Distinct Continuous Maps with All Riemann Sums Equal

Krzysztof Chris Ciesielski and Yichen Liu

Abstract. Do examples as in the title exist? It depends on how the term *Riemann sum* is understood. For the standard, left, or right Riemann sums such examples do not exist. However, as we will see, they do exist for the lower and upper Riemann sums. Nevertheless, there are only a few examples of such pairs and they have a very simple structure. In this article, we describe all such pairs among Riemann integrable functions from an interval $[a, b]$ into $\mathbb{R}$. We also show that such pairs have an especially nice format when we restrict our attention to continuous maps. All the arguments presented are elementary. In particular, the part concerning continuous functions is self-contained and presented in a format accessible to good undergraduate students.

1. INTRODUCTION. The most commonly taught definition of a definite integral of a function $f : [a, b] \to \mathbb{R}$ is due to Bernhard Riemann (1826–1866). It was presented to the faculty at the University of Göttingen in 1854, but was not published in a journal until 1868; see [**5**]. It relies on a notion of a *Riemann sum* which, for a partition $P := \{x_0, x_1, \dots, x_n\}$ of an interval $I := [a, b]$ (such that $a = x_0 < x_1 < \cdots < x_n = b$) and a sequence $\vec{t} := \langle t_i \in [x_i, x_{i+1}] : i < n\rangle$ of *tags*, is defined as

$$S\left(f, P, \vec{t}\right) := \sum_{i<n} (x_{i+1} - x_i) f(t_i).$$

A map $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* provided there exists a unique number $\sigma \in \mathbb{R}$, denoted $\int_a^b f(x)\,dx$ and referred to as the *Riemann integral* of f, such that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\left|S\left(f, P, \vec{t}\right) - \sigma\right| < \varepsilon \tag{1}$$

whenever the number $|P| := \max\{x_{i+1} - x_i : i < n\}$ is less than δ.

A somewhat unpleasant part of this definition is its use of (essentially arbitrary) tags $\vec{t}$. There are several ways to avoid their use while still getting the same notion of integrability. One way is to use the notions of *lower* and *upper Darboux sums*, also known as *lower* and *upper Riemann sums*, and defined, respectively, as

$$L(f, P) := \sum_{i<n} (x_{i+1} - x_i) \inf_{t\in[x_i,x_{i+1}]} f(t)$$

and

$$U(f, P) := \sum_{i<n} (x_{i+1} - x_i) \sup_{t\in[x_i,x_{i+1}]} f(t).$$

If, in the above definition, we replace (1) with

$$|L(f, P) - \sigma| < \varepsilon \quad \text{and} \quad |U(f, P) - \sigma| < \varepsilon,$$

then we obtain a notion of the *Darboux integral*, first introduced by Jean-Gaston Darboux (1842–1917) in his 1875 *Mémoire* [**2**]. It is well known and easy to see that these two notions of integrals are actually equivalent, since $L(f,P) \le S\left(f,P,\vec{t}\right) \le U(f,P)$ and each of the sums $L(f,P)$ and $U(f,P)$ is arbitrarily close to $S\left(f,P,\vec{t}\right)$ for an appropriate choice of $\vec{t}$.

Yet another way to avoid tags in (1) is to replace $S\left(f,P,\vec{t}\right)$ with the *right Riemann sum* (or another uniformly defined notion of tags) defined as

$$R(f,P) := S\left(f,P,\vec{r}\,\right),$$

where, for a fixed partition $P := \{x_0, x_1, \ldots, x_n\}$, we define $\vec{r} := \langle x_{i+1} \colon i < n\rangle$. Once again, this leads to the same notion of an integral. (For a proof in the case of continuous maps, see, e.g., [**3**].)

