ALGEBRAS OF MEASURABLE EXTENDABLE FUNCTIONS OF MAXIMAL CARDINALITY

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ABSTRACT. The class Ext of all extendable functions from $\mathbb{R}$ to $\mathbb{R}$ is the smallest among all Darboux-like classes of functions, which constitute different natural generalizations of the class of usual continuous functions. The goal of this paper is to construct, within Ext, an algebra $\mathfrak{A}$ which has maximal possible cardinality, that is, $2^{\mathfrak{c}}$. This, in particular, would confirm a conjecture of T. Natkaniec from 2013. Moreover, the constructed algebra $\mathfrak{A}$ consists only of functions that are both Baire and Lebesgue measurable.

1. PRELIMINARIES

The work presented here is a contribution to a recent ongoing research concerning the following general question: *For an arbitrary subset M of a vector space (or algebra) W, how big can a vector subspace (or algebra) V contained in $M \cup \{0\}$ be?* The current state of the art concerning this topic is described in [1, 5]. Given a cardinal number α and a vector space X, we say that $M \subset X$ is *α-lineable* if $M \cup \{0\}$ contains a vector subspace of X of dimension α. Moreover, provided that X is a vector space contained in some (linear) algebra, then A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$. Recall that if X is contained in a commutative algebra, then a set $B \subset X$ is a generating set of some free algebra contained in A if, and only if, for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$.

The notion of simple *α-algebrability* is defined in a similar fashion. Of course, strong α-algebrability implies α-algebrability, which implies α-lineability. However, in general, the converse implications do not hold, see, e.g., [1, 5].

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For a topological space X and $A \subset X$ we will use the symbols $\mathrm{int}(A)$, $\mathrm{cl}(A)$, and $\mathrm{bd}(A)$ to denote the interior, the closure, and the boundary of A in X, respectively. To define the class of extendable functions, recall that for the topological spaces X and Y a function $f\colon X \to Y$ is

- *connectivity* provided the graph of $f \restriction C$, the restriction of f to C, is connected in $X \times Y$ for every connected subset C of X.
- *extendable* provided there is a connectivity map $F\colon X \times [0,1] \to Y$ with $f(x) = F(x,0)$ for all $x \in X$.
- *peripherally continuous* provided for every $x \in X$, open $U \subset X$ containing x, and open $V \subset Y$ containing $f(x)$, there exists an open $W \subset X$ such that $x \in W \subset \mathrm{cl}(W) \subset U$ and $f[\mathrm{bd}(W)] \subset V$.

These classes are denoted, respectively, as $\mathrm{Conn}(X,Y)$, $\mathrm{Ext}(X,Y)$, and $\mathrm{PC}(X,Y)$. We will write simply Conn, Ext, and PC when $X = Y = \mathbb{R}$.

The notion of connectivity functions can be traced back to 1922 paper [21] of K. Kuratowski and W. Sierpiński. However, the term "connectivity map" seems to be first used in 1956 note [23] of J. Nash, see [19]. The notion of peripherally continuous functions was introduced in 1957 paper [19] of O. H. Hamilton, see also [26]. The notion of extendable functions (without the name) first appeared in 1959 paper [26] of J. Stallings, where he asks a question whether every connectivity function defined on $[0,1]$ is extendable.

For what follows, we will need the following 2001 result.

Proposition 1.1 (K. Ciesielski, T. Natkaniec, and J. Wojciechowski [11])**.** *If* $n \geq 2$*, then* $\mathrm{Ext}(\mathbb{R}^n,\mathbb{R}) = \mathrm{Conn}(\mathbb{R}^n,\mathbb{R}) = \mathrm{PC}(\mathbb{R}^n,\mathbb{R})$.

