"Big" Continuous Restrictions of Arbitrary Functions

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Abstract. We discuss an amazing 1922 result of Henry Blumberg that *for an arbitrary* $f\colon \mathbb{R} \to \mathbb{R}$*, there is a dense* $D \subset \mathbb{R}$ *such that the restriction* $f \restriction D$ *is continuous*. In particular, we provide a new short proof of this theorem.

1. INTRODUCTION. As soon as a student is introduced to the notion of continuity for one variable real-valued functions, $f\colon \mathbb{R} \to \mathbb{R}$, it is natural to note that not all such maps are (everywhere) continuous. Perhaps the most natural examples illustrating this are maps having just a single jump discontinuity, such as the famous characteristic function $\chi_{(0,\infty)}\colon \mathbb{R} \to \{0,1\}$ of $(0,\infty)$. Most undergraduate students are, usually, pleased after learning such examples, without even wondering whether anything "worse" could happen. However, some students may inquire if an arbitrary $f\colon \mathbb{R} \to \mathbb{R}$ must have "a lot" of points of continuity, as $\chi_{(0,\infty)}$ does. Fortunately, there is yet another simple example of a function f that is, actually, discontinuous at every point: the characteristic function $\chi_{\mathbb{Q}}$ of the set $\mathbb{Q}$ of all rational numbers, known as the Dirichlet function, and named after P. Dirichlet (1805–1859). This example would surely satisfy all but the most curious students. However, such extremely curious (probably graduate) students may notice that the restriction $f \restriction \mathbb{Q}^c$ of $f = \chi_{\mathbb{Q}}$ to the (very big) set $\mathbb{Q}^c := \mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is still continuous. A natural question arises: *Must something like this be true for every function* $f\colon \mathbb{R} \to \mathbb{R}$*?*

In the early 20th century Henry Blumberg (1886–1950, see Figure 1), a Russian-American mathematician, proved the following astonishing result [**2**].

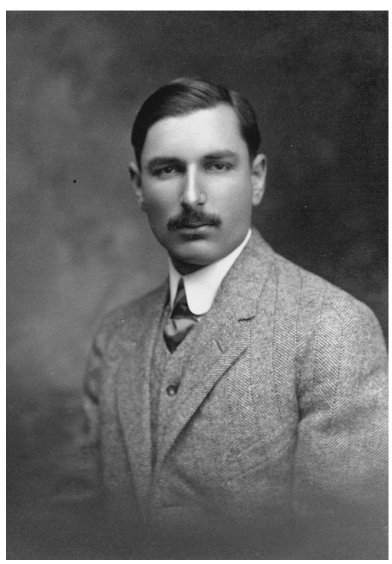

Figure 1. H. Blumberg in 1914 (courtesy of Dr. George Blumberg and the Blumberg family).

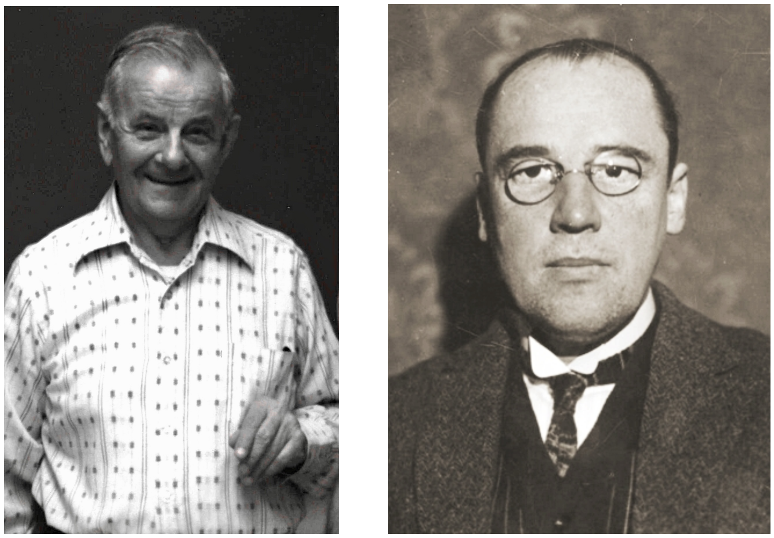

Figure 2. A. Zygmund in 1980 Summer Symposium in Real Analysis (courtesy of the *Real Analysis Exchange*) and W. Sierpiński

Theorem 1. *For every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f \restriction D$ is continuous.*

Of course, the key property of the set D in Theorem 1 is that it is "big," in the sense that it is dense in $\mathbb{R}$. However, the set D provided in the construction is just countable. Consequently, a natural question is whether the existence of an even bigger set D in the theorem above can always be ensured.

