

The next lemma concerns the possible values of numbers k_n.

Lemma 5. *For every $n \geq 2$ we have*

$$k_n \in \left(\frac{1}{e^2-1}n - 1, \frac{1}{e^2-1}n + 1\right). \tag{7}$$

In particular, (5) holds. Moreover, $\lim_{n\to\infty} \frac{k_n}{\frac{1}{e^2-1}n} = 1$.

Proof. Clearly $1 + x < \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$ for all $x > 0$. In particular, for any $x = \frac{1}{k+i}$ with $k \geq 0$ and $i \geq 1$, we get $\frac{k+i+1}{k+i} = 1 + \frac{1}{k+i} < e^{\frac{1}{k+i}}$. So,

$$\frac{k+n+1}{k+1} = \frac{k+2}{k+1} \cdot \frac{k+3}{k+2} \cdot \dots \cdot \frac{k+n+1}{k+n} = \prod_{i=1}^{n}\left(1 + \frac{1}{k+i}\right) < \prod_{i=1}^{n} e^{\frac{1}{k+i}} = e^{\sum_{i=1}^{n} \frac{1}{k+i}}.$$

Using this with $k = k_n$, we obtain $\sum_{i=1}^{n} \frac{1}{k+i} \leq 2$ and $1 + \frac{n}{k_n+1} = \frac{k_n+n+1}{k_n+1} < e^2$. Hence, $k_n > \frac{n}{e^2-1} - 1$.

Similarly, $1 - x < \sum_{i=0}^{\infty} \frac{(-x)^i}{i!} = e^{-x}$ for all $x \in (0,1]$. In particular, for any $x = \frac{1}{k+i}$ with $k \geq 0$ and $i \geq 1$, we get $\frac{k+i-1}{k+i} = 1 - \frac{1}{k+i} < e^{-\frac{1}{k+i}}$. So,

$$\frac{k}{k+n} = \frac{k}{k+1} \cdot \frac{k+1}{k+2} \cdot \dots \cdot \frac{k+n-1}{k+n} = \prod_{i=1}^{n}\left(1 - \frac{1}{k+i}\right) < \prod_{i=1}^{n} e^{-\frac{1}{k+i}} = e^{-\sum_{i=1}^{n} \frac{1}{k+i}}.$$

Using this with $k = k_n - 1$ for which $k_n > 0$, we obtain $\sum_{i=1}^n \frac{1}{k+i} > 2$ and $\frac{k_n-1}{k_n-1+n} < e^{-\sum_{i=1}^n \frac{1}{k+i}} < e^{-2}$. Hence $1 + \frac{n}{k_n-1} = \frac{k_n-1+n}{k_n-1} > e^2$ and $k_n < \frac{n}{e^2-1} + 1$. Since this last inequality is also true for $k_n = 0$, we conclude that $\frac{n}{e^2-1} - 1 < k_n < \frac{n}{e^2-1} + 1$, the desired property (7), holds for every $n \ge 2$. Moreover, $1 - \frac{e^2-1}{n} < \frac{k_n}{\frac{1}{e^2-1}n} < 1 + \frac{e^2-1}{n}$. So, by the squeeze theorem, $\lim_{n\to\infty} \frac{k_n}{\frac{1}{e^2-1}n} = 1$. ∎

Proof of Theorem 2. Clearly (5) follows from Lemma 5. Next, we will prove (6), which follows from the inequality $2(k_n + n) \le D(n) \le 2(k_n + n) + 2$.

First, we will justify that $D(n) \ge 2(k_n + n)$. To see this, let g be a GPE given by (2) of minimal degree, that is, with $D(n) = \deg(g) = \beta_n = 2m$ for some natural number m. Then, by Theorem 1, the numbers $\beta_1 < \cdots < \beta_n$ are even and $\sum_{i=1}^n \frac{\alpha_i}{\beta_i} \le 1$. Hence $\beta_{n-i} \le 2(m-i)$ and

$$1 \ge \sum_{i=1}^n \frac{\alpha_i}{\beta_i} \ge \sum_{i=1}^n \frac{1}{\beta_i} \ge \sum_{i=0}^{n-1} \frac{1}{2(m-i)} = \frac{1}{2}\sum_{i=1}^n \frac{1}{(m-n)+i}.$$

Thus, $\sum_{i=1}^n \frac{1}{(m-n)+i} \le 2$, that is, the number $k = m - n$ satisfies $\sum_{i=1}^n \frac{1}{k+i} \le 2$. Hence, by the minimality of k_n, we have $k_n \le m - n$. Therefore, we also have $D(n) = 2m \ge 2(k_n + n)$, as needed.