The main contribution of [11] is the proof that $\mathrm{Conn}(\mathbb{R}^n,\mathbb{R}) \subset \mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$, since the inclusion $\mathrm{Ext}(\mathbb{R}^n,\mathbb{R}) \subset \mathrm{Conn}(\mathbb{R}^n,\mathbb{R})$ is obvious, $\mathrm{Conn}(\mathbb{R}^n,\mathbb{R}) \subset \mathrm{PC}(\mathbb{R}^n,\mathbb{R})$ was proved by J. Stallings [26] (see also O. H. Hamilton [19]), and $\mathrm{PC}(\mathbb{R}^n,\mathbb{R}) \subset \mathrm{Conn}(\mathbb{R}^n,\mathbb{R})$ was proved by M. R. Hagan [18]. (See also G. T. Whyburn [27] and R. G. Gibson, F. Roush [16, theorem 8.1].)

The relations between these classes for the functions from $\mathbb{R}$ to $\mathbb{R}$ look considerably different. To give a fuller picture of this, we also recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is

- *Darboux* provided $f[C]$ is connected (i.e., an interval) for every connected $C \subset \mathbb{R}$. That is, f is Darboux if, and only if, it has the intermediate value property. The name is in honor of J.-G. Darboux who in his 1875 paper [13] studied this property and shown that every derivative (even discontinuous) must have the intermediate value property.
- *almost continuous* (in the sense of Stallings) provided every open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. This class was introduced in a 1959 paper [26] by J. Stallings, where it is proved that every such function from $[0,1]$ onto $[0,1]$ must have a fixed point.

- *SCIVP* (or *has the Strong Cantor Intermediate Value Property*) provided for every $p, q \in \mathbb{R}$ with $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f[C] \subset K$ and $f \restriction C$ is continuous. This notion was introduced in a 1992 paper [25] of H. Rosen, R. G. Gibson, and F. Roush to help distinguish extendable and connectivity functions on $\mathbb{R}$.

These classes of functions are denoted, respectively, as D, AC, and SCIVP. The relation between all these classes, commonly referred to as *Darboux-like* classes of functions, are presented in Figure 1. More information about these classes can be found in the papers [17], [6], and [10].

$$\begin{tikzcd} \mathrm{Ext} \arrow[r] \arrow[drr] & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[r] & \mathrm{PC} \\ & & \mathrm{SCIVP} \arrow[urr] & & \end{tikzcd}$$

FIGURE 1. The strict inclusions, indicated by arrows, among the classes of Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$.

The lineability of the above defined classes has been thoroughly studied by several authors in the past several years (see, e.g., [3, 7–9, 24]). Thus, it has been proved in [9, theorem 4.2] that, for $n > 1$, the class $\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ is not $\mathfrak{c}^+$-lineable, where $\mathfrak{c}$ stands for *continuum*, that is, the cardinality of $\mathbb{R}$. On the other hand $\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ is clearly $\mathfrak{c}$-algebrable, as justified by the class of all continuous functions. The class Ext is $2^{\mathfrak{c}}$-lineable, as shown independently in a 2013 paper [24, corollary 12] and 2014 paper [9, theorem 3.1].

The difficulty with proving the strong $2^{\mathfrak{c}}$-algebrability for the functions from $\mathbb{R}$ to $\mathbb{R}$ is that many polynomials (e.g., $p(x) = x^2$) are not surjective. (The algebrability of the classes of functions from $\mathbb{C}$ to $\mathbb{C}$ is handled considerably easier, see e.g. [2–4].) Nevertheless, in a 2013 paper T. Natkaniec partially overcame these difficulties showing [24, corollary 10] that the class $\mathrm{AC}(\mathbb{R}^n, \mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable for every $n \geq 1$. He also asks [24, problem 11] whether the class Ext is (strongly) $2^{\mathfrak{c}}$-algebrable. The goal of this article is to give an affirmative answer to this question. It is presented in Theorem 3.1 and Corollary 3.2, whose proofs employ the ideas that come from [3]. Also, in what follows, for $f, g \colon \mathbb{R}^n \to \mathbb{R}$ we define $[f = g] := \{x \in \mathbb{R}^n \colon f(x) = g(x)\}$.