A negative answer to this last question was given only a year later, in the 1923 paper [**14**] by two Polish mathematicians, Wacław Sierpiński (1882–1969) and Antoni Zygmund[1] (1900–1992); see Figure 2. More particularly, they proved the following result (where $\mathfrak{c}$ denotes the cardinality of the continuum, that is, of $\mathbb{R}$). Any function as in the following theorem is nowadays called a *Sierpiński–Zygmund* (or just *SZ-*) *function*.

Theorem 2. *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is discontinuous for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.*

Thus, by Theorem 2, the countable set D constructed in the proof of Theorem 1 is the best we can do within the standard axiom system ZFC (the Zermelo–Fraenkel axioms with the axiom of choice) of set theory. Indeed, under the *continuum hypothesis CH*,[2] if f is an SZ-function, then any set D with continuous $f \restriction D$ must be countable, as it has cardinality less than $\mathfrak{c}$. Still, one might wonder if under the negation of the continuum hypothesis something more can be said about the cardinality of the set D from Blumberg's theorem. However, even ¬CH does not decide anything definitive on the possible size of D. Specifically, this follows from the following two results.

(1) In a model of ZFC obtained by adding at least ω_2 Cohen reals, the continuum hypothesis fails, while *there exists an $f\colon \mathbb{R} \to \mathbb{R}$ for which $f \restriction X$ is discontinuous for every uncountable $X \subset \mathbb{R}$*. This has been proved by Gruenhage (see the work of

[1]After the Second World War Zygmund worked in the United States.

[2]Recall that CH, the statement that *there is no cardinal number between $\mathfrak{c}$ and ω* (where ω is the cardinality of $\mathbb{N}$), is independent of the usual axioms ZFC of set theory.

Recław [**11**, Theorem 4]) and Shelah [**13**, §2]. Of course, in such a model of ZFC the set D from Blumberg's theorem can be at most countable, while ¬CH holds.

(2) Under Martin's axiom MA, *for every function* $f\colon \mathbb{R} \to \mathbb{R}$ *and every infinite cardinal* $\kappa < \mathfrak{c}$ *there exists a* κ*-dense set* $X \subset \mathbb{R}$ *(i.e., such that* $X \cap (a, b)$ *has cardinality* κ *for every* $a < b$*) for which* $f \restriction X$ *is continuous*. This was proved by Baldwin [**1**]. In particular, under MA+¬CH, which is consistent with ZFC, the set D from Blumberg's theorem can actually be ω_1-dense.

Another possible generalization of Theorem 1 studied in the literature is whether there is a model of ZFC in which the set D (not necessarily dense) can be always chosen either of second category or of positive Lebesgue outer measure. Of course, neither of these holds either in the model from (1) or under MA (since, under MA, every set of cardinality less than $\mathfrak{c}$ is both meager and of measure 0). But each of these questions has a positive answer. A model of ZFC in which *for every* $f\colon \mathbb{R} \to \mathbb{R}$ *there exists a second category set* D *with* $f \restriction D$ *continuous* is constructed in a 1995 paper [**13**] of Shelah. It is easy to see that this property implies that the set D can also be of second category in every nonempty open set in $\mathbb{R}$ (see, e.g., [**5**, Theorem 2.10]). In the measure case, Rosłanowski and Shelah proved, in a 2006 paper [**12**], that it is consistent with ZFC that *for every* $f\colon \mathbb{R} \to \mathbb{R}$ *there exists a continuous function* $g\colon \mathbb{R} \to \mathbb{R}$ *that agrees with* f *on a set* D *of positive Lebesgue outer measure*. Of course, for $f = \chi_{(0,\infty)}$, this last set D cannot be dense. But, even if we require only that $f \restriction D$ be continuous, such a D cannot be expected to be of positive outer measure in every nonempty open set in $\mathbb{R}$. This is prevented by an example of Brown [**4**]. (Compare also [**5**, Theorem 2.11].)

There is also a multitude of other generalizations of Blumberg's theorem (e.g., concerning functions between topological spaces X and Y). See, for example, [**3, 6, 7, 10**]. To see these results from a more general real analysis perspective, see [**5, 9**].