Now, assume that for the Riemann integrable functions $f, g\colon [a,b] \to \mathbb{R}$ we have

$$L(f,P) = L(g,P) \quad \text{for every partition } P \text{ of } [a,b]. \tag{2}$$

Does this imply that

$$\int_c^d f(x)dx = \int_c^d g(x)dx \quad \text{for all} \quad [c,d] \subseteq [a,b]? \tag{3}$$

In the textbook [**4**] the authors actually ask, in Exercise 14.32, for a proof that indeed (2) implies (3). In fact, such an implication does not hold. However, before we show this, let us first indicate that the implication (2)⇒(3) actually seems very intuitive. We argue for this by showing that a slight variation of (2), namely

$$R(f,P) = R(g,P) \quad \text{for every partition } P \text{ of } [a,b] \tag{4}$$

does imply (3) (for Riemann integrable f and g). In other words, the examples suggested by the title do not exist when the term "Riemann sum" is understood as "right Riemann sum"—validating our claim from the abstract. (The arguments for the left and standard Riemann sums are similar.)

Indeed, (4) implies that $R(f-g,P) = R(f,P) - R(g,P) = 0$ for every partition P. Using this with $P = \{a,b\}$ we get $(b-a)(f-g)(b) = R(f-g,P) = 0$, that is, $(f-g)(b) = 0$. Moreover, for every $x \in (a,b)$, if $P = \{a,x,b\}$, then

$$(x-a)(f-g)(x) = (x-a)(f-g)(x) + (b-x)(f-g)(b) = R(f-g,P) = 0.$$

In other words, (4) implies that $f = g$ on $(a,b]$ and so we have (3).

Why does a similar argument not work for the lower Riemann sums (2) in place of the right Riemann sums (4)? A simple answer is that, in general, the equation $L(f-g,P) = L(f,P) - L(g,P)$ does *not* hold. Specifically, if f and g are continuous and f is increasing while g is decreasing, then $L(g,P) = S\left(g,P,\vec{r}\,\right)$, while $L(f,P) = S\left(f,P,\vec{\ell}\right)$ with $\vec{\ell} = \langle x_i \colon i < n\rangle \ne \vec{r}$. This gives us all the elbow room needed to find a counterexample to the implication (2)⇒(3).

Proposition. *Let $I = [a,b]$, fix $m, \beta, \gamma \in \mathbb{R}$, and let $f, g\colon I \to \mathbb{R}$ be defined, for every $x \in I$, as $f(x) = mx + \beta$ and $g(x) = -mx + \gamma$, respectively. If $f(a) = g(b)$ (see Figure 1), then $L(f,P) = L(g,P)$ for every partition P of I. In particular, if $m \ne 0$, then (3) fails for $[c,d] = \left[\frac{a+b}{2}, b\right]$.*

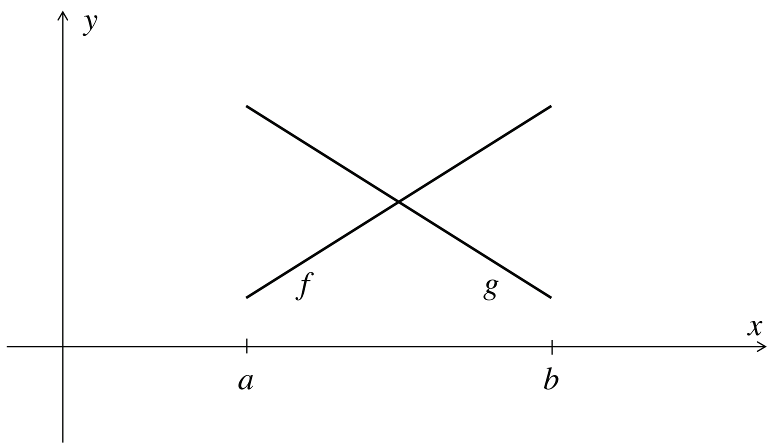

Figure 1. Graphs of maps f and g from Proposition for $m > 0$.