2. Massive sets for functions with interval range

Let $\mathcal{F} \subset \mathbb{R}^{(\mathbb{R}^n)}$ and $J \subset \mathbb{R}$ be nonempty. Following Natkaniec [24], we say that a set $M \subset \mathbb{R}^n$ is $\langle \mathcal{F}, J \rangle$-*massive* provided there exists a map $g \colon \mathbb{R}^n \to J$ such that for every $f \colon \mathbb{R}^n \to J$, if $M \subset [f = g]$, then $f \in \mathcal{F}$. Notice that such a *witness* map $g \colon \mathbb{R}^n \to J$ belongs to $\mathcal{F}$.

Our main result heavy relies on the following 1996 result.

Proposition 2.1 (K. Ciesielski and I. Recław [12, theorem 3.3])**.** *There exists a* $\varphi \in \mathrm{Conn}(\mathbb{R}^2, \mathbb{R})$ *and a meager subset* $\overline{M}$ *of* $\mathbb{R}^2$ *such that for every* $\psi \colon \mathbb{R}^2 \to \mathbb{R}$*, if* $\overline{M} \subset [\psi = \varphi]$*, then* $\psi \in \mathrm{Conn}(\mathbb{R}^2, \mathbb{R})$*. In particular,* $\overline{M}$ *is* $\langle \mathrm{Conn}(\mathbb{R}^2, \mathbb{R}), \mathbb{R}\rangle$*-massive and, by Proposition 1.1, also* $\langle \mathrm{PC}(\mathbb{R}^2, \mathbb{R}), \mathbb{R}\rangle$*-massive.*

The set $\overline{M}$ from Proposition 2.1 constructed in [12] is of the form of $\bigcup_{B \in \mathcal{B}} \mathrm{bd}(B)$, where $\mathcal{B}$ is a countable basis of $\mathbb{R}^2$ from the following lemma. It is interesting to see that actually any set $\overline{M}$ from Proposition 2.1 must contain a subset of the form $\bigcup_{B \in \mathcal{B}} \mathrm{bd}(B)$.

Lemma 2.2. *Let* $\overline{M}$ *and* φ *be as in Proposition 2.1. Then there exists a countable basis* $\mathcal{B}$ *of* $\mathbb{R}^2$ *such that*

(i) $\mathrm{bd}(B) \subset \overline{M}$ *for every* $B \in \mathcal{B}$*, and*
(ii) *for every* $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ *with* $\overline{M} \subset [\psi = \varphi]$*, the sets* W *justifying the definition of peripheral continuity of* ψ *can be chosen from* $\mathcal{B}$*.*

Proof. Let $\mathcal{B}_0$ be the family of all open sets W in $\mathbb{R}^2$ with $\mathrm{bd}(W) \subset \overline{M}$ and notice that this is a basis for $\mathbb{R}^2$.

To see this, fix an open $U \subset \mathbb{R}^2$ and an $x \in U$. By way of contradiction, assume that $\mathrm{bd}(W) \setminus \overline{M} \neq \emptyset$ for every open $W \subset \mathbb{R}^2$ with $x \in W \subset \mathrm{cl}(W) \subset U$. Let $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ be an extension of $\varphi \restriction \overline{M}$ such that $\psi(y) = \psi(x) + 1$ for every $y \notin \{x\} \cup \overline{M}$. Then $\overline{M} \subset [\psi = \varphi]$, so ψ should belong to $\mathrm{Conn}(\mathbb{R}^2, \mathbb{R}) = \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. However, ψ fails the definition of peripheral continuity for $V = (\psi(x) - 1, \psi(x) + 1)$, x, and U, a contradiction.

Now, since $\mathcal{B}_0$ is a basis for $\mathbb{R}^2$, which is second countable, it contains a countable subfamily $\mathcal{B}$ which forms a basis for $\mathbb{R}^2$. (See e.g., [22, exercise 2, p. 194].) □

We will also need the following variant of Proposition 2.1.