Next, we will justify the inequality $D(n) \le 2(k_n + n) + 2$. First, we will show this assuming that the number n does not belong the following set of exceptions: $E = \{2, 3, 4, 5, 6, 7, 10, 11\}$.

So, assume that $n \notin E$ and put $k = k_n + 1$. We will show that such numbers satisfy the assumptions of Proposition 4. This will give the desired inequality, since then $D(n) \le 2(n + k) = 2(n + k_n) + 2$, as needed.

To see that the inequality $n \ge k+2$, needed for Proposition 4, holds for $n \notin E$ notice that for $k = k_n + 1$ it becomes $n \ge k_n + 3$, that is, $n - k_n \ge 3$. But this holds for any $n \ge 8$, since, by the inequalities $k_n < \frac{1}{e^2-1}n + 1$, proved in Lemma 5, and $\frac{1}{e^2-1} < 0.2$, we have $n - k_n > n - \left(\frac{1}{e^2-1}n + 1\right) > 0.8n - 1 \ge 0.8 \cdot 8 - 1 > 3$.

To see that, for $\beta_i = 2(k+i)$, we have $\sum_{i=1}^n \frac{1}{\beta_i} + \frac{4}{\beta_n} \le 1$, the other requirement of Proposition 4, first notice that $\sum_{i=1}^n \frac{1}{k_n+i} \le 2$. This, in particular, implies that $\sum_{i=1}^n \frac{1}{k+i} = \sum_{i=1}^n \frac{1}{k_n+1+i} = \sum_{i=1}^n \frac{1}{k_n+i} + \frac{1}{k_n+1+n} - \frac{1}{k_n+1} \le 2 - \left(\frac{1}{k_n+1} - \frac{1}{k_n+1+n}\right)$. By this and the equality $\frac{1}{k_n+1} - \frac{1}{k_n+1+n} = \left(\frac{n}{k_n+1}\right)\frac{1}{k_n+1+n} = \left(\frac{n}{k_n+1}\right)\frac{2}{\beta_n}$ we see that $\sum_{i=1}^n \frac{1}{\beta_i} = \frac{1}{2}\sum_{i=1}^n \frac{1}{k+i} \le 1 - \left(\frac{n}{k_n+1}\right)\frac{1}{\beta_n}$, that is, $\sum_{i=1}^n \frac{1}{\beta_i} + \left(\frac{n}{k_n+1}\right)\frac{1}{\beta_n} \le 1$. Therefore, to prove that $\sum_{i=1}^n \frac{1}{\beta_i} + \frac{4}{\beta_n} \le 1$ holds for any $n \notin E$, it is enough to show that

$$\frac{n}{k_n + 1} \ge 4 \quad \text{for any } n \notin E, \tag{8}$$

since then, for $n \notin E$, we have $\sum_{i=1}^{n} \frac{1}{\beta_i} + \frac{4}{\beta_n} \le \sum_{i=1}^{n} \frac{1}{\beta_i} + \left(\frac{n}{k_n+1}\right)\frac{1}{\beta_n} \le 1$, as needed.

To see (8), first notice that $\frac{n}{\frac{1}{e^2-1}n+2} \ge 4$ is equivalent to $n \ge 4\frac{1}{e^2-1}n + 8$ and to $n \ge 8/\left(1 - 4\frac{1}{e^2-1}\right)$, which holds for $n \ge 22$, since $22 > 21.4 > 8/\left(1-4\frac{1}{e^2-1}\right)$. Hence, (8) holds for any $n \ge 22$ as, in this case, using the inequality $k_n < \frac{1}{e^2-1}n + 1$ shown in Lemma 5, we have $\frac{n}{k_n+1} > \frac{n}{\left(\frac{1}{e^2-1}n+1\right)+1} = \frac{n}{\frac{1}{e^2-1}n+2} \ge 4$, as needed. To see that (8) holds for the remaining values of $n \notin E$ we use the values of k_n presented in Table 1: for $n \in \{8,9\}$ we have $\frac{n}{k_n+1} = \frac{n}{2} \ge 4$, for $12 \le n \le 16$ we obtain $\frac{n}{k_n+1} = \frac{n}{3} \ge 4$, while for $17 \le n \le 21$ we get $\frac{n}{k_n+1} = \frac{n}{4} > 4$.