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $I = [a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$. Exchanging f with g, if necessary, we can assume that $m \geq 0$. Then

$$\begin{aligned}
&L(f,P) - L(g,P)\\
&\qquad = \sum_{k=1}^{n} L(f,\{x_{k-1},x_k\}) - \sum_{k=1}^{n} L(g,\{x_{k-1},x_k\})\\
&\qquad = \sum_{k=1}^{n}(x_k - x_{k-1})(mx_{k-1}+\beta) - \sum_{k=1}^{n}(x_k - x_{k-1})(-mx_k+\gamma))\\
&\qquad = \sum_{k=1}^{n}\big(m(x_k^2 - x_{k-1}^2) + (\beta-\gamma)(x_k - x_{k-1})\big)\\
&\qquad = m(x_n^2 - x_0^2) + (\beta-\gamma)(x_n - x_0)\\
&\qquad = (x_n - x_0)\big((mx_0+\beta) - (-mx_n+\gamma)\big) = (b-a)(f(a)-g(b)) = 0
\end{aligned}$$

as needed.

Clearly the implication (2)⇒(3) fails for $[c, d] = \left[\frac{a+b}{2}, b\right]$ since, for our choice of functions f and g, we have $\int_c^d f(x)dx - \int_c^d g(x)dx = \frac{(b-a)^2}{4} m \neq 0$. ∎

2. THE CASE OF CONTINUOUS FUNCTIONS. We will refer to any pair $\langle f, g\rangle$ of functions as in Proposition (and Figure 1) with $m \neq 0$ as an $\times$-*pair*. As we indicated earlier, the existence of such examples seems quite counterintuitive. But, perhaps, even more surprising is the fact that the $\times$-pairs are the *only* such examples within the class of continuous functions, as stated in the following theorem.

Theorem 1. *For every continuous $f, g\colon I \to \mathbb{R}$ the following conditions are equivalent:*

(a_1**)** $L(f, P) = L(g, P)$ *for every partition P of I and $f \neq g$.*
(b_1**)** $\langle f, g\rangle$ *is an $\times$-pair.*

In Theorem 2, presented in the next section, we will generalize the above characterization to the case when the functions f and g are assumed only to be Riemann integrable. Since Theorem 1 follows easily from Theorem 2, we could have opted to skip a direct proof of Theorem 1 in favor of deducing it from Theorem 2. Nevertheless, we decided against such an approach and we start with a proof of Theorem 1. There are two advantages of doing so. First, the proof of Theorem 1 we present is self-contained and at the level accessible to good undergraduate students, while even the statement of Theorem 2 uses more advanced mathematical language (of functions equal almost everywhere). Thus, the proof of Theorem 1 will be accessible to a considerably wider audience than that of Theorem 2. Second, the proof in the case of continuous maps emphasizes the idea behind the proof of the general case without getting too cluttered with the technical details needed for the argument. Since the format of the proof of Theorem 2 is quite close to that of Theorem 1, this will allow us to just sketch the proof of the general case, emphasizing only the differences between the two cases.

Proof of Theorem 1. Clearly, by Proposition, any $\times$-pair satisfies the condition (a_1), that is, (b_1) implies (a_1). Thus, we only need to prove the converse. For this, fix maps $f, g\colon I \to \mathbb{R}$ satisfying (a_1). We will show that $\langle f, g\rangle$ constitutes an $\times$-pair.

To see this, first notice that, for the partition $P = \{a, b\}$,

$$(b-a)\min_{t\in I} f(t) = L(f,P) = L(g,P) = (b-a)\min_{t\in I} g(t)$$

and let

$$\mu := \min_{t\in I} f(t) = \min_{t\in I} g(t).$$

We start with the following fact.

Fact 2.1. $f^{-1}(\mu) \cap g^{-1}(\mu) = \emptyset$.