Lemma 2.3. *Let* $I \subset \mathbb{R}$ *be a non-degenerate interval and* $h \colon \mathbb{R} \to \mathrm{int}(I)$ *be a homeomorphism. If* φ *and* $\overline{M}$ *are as in Proposition 2.1, then for every* $\psi \colon \mathbb{R}^2 \to I$ *with* $\overline{M} \subset [\psi = h \circ \varphi]$ *we have* $\psi \in \mathrm{Conn}(\mathbb{R}^2, I)$*. In particular,* $h \circ \varphi$ *witnesses that* $\overline{M}$ *is* $\langle \mathrm{Conn}(\mathbb{R}^2, \mathbb{R}), I\rangle$*- and* $\langle \mathrm{PC}(\mathbb{R}^2, \mathbb{R}), I\rangle$*-massive.*

Proof. Let $\psi \colon \mathbb{R}^2 \to I$ be such that $\overline{M} \subset [\psi = h \circ \varphi]$. It is enough to show that ψ is peripherally continuous. To see this, fix an $x \in \mathbb{R}^2$, an open $U \subset \mathbb{R}^2$ containing x, and an open $V \subset I$ containing $\psi(x)$. We need to find an open $W \subset \mathbb{R}^2$ such that $x \in W \subset \mathrm{cl}(W) \subset U$ and $\psi[\mathrm{bd}(W)] \subset V$.

Let $D = \psi^{-1}(\mathrm{int}(I))$ and notice that $\overline{M} \subset D$, since for every $y \in \overline{M} \subset [\psi = h \circ \varphi]$ we have $\psi(y) = (h \circ \varphi)(y) \in \mathrm{int}(I)$. Let $\hat{\psi} \colon \mathbb{R}^2 \to \mathrm{int}(I)$ be an extension of $\psi \restriction D$ such that $\hat{\psi}(x) \in V_1 := V \cap \mathrm{int}(I)$. Then, we have $\overline{M} \subset [\hat{\psi} = h \circ \varphi] = [h^{-1} \circ \hat{\psi} = \varphi]$ and so $h^{-1} \circ \hat{\psi} \in \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. Let $\mathcal{B}$ be as in Lemma 2.2. Since $h^{-1} \circ \hat{\psi}(x) \in h^{-1}(V_1)$, there exists a $W \in \mathcal{B}$ with $x \in W \subset \mathrm{cl}(W) \subset U$ and $h^{-1} \circ \hat{\psi}[\mathrm{bd}(W)] \subset h^{-1}(V_1)$. But we have $\mathrm{bd}(W) \subset \overline{M} \subset D$, so that $\hat{\psi}[\mathrm{bd}(W)] = \psi[\mathrm{bd}(W)]$. Therefore, also $\psi[\mathrm{bd}(W)] = \hat{\psi}[\mathrm{bd}(W)] \subset V_1 \subset V$. This means that W is as needed. □

From Lemma 2.3 we deduce the following corollary, which constitutes a slight strengthening of [12, corollary 3.4].

Corollary 2.4. *There exists a $g \in \mathrm{Ext}$ and a meager subset M of $\mathbb{R}$ such that for every non-degenerate interval I the set M is $\langle \mathrm{Ext}, I\rangle$-massive, what is witnessed by $h \circ g$, where h is a homeomorphism from $\mathbb{R}$ onto $\mathrm{int}(I)$.*

Proof. Let φ and $\overline{M}$ be as in Proposition 2.1. By Kuratowski-Ulam theorem (see, e.g., [20]), there is a $y \in \mathbb{R}$ such that the set $M := \{x \in \mathbb{R} \colon (x, y) \in \overline{M}\}$ is meager in $\mathbb{R}$. Define g by letting $g(x) = \varphi(x, y)$ for every $x \in \mathbb{R}$. We claim that g and M are as needed.

To see this, let I be a non-degenerate interval and h be a homeomorphism from $\mathbb{R}$ onto $\mathrm{int}(I)$. Let $f \colon \mathbb{R} \to I$ be such that $M \subset [f = h \circ g]$. It is enough to find a connectivity function $F \colon \mathbb{R}^2 \to I$ with $f(x) = F(x, 0)$ for all $x \in \mathbb{R}$.