2. THE PROOFS. The proof of Blumberg's theorem relies on the following lemma from [**2**]. (See also [**9**].) For $f\colon \mathbb{R} \to \mathbb{R}$, a point $x \in \mathbb{R}$ is said to be f*-pleasant* provided for every open $B \ni f(x)$ there is an open $U_x^B \ni x$ such that the set $f^{-1}(B)$ is categorically dense in U_x^B (i.e., $f^{-1}(B) \cap V$ is of second category for every nonempty open $V \subset U_x^B$).

Lemma 3. *For every* $f\colon \mathbb{R} \to \mathbb{R}$ *the set* P_f *of all* f*-pleasant points is residual (i.e., it contains an intersection of countably many dense open sets) in* $\mathbb{R}$.

Proof. Let $\mathcal{B}$ be a countable basis for $\mathbb{R}$. For every $B \in \mathcal{B}$ let

$$E_B := \left\{x \in f^{-1}(B)\colon\ f^{-1}(B) \text{ is not categorically dense in any open } U \ni x\right\}$$

and notice that E_B is of first category. Indeed, it is a union of two first category sets: $W \cap E_B$, where $W = \bigcup\{V \in \mathcal{B}\colon V \cap E_B \text{ is of first category}\}$, and $\mathrm{bd}(W) \cap E_B$ (where $\mathrm{bd}(W)$ is the boundary of W).

Since $E := \bigcup_{B\in\mathcal{B}} E_B$ is of first category, it is enough to show that $\mathbb{R} \setminus E \subset P_f$. To see this, fix an $x \in \mathbb{R} \setminus E$ and an open $W \ni f(x)$. Choose $B \in \mathcal{B}$ with $f(x) \in B \subset W$. Since $x \notin E_B$, there is an open $U_x^B \ni x$ such that $f^{-1}(B)$ is categorically dense in U_x^B. Then $f^{-1}(W) \supset f^{-1}(B)$ is also categorically dense in U_x^B; that is, $U_x^W := U_x^B$ is as needed. ■

New proof of Blumberg's Theorem. Let $\mathcal{B} = \{B_n\colon n < \omega\}$ be a basis for $\mathbb{R}$. We construct, by induction on $n < \omega$, the sequences $\langle x_n \in B_n \cap P_f \setminus \{x_i\colon i < n\}\colon n < \omega\rangle$

and $\langle\langle U_k^n, V_k^n\rangle \in \mathcal{B}^2\colon k \le n < \omega\rangle$, aiming for $D := \{x_n\colon n < \omega\}$ to be our desired set. The continuity of $f \restriction D$ is ensured by the properties of the constructed sets U_k^n and V_k^n: each family $\{V_i^n\colon n < \omega\}$ will form a basis of $\mathbb{R}$ at $f(x_i)$ and each D-open $U_i^j \cap D \ni x_i$ will be contained in $f^{-1}(V_i^j)$.

To ensure this, we will assume that for every $n < \omega$ and $i \le j \le n$, $k \le \ell \le n$ with $j \le \ell$:

(a_n) $f^{-1}(V_i^j)$ is categorically dense in U_i^j, $x_i \in U_i^j \cap f^{-1}(V_i^j)$, and V_i^j has diameter less than 2^{-j};
(b_n) if $U_i^j \cap U_k^\ell \neq \emptyset$ and $\langle i, j\rangle \neq \langle k, \ell\rangle$, then $j < \ell$ and $U_k^\ell \times V_k^\ell \subset U_i^j \times V_i^j$.

These properties guarantee that $D := \{x_n\colon n < \omega\}$ is as needed. Indeed, D is dense, since it intersects every $B_n \in \mathcal{B}$. Each family $\{V_i^n\colon n < \omega\}$ will form a basis of $\mathbb{R}$ at $f(x_i)$, since each open set V_i^n contains x_i and the diameters of V_i^j go to 0 as $j \to \infty$. Thus, to show that $f \restriction D$ is continuous at x_i, it is enough to show that f maps each D-open $U_i^j \cap D \ni x_i$ into V_i^j. To see this, fix an $x_k \in D \cap U_i^j$. We cannot have $k < i$, since then x_k would belong to disjoint U_k^j and U_i^j. By (a_j), we have $f(x_i) \in V_i^j$. Thus, assume that $i < k$. Then $x_k \in U_i^j \cap U_k^k$ and, by (b_k), $f(x_k) \in V_k^k \subset V_i^j$, as needed.