Proof. To see this, by way of contradiction assume that $f(x) = g(x) = \mu$ for some $x \in I$. We will show that this implies that $f = g$, contradicting (a_1). So, let $t \in (a, b)$. We need to show that $f(t) = g(t)$. We can assume that $t > x$, the case $t < x$ being similar. Now, using $L(f, P) = L(g, P)$ with $P = \{a, t, b\}$, we see that

$$\min_{u\in[t,b]} f(u) = \min_{u\in[t,b]} g(u), \tag{5}$$

as $\mu(t-a) + (b-t)\min_{u\in[t,b]} f(u) = \mu(t-a) + (b-t)\min_{u\in[t,b]} g(u)$. In particular, for every $s \in (x, t)$, using $L(f, P) = L(g, P)$ with $P = \{a, s, t, b\}$, we get

$$\mu(s-a) + (t-s)\min_{u\in[s,t]} f(u) + (b-t)\min_{u\in[t,b]} f(u) \\ = \mu(s-a) + (t-s)\min_{u\in[s,t]} g(u) + (b-t)\min_{u\in[t,b]} g(u)$$

which, by (5), implies that $\min_{u\in[s,t]} f(u) = \min_{u\in[s,t]} g(u)$. Therefore,

$$f(t) = \lim_{s\to t^-} \left(\min_{u\in[s,t]} f(u)\right) = \lim_{s\to t^-} \left(\min_{u\in[s,t]} g(u)\right) = g(t).$$

So, $f = g$ on (a, b) and, since the functions are continuous, also on $[a, b]$. ■

Notice that, by Fact 2.1, we have $\min f^{-1}(\mu) \neq \min g^{-1}(\mu)$.

Fact 2.2. *If* $\min f^{-1}(\mu) < \min g^{-1}(\mu)$ *and* (a_1) *holds, then* $f^{-1}(\mu) = \{a\}$.

Proof. First, we notice that $x := \min f^{-1}(\mu)$ equals a. Indeed, otherwise we would have $x - a > 0$. But $\min f^{-1}(\mu) < \min g^{-1}(\mu)$ implies that $\mu < \min_{u\in[a,x]} g(u)$ and so, using $P = \{a, x, b\}$, we get

$$\begin{aligned} L(f, P) &= \mu(x - a) + \mu(b - x) \\ &< \min_{u\in[a,x]} g(u)(x - a) + \min_{u\in[x,b]} g(u)(b - x) = L(g, P), \end{aligned}$$

which contradicts (a_1).

To finish the proof we show that existence of an $x \in (a, b]$ with $f(x) = \mu$ leads to a contradiction. Indeed, by Fact 2.1, we have $g(x) > \mu$. So, by the continuity of g, there exists a $t \in (a, x)$ with $\min_{u\in[t,x]} g(u) > \mu$. But then, using $P = \{a, t, x, b\}$, we get

$$\begin{aligned} L(f, P) &= \mu(t - a) + \mu(x - t) + \mu(b - x) \\ &< \min_{u\in[a,t]} g(u)(t - a) + \min_{u\in[t,x]} g(u)(x - t) + \min_{u\in[x,b]} g(u)(b - x) = L(g, P), \end{aligned}$$

which contradicts (a_1). ■

In the remainder of the proof of Theorem 1 we will assume that $\min f^{-1}(\mu) < \min g^{-1}(\mu)$, the other case being symmetric. Thus, by Fact 2.2, we have $f^{-1}(\mu) = \{a\}$. Notice also that $g^{-1}(\mu) = \{b\}$. This can be deduced either by a similar argument or by applying Fact 2.2 to the functions $f(-x)$ and $g(-x)$. Therefore, in what follows we will assume that

$$f^{-1}(\mu) = \{a\} \quad \text{and} \quad g^{-1}(\mu) = \{b\}. \tag{6}$$

In the following fact, and in the rest of this article, the term “increasing” need not mean strictly increasing, and similarly for the term “decreasing.”

Fact 2.3. *If (6) and* (a_1) *hold, then* f *is increasing and* g *is decreasing.*

Proof. We prove only the monotonicity of f, the argument for g being similar. So, by way of contradiction, assume that f is not increasing. Then there are $s < t$ in $(a, b]$ so that $f(s) > f(t)$ and so $y := \min_{u\in[s,b]} f(u) < f(s)$.