But, by Lemma 2.3, $h \circ \varphi$ witnesses that $\overline{M}$ is $\langle \mathrm{PC}(\mathbb{R}^2, \mathbb{R}), I\rangle$-massive. Let $\psi \colon \mathbb{R}^2 \to I$ be an extension of $h \circ \varphi \restriction \overline{M}$ such that $\psi(x, y) = h \circ \varphi(x, y) = f(x)$ for every $x \in \mathbb{R}$. This is possible, since $h \circ \varphi(x, y) = h \circ g(x) = f(x)$ whenever $(x, y) \in \overline{M}$. Then ψ is a connectivity function, as $\overline{M} \subset [\psi = h \circ \varphi]$. So, the map $F \colon \mathbb{R}^2 \to I$ defined for every $z \in \mathbb{R}$ as $F(x, z) = \psi(x, y + z)$ is connectivity as well. To finish the proof, it is enough to notice that $F(x, 0) = \psi(x, y) = f(x)$ for all $x \in \mathbb{R}$. □

We finish with the following simple implication of Corollary 2.4, which is a strengthening of [12, lemma 3.2].

Lemma 2.5. *There exists a family $\mathcal{K}$ of $\mathfrak{c}$-many pairwise disjoint sets such that for every non-degenerate interval I the set $K \in \mathcal{K}$ is $\langle \mathrm{Ext}, I\rangle$-massive and the union $\bigcup \mathcal{K}$ has measure zero and is also meager. In particular, every set in $\mathcal{K}$ has also measure zero and is meager.*

Proof. First notice that there exists a family $\{K_\xi \colon \xi < \mathfrak{c}\}$ of pairwise disjoint $\mathfrak{c}$-dense F_σ-sets such that $\bigcup_{\xi<\mathfrak{c}} K_\xi$ is meager and of measure zero. To see this, fix a countable basis $\mathcal{B}$ for $\mathbb{R}$. For every $B \in \mathcal{B}$ choose a measure zero Cantor set $C_B \subset B$ (in such a way that these C_B's are pairwise disjoint for $B \in \mathcal{B}$) and let $\{C_B^\xi \colon \xi < \mathfrak{c}\}$ be a partition of C_B into Cantor subsets. For every $\xi < \mathfrak{c}$ let $K_\xi := \bigcup_{B \in \mathcal{B}} C_B^\xi$. Then the family $\{K^\xi \colon \xi < \mathfrak{c}\}$ is as needed.

Indeed, for any $\xi < \mathfrak{c}$ and $B \in \mathcal{B}$, C_B^ξ is a closed set of measure zero. In particular, every K_ξ is a $\mathfrak{c}$-dense F_σ-set also of measure zero. Also, by construction, sets K_ξ are pairwise disjoint. Finally, notice that

$$\bigcup_{\xi<\mathfrak{c}} K_\xi = \bigcup_{\xi<\mathfrak{c}} \bigcup_{B\in\mathcal{B}} C_B^\xi = \bigcup_{B\in\mathcal{B}} C_B$$

is a countable union of $\mathfrak{c}$-dense F_σ-set that has measure zero.

Now, let $g \in \mathrm{Ext}$ and M be as in Corollary 2.4. Then, for every $\xi < \mathfrak{c}$, there exists an auto-homeomorphisms h_ξ of $\mathbb{R}$ such that $h_\xi[M] \subset K_\xi$. (See e.g. [15].) Then the family $\mathcal{K} = \{h_\xi[M] \colon \xi < \mathfrak{c}\}$ is as needed. This is the case, since, for every $\xi < \mathfrak{c}$, the map $g \circ h_\xi^{-1}$ and the set $h_\xi[M]$ satisfy Corollary 2.4. □

3. The strongly $2^{\mathfrak{c}}$-algebrability of Ext

Let $\mathcal{H}^n$ be a family of all polynomials from $\mathbb{R}^n$ into $\mathbb{R}$.