n	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\left\lfloor\frac{n}{e^2-1}\right\rfloor$	1	1	1	1	1	2	2	2	2	2	2	2	3	3
k_n	1	1	2	2	2	2	2	2	2	3	3	3	3	3

Table 1: The values of k_n for $8 \le n \le 21$, checked by simple arithmetic. Note that $k_n = \left\lfloor\frac{n}{e^2-1}\right\rfloor$ only in some cases.

To finish the proof of (6) it is enough show that $D(n) \le 2(k_n + n) + 2$ holds for every $n \in E = \{2,3,4,5,6,7,10,11\}$. But this is justified by the entries in Table 2.

To prove the additional part of the theorem on $D(n)$, notice that, by Lemma 5, $k_n \in \left(\frac{1}{e^2-1}n - 1, \frac{1}{e^2-1}n + 1\right)$ while, by (6), $2(n+k_n) \le D(n) \le 2(1 + n + k_n)$. Hence

$$\begin{aligned} D(n) &\in 2\left(n + \frac{1}{e^2-1}n - 1, 1 + n + \frac{1}{e^2-1}n + 1\right) \\ &= \left(\frac{2e^2}{e^2-1}n - 2, \frac{2e^2}{e^2-1}n + 4\right) \subset (2.31n - 2, 2.32n + 4), \end{aligned}$$

where the last inclusion follows from the fact that $2.31 < \frac{2e^2}{e^2-1} < 2.32$. This clearly implies that $D(n) = 2\left\lfloor\frac{e^2}{e^2-1}n\right\rfloor + i$ for some $i \in \{0,2,4\}$.

To finish the proof, it is enough to show the last part of the theorem, concerning existence of GPEs $\frac{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}{x_1^{\beta_1}+x_2^{\beta_2}+\cdots+x_n^{\beta_n}}$ of minimal degrees (i.e., with $D(n) = \beta_n$) having α_is and β_is of the right format. To see this, fix an $n \ge 2$.

For $n \in E$, a GPE of a correct format exists, as shown in Table 2. We just need to note that each of these examples is of minimal degree. Indeed, this is clearly the case when $n \ne 3$, since for such values $\deg(g) = 2(k_n + n)$ is, by (6), as small as it can be. For $n = 3$ this argument does not work. However,

n	k_n	a GPE g of n variables	deg(g)	$2(k_n+n)+2$
2	0	$\frac{x_1^1 x_2^2}{x_1^2+x_2^4}$	4	6
3	0	$\frac{x_1^1 x_2^3 x_3^2}{x_1^4+x_2^6+x_3^8}$	8	8
4	1	$\frac{x_1^2 x_2^1 x_3^1 x_4^2}{x_1^4+x_2^6+x_3^8+x_4^{10}}$	10	12
5	1	$\frac{x_1^1 x_2^1 x_3^1 x_4^2 x_5^3}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}}$	12	14
6	1	$\frac{x_1^1 x_2^1 x_3^2 x_4^1 x_5^1 x_6^2}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}+x_6^{14}}$	14	16
7	1	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^1 x_6^2 x_7^2}{x_1^4+x_2^6+x_3^8+x_4^{10}+x_5^{12}+x_6^{14}+x_7^{16}}$	16	18
10	2	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^2 x_6^1 x_7^1 x_8^1 x_9^1 x_{10}^4}{x_1^6+x_2^8+x_3^{10}+x_4^{12}+x_5^{14}+x_6^{16}+x_7^{18}+x_8^{20}+x_9^{22}+x_{10}^{24}}$	24	26
11	2	$\frac{x_1^1 x_2^1 x_3^1 x_4^1 x_5^1 x_6^1 x_7^1 x_8^1 x_9^1 x_{10}^2 x_{11}^4}{x_1^6+x_2^8+x_3^{10}+x_4^{12}+x_5^{14}+x_6^{16}+x_7^{18}+x_8^{20}+x_9^{22}+x_{10}^{24}+x_{11}^{26}}$	26	28