To make the nth step in our construction, choose a nonempty interval $\hat{B}_n \subset B_n$ such that, for every $i \le j < n$, $\hat{B}_n$ is either contained in U_i^j or it is disjoint from U_i^j. Let

$$\mathcal{F}_n := \{U_i^j\colon i \le j < n \;\&\; \hat{B}_n \subset U_i^j\}.$$

If $\mathcal{F}_n \neq \emptyset$, then $n > 0$ and, by (b_{n-1}), $\mathcal{F}_n$ contains a smallest element, say U_κ^μ. We choose

$$x_n \in \hat{B}_n \cap P_f \cap f^{-1}(V_\kappa^\mu) \setminus \{x_i\colon i < n\}.$$

This choice can be made since $\hat{B}_n \subset U_\kappa^\mu$ is open and nonempty, $f^{-1}(V_\kappa^\mu)$ is categorically dense in U_κ^μ, and $P_f \setminus \{x_i\colon i < n\}$ is residual. If $\mathcal{F}_n = \emptyset$, take $x_n \in \hat{B}_n \cap P_f \setminus \{x_i\colon i < n\}$.

To finish the construction we first choose, for each $k \le n$, a V_k^n as an open interval containing $f(x_k)$ of length less than 2^{-k} small enough such that if $f(x_k) \in V_i^j$ for some $i \le j < n$, then $V_k^n \subset V_i^j$. The existence of sets U_k^n, $k \le n$, satisfying (a_n) follows from $\{x_i\colon i \le n\} \subset P_f$. Shrinking them if necessary, we can also ensure that they are pairwise disjoint and that if, for some $i \le j < n$, $x_k \in U_i^j$, then $U_k^n \subset U_i^j$. These choices ensure that (a_n) and (b_n) are satisfied. ■

Construction of a Sierpiński–Zygmund function. The key fact needed in the construction is the following result of Kuratowski (1896–1980), see, e.g., [**8**, p. 16]:

(E) For every continuous g from an $S \subset \mathbb{R}$ into $\mathbb{R}$ there exists a G_δ-set $G \supset S$ and a continuous extension $\bar{g}\colon G \to \mathbb{R}$ of g. In particular, g admits a Borel extension $\hat{g}\colon \mathbb{R} \to \mathbb{R}$.

Indeed, for every $x \in \mathrm{cl}(S)$ define

$$\mathrm{osc}_g(x) := \inf\{\mathrm{diam}(g[U \cap S])\colon U \ni x \text{ is open}\}$$

and notice that $G := \{x \in \mathrm{cl}(S) \colon \mathrm{osc}_g(x) = 0\}$ contains S and is a G_δ-set in $\mathbb{R}$ since $G := \bigcap_{n\in\mathbb{N}} W_n$, where each set $W_n := \{x \in \mathrm{cl}(S) \colon \mathrm{osc}_g(x) < 1/n\}$ is open. Now, if $\mathrm{cl}(g)$ is the closure in $\mathbb{R}^2$ of the graph of g, then $\bar{g} = \mathrm{cl}(g) \cap (G \times \mathbb{R})$ is the graph of our desired function $\bar{g}$. A Borel extension $\hat{g}$ of $\bar{g}$ can be defined to be 0 on $\mathbb{R} \setminus G$.

To construct a Sierpiński–Zygmund function $f \colon \mathbb{R} \to \mathbb{R}$, let $\{x_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathbb{R}$ and let $\{\hat{g}_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all Borel functions from $\mathbb{R}$ to $\mathbb{R}$. For every $\xi < \mathfrak{c}$ define $f(x_\xi)$ so that

$$f(x_\xi) \in \mathbb{R} \setminus \{\hat{g}_\zeta(x_\xi) \colon \zeta < \xi\}.$$

This defines our SZ-function. Indeed, if $f \restriction S$ is continuous for some $S \subset \mathbb{R}$ then, by (E), there exists a Borel extension $\hat{g} \colon \mathbb{R} \to \mathbb{R}$ of $f \restriction S$. Let $\zeta < \mathfrak{c}$ be such that $\hat{g}_\zeta = \hat{g}$. Then $S \subset \{x_\xi \colon \xi \le \zeta\}$, since $f(x_\xi) \neq \hat{g}_\zeta(x_\xi) = \hat{g}(x_\xi)$ for every $\xi > \zeta$. Thus, S has cardinality $< \mathfrak{c}$, as needed, and we are done. ■

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