Theorem 3.1. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of cardinality $2^{\mathfrak{c}}$ such that for every $h \in \mathcal{H}^n$ and every sequence $\langle f_1, \ldots, f_n\rangle$ of distinct elements of $\mathcal{F}$ the map $h(f_1, \ldots, f_n)$ is non-zero, extendable, and both Baire and Lebesgue measurable.*

First, notice that this immediately implies our main result.

Corollary 3.2. *The algebra generated by the family $\mathcal{F}$ from Theorem 3.1 consist only of functions that are extendable and both Baire and Lebesgue measurable. In particular, all classes of Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$ (see Figure 1), are strongly $2^{\mathfrak{c}}$-algebrable.*

Proof. The statement follows immediately from the fact that $h(f_1, \ldots, f_n) \in$ Ext for all sequences $\langle f_1, \ldots, f_n\rangle$ in $\mathcal{F}$ (not only those of distinct element). This is the case, since for every sequence $\langle f_1, \ldots, f_n\rangle$ in $\mathcal{F}$ there is a sequence $\langle g_1, \ldots, g_m\rangle$ of distinct elements of $\mathcal{F}$ and a function $j \colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ with the property that $\langle f_1, \ldots, f_n\rangle = \langle g_{j(1)}, \ldots, g_{j(n)}\rangle$. Then $\hat{h}(x_1, \ldots, x_m) := h(x_{j(1)}, \ldots, x_{j(n)}) \in \mathcal{H}^m$ and, by Theorem 3.1, we have $h(f_1, \ldots, f_n) = \hat{h}(g_1, \ldots, g_m) \in$ Ext, as needed. □

Proof of Theorem 3.1. Let $\mathcal{H} := \bigcup_{n=1}^{\infty} \mathcal{H}^n \times n^\omega$, where symbol n^ω denotes $\{0, \ldots, n-1\}^\omega$. Since $|\mathcal{H}| = \mathfrak{c}$, we can enumerate, with no repetition, the family $\mathcal{K}$ from Lemma 2.5 as $\{K_{h,p} \colon \langle h, p\rangle \in \mathcal{H}\}$.

For every $\langle h, p\rangle \in \mathcal{H}^n \times n^\omega$, let $g_{h,p} \colon \mathbb{R} \to \operatorname{int}(h[\mathbb{R}^n])$ witness the fact that $K_{h,p}$ is $\langle \mathrm{Ext}, h[\mathbb{R}^n]\rangle$-massive. For each $x \in \mathbb{R}$, let

$$\vec{v}_{h,p}(x) = \langle \vec{v}_{h,p}(x)_0, \ldots, \vec{v}_{h,p}(x)_{n-1}\rangle \in \mathbb{R}^n$$

be such that $h(\vec{v}_{h,p}(x)) = g_{h,p}(x)$. Also, let $\bar{p} \colon \beta\omega \to n$ be a continuous extension of p to the Stone-Čech compactification $\beta\omega$ of ω. (We refer the interested reader to [14] for an extensive account of the properties of the Stone-Čech compactification.)

For every ultrafilter $\mathcal{U} \in \beta\omega$ and every $x \in K_{h,p}$ let $f_{\mathcal{U}}(x) := \vec{v}_{h,p}(x)_{\bar{p}(\mathcal{U})} \in \mathbb{R}$. For $x \notin \bigcup_{\langle h,p\rangle \in \mathcal{H}} K_{h,p}$ we can define $f_{\mathcal{U}}(x) := 0$. Then, the family $\mathcal{F} := \{f_{\mathcal{U}} \colon \mathcal{U} \in \beta\omega\}$ is as needed.

To see this, first choose distinct $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1} \in \beta\omega$ and an $h \in \mathcal{H}^n$. We will show that $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}}) \in$ Ext and is Baire and Lebesgue measurable.