Table 2: The examples of GPEs of n-variables, for $n \in E$, with the degrees $\le 2(k_n+n)+2$. Each of these functions is GPE since it satisfies the assumptions of Proposition 3, as an easy calculation shows. The given values of the number k_n can be easily checked.

it is an easy exercise to check that actually $D(3) = 8 = \deg(g)$. (See e.g., [4].) Thus, for the rest of the argument, we will assume that $n \notin E$.

In such case, as we shown above, the numbers n and $k = k_n + 1$ satisfy the assumptions of Proposition 4. In particular, if $D(n) = 2(k_n + n) + 2$, then GPE from Proposition 4 is of minimal degree and of the right format. So, assume that $D(n) < 2(k_n + n) + 2$. In this case, $D(n) = 2(k_n + n)$ and we are only concern about the format of α_is. Let $k = k_n$. If $\sum_{i=1}^n \frac{1}{2(k+i)} + \frac{4}{2(k+n)} \le 1$, then the numbers n and k still satisfy the assumptions of Proposition 4 and, once again, the GPE from Proposition 4 is of minimal degree and of the right format. So, assume that $\sum_{i=1}^n \frac{1}{2(k+i)} + \frac{4}{2(k+n)} > 1$, let g be a GPE given by (2) of minimal degree, and let m be the number of α_is in g greater than 1. Then, $\sum_{i=1}^n \frac{1}{2(k+i)} + \frac{m}{2(k+n)} \le \sum_{i=1}^n \frac{\alpha_i}{\beta_i} \le 1 < \sum_{i=1}^n \frac{1}{2(k+i)} + \frac{4}{2(k+n)}$, that is, $m < 4$, as needed. ∎

4 Remarks and open problems

Although Theorem 2 gives a lot of information about the size of numbers $D(n)$ and the structure of minimal GPEs, there is still a lot that is unknown. For example, we do not know, if for every $n \geq 2$ there exists a minimal GPE of n variables with β_is being consecutive even numbers. Note that not all minimal GPE must have this property, since both $\frac{x_1x_2x_3x_4^2}{x_1^2+x_2^6+x_3^8+x_4^{10}}$ and $\frac{x_1x_2x_3^2x_4^3}{x_1^4+x_2^6+x_3^8+x_4^{10}}$ are minimal GPEs. (A verification can be found in [4].)

The sequence $\langle D(n): n \geq 2 \rangle$ is non-decreasing, since, by Theorem 2, for every $n \geq 2$ we have $D(n) \leq 2(k_n + n + 1) \leq 2(k_{n+1} + (n+1)) \leq D(n+1)$. Also, an inspection of the actual values of $D(n)$ for the first dozen of numbers n, compare Table 3, suggests that this sequence is strictly increasing. However, this is not the case, as $D(14) = D(15) = 34$. Indeed, the map $\frac{x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13}x_{14}x_{15}^2}{x_1^6+x_2^8+x_3^{10}+x_4^{12}+x_5^{14}+x_6^{16}+x_7^{18}+x_8^{20}+x_9^{22}+x_{10}^{24}+x_{11}^{26}+x_{12}^{28}+x_{13}^{30}+x_{14}^{32}+x_{15}^{34}}$ justifies $D(15) = 34 = 2(k_{15} + 15)$, as it satisfies the assumptions of Proposition 3. On the other hand, it requires checking just few cases that $D(14)$ cannot be smaller than $34 = 2(k_{14} + 14) + 2$.

Problem 1. How big is the set $T = \{n \geq 2: D(n) = D(n+1)\}$? Is it infinite?

Currently 14 is the only element of T that we identified.

By Theorem 2, any $n \geq 2$ belongs to either $L = \{n \geq 2: D(n) = 2(k_n + n)\}$ or $H = \{n \geq 2: D(n) = 2(k_n + n) + 2\}$.