To see this, choose a partition $\{U_0, \ldots, U_{n-1}\}$ of ω such that $U_i \in \mathcal{U}_j$ if, and only if, $i = j$. Let $p \in n^\omega$ be such that $p^{-1}(i) = U_i$ for every $i \in n$.

Then $\bar{p}\colon \beta\omega \to n$ has the property $\bar{p}(\mathcal{U}_i) = i$ for every $i \in n$. We claim that $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}}) = g_{h,p}$ on $K_{h,p}$, ensuring $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}}) \in \mathrm{Ext}$. Indeed, for every $x \in K_{h,p}$ we have

$$\begin{aligned} h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}})(x) &= h(f_{\mathcal{U}_0}(x), \ldots, f_{\mathcal{U}_{n-1}}(x)) \\ &= h(\vec{v}_{h,p}(x)_{\bar{p}(\mathcal{U}_0)}, \ldots, \vec{v}_{h,p}(x)_{\bar{p}(\mathcal{U}_{n-1})}) \\ &= h(\vec{v}_{h,p}(x)_0, \ldots, \vec{v}_{h,p}(x)_{n-1}) \\ &= h(\vec{v}_{h,p}(x)) = g_{h,p}(x) \end{aligned}$$

as needed. Thus, indeed $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}}) \in \mathrm{Ext}$. Notice also that, by construction, $K\colon = \bigcup_{\langle h,p\rangle \in \mathcal{H}} K_{h,p}$ is meager and has measure zero. Moreover, on $\mathbb{R} \setminus K$, the values of $f_{\mathcal{U}}$ are always the same, equal 0. Thus, on $\mathbb{R} \setminus K$, $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}})$ is constant equal $h(0, \ldots, 0)$. So, indeed $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}})$ is Baire and Lebesgue measurable.

Next, notice that for every non-constant $h \in \mathcal{H}^n$ there exists an $x \in K_{h,p}$ such that $g_{h,p}(x) \neq 0$. To see this, choose a nonzero $c \in h[\mathbb{R}^n]$ and let $\psi\colon \mathbb{R} \to \mathbb{R}$ extend $g_{h,p} \restriction K_{h,p}$ be such that $\psi(x) = c$ for every $x \in \mathbb{R} \setminus K_{h,p}$. Then, ψ is extendable, so Darboux. Thus, it is impossible for $g_{h,p} \restriction K_{h,p}$ to attain only value 0.

We use this fact to show that $\mathcal{F}$ has indeed cardinality $2^{\mathfrak{c}}$. Since $\beta\omega$ has such cardinality, it is enough to show that all maps $f_{\mathcal{U}}$ are distinct. To see this, choose distinct $\mathcal{U}_0, \mathcal{U}_1 \in \beta\omega$ and let $h(s,t) = s - t$. Choose p with $\bar{p}(\mathcal{U}_i) = i$ for every $i < 2$ and pick $x \in K_{h,p}$ with $g_{h,p}(x) \neq 0$. Then, $f_{\mathcal{U}_0}(x) - f_{\mathcal{U}_1}(x) = \vec{v}_{h,p}(x)_0 - \vec{v}_{h,p}(x)_1 = h(\vec{v}_{h,p}(x)) = g_{h,p}(x) \neq 0$, so that $f_{\mathcal{U}_0} \neq f_{\mathcal{U}_1}$, as needed.

Finally, we need to show that our algebra is strongly $2^{\mathfrak{c}}$-algebrable. To see this, let $h \in \mathcal{H}^n$ be non-constant and choose distinct $\mathcal{U}_0, \ldots, \mathcal{U}_{n-1} \in \beta\omega$. We need to find an x such that $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}})(x) \neq 0$. For this, choose $p \in n^\omega$ with $\bar{p}[\mathcal{U}_i] = \{i\}$ for every $i < n$ and pick $x \in K_{h,p}$ with $g_{h,p}(x) \neq 0$. Then, $h(f_{\mathcal{U}_0}, \ldots, f_{\mathcal{U}_{n-1}})(x) = g_{h,p}(x) \neq 0$, as needed. $\square$

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