Problem 2. Are both sets L and H infinite?

The numerical data from Table 3 suggests that for every $i \in N$, we have $\min\{n \geq 2: k_n = i\} \in L$ and $\max\{n \geq 2: k_n = i\} \in H$. Is this true in general?

Numbers $D_b(n)$: the minimal bounded GPEs The original Genocchi-Peano example, given by (1), is a bounded function. The formula (4) shows that such bounded examples exist for every $n \geq 2$. However, the only known global boundaries for these numbers are given in the following result.[1]

Proposition 6. *$D(n) \leq D_b(n) \leq \min\{2^n, n(n+1)\}$ for all $n \geq 2$.*

[1] The proof of Proposition 6 actually shows that $D_b(n) \leq n^2$ for even n and $D_b(n) \leq n^2+n$ for odd n. In fact, for the odd numbers of the form $n = 4k-1$ this upper bound can be further improved to $D_b(n) \leq n^2 - 1$, since the functions $\frac{x_1^1 \cdot x_2^1 \cdots x_i^i \cdots x_n^{n-1}}{x_1^{2k}+x_2^{n+1}+\cdots+x_i^{(i-1)(n+1)}+\cdots+x_n^{(n-1)(n+1)}}$ are bounded GPEs, as $\frac{1}{2k} + \sum_{i=2}^n \frac{i-1}{(i-1)(n+1)} = \frac{2}{n+1} + (n-1)\frac{1}{n+1} = 1$.

n	k_n	$2(k_n+n)$	$D(n)$	$D_b(n)$	n	k_n	$2(k_n+n)$	$D(n)$
					16	2	36	38
2	0	4	4	4	17	3	40	40
3	0	6	8	8	18	3	42	42
4	1	10	10	12	19	3	44	44
5	1	12	12	16	20	3	46	46
6	1	14	14	18	21	3	48	50
7	1	16	16	20	22	3	50	52
8	1	18	18	24	23	4	54	54
9	1	20	22		24	4	56	56
10	2	24	24		25	4	58	58
11	2	26	26		26	4	60	60
12	2	28	28		27	4	62	62
13	2	30	30		28	4	64	66
14	2	32	34		29	5	68	68
15	2	34	34		30	5	70	70

Table 3: Computer calculated values of k_n, $2(k_n+n)$, $D(n)$, and (some of) $D_b(n)$ for $2 \le n \le 30$.

Proof. The inequality $D(n) \le D_b(n)$ is obvious. The upper bound of $D_b(n)$ is justified by the left example from (4) and the following modifications of the right example from (4): for odd n by $\frac{x_1^1\cdots x_i^i\cdots x_{n-1}^{n-1}x_n^{2n}}{x_1^{n+1}+\cdots+x_i^{i(n+1)}+\cdots+x_n^{n(n+1)}}$; for even n by $\frac{x_1^1\cdots x_i^i\cdots x_n^n}{x_1^n+\cdots+x_i^{in}+\cdots+x_n^{n^2}}$. ∎

Problem 3. Find the bounds for the numbers $D_b(n)$ better than those given in Proposition 6.

We do not know, if the set $Z = \{n \geq 2 \colon D_b(n) = D(n)\}$ is finite. (By Table 3, we have $2, 3 \in Z$ and $4, 5, 6, 7, 8 \notin Z$.) Notice, that the examples justifying the values of $D_b(n)$ cannot be of the form of those constructed in Proposition 4, since, by Bertrand's Postulate, in these examples there exists an $i \in \{1, \dots, n\}$ with $\beta_i/2$ being a prime number and $2\beta_i > \beta_n$. However, this precludes the equality $\sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = 1$.

Problem 4. What can be shown about the set $Z = \{n \geq 2 \colon D_b(n) = D(n)\}$? In particular, is it finite? infinite?

We mentioned above that the sequence $\langle D(n) \colon n \geq 2 \rangle$ increasing. Is the similar result true for numbers $D_b(n)$?

Problem 5. Is the sequence $\langle D_b(n) \colon n \geq 2 \rangle$ increasing? strictly increasing?

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