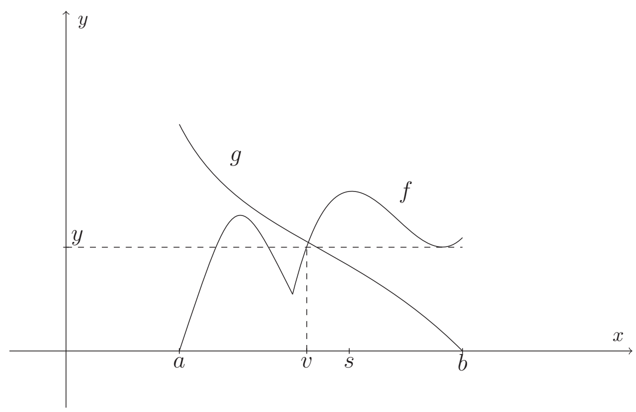

Figure 2. Illustration for the proof of Fact 2.3.

By (6), we have $f(a) = \mu < y < f(s)$. So, by the intermediate value theorem, there exists a largest number $v \in [a, s]$ so that $f(v) = y$. See Figure 2. Define $P = \{a, v, s, b\}$ and $Q = \{a, v, b\}$ and notice that

$$L(f, P) = \mu(v - a) + y(s - v) + y(b - s) = \mu(v - a) + y(b - v) = L(f, Q).$$

So, by (a_1), we have $L(g, P) = L(f, P) = L(f, Q) = L(g, Q)$. At the same time, by (6), we have $\min_{u \in [v,s]} g(u) > \mu$ so that

$$\begin{aligned} L(g, P) - L(g, Q) &= \left(\min_{u \in [v,s]} g(u)(s - v) + \mu(b - s) \right) - \mu(b - v) \\ &= \left(\min_{u \in [v,s]} g(u) - \mu \right) (s - v) > 0, \end{aligned}$$

a contradiction. ■

With the above results, the proof of Theorem 1 is completed with a proof of the following lemma. Notice that in its statement we do not assume that either f or g is continuous. This is important, as we will also use Lemma in the proof of Theorem 2.

Lemma. *Assume that $f, g \colon [a, b] \to \mathbb{R}$ satisfy property (a_1). If f is increasing and continuous at b and g is decreasing and continuous at a, then $\langle f, g \rangle$ is an $\bigtimes$-pair.*

Proof. Let $a < x < c < b$, $P = \{a, b\}$, and $Q = \{a, c, b\}$. See Figure 3. Then

$$\begin{aligned} L(f, Q) - L(f, P) &= f(a)(c - a) + f(c)(b - c) - f(a)(b - a) \\ &= (f(c) - f(a))(b - c) \end{aligned}$$

and

$$\begin{aligned} L(g, Q) - L(g, P) &= g(c)(c - a) + g(b)(b - c) - g(b)(b - a) \\ &= (g(c) - g(b))(c - a). \end{aligned}$$

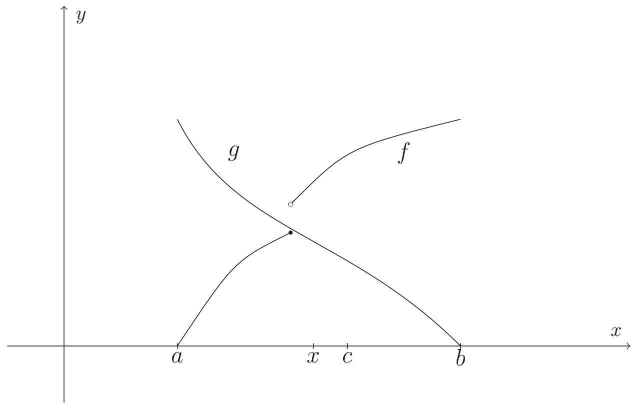

Figure 3. Illustration for the proof of Lemma.

Since, by (a_1), these two quantities are equal, we obtain

$$\frac{g(c)-g(b)}{b-c} = \frac{f(c)-f(a)}{c-a}. \tag{7}$$

Similarly, using $P = \{a, x, b\}$ and $Q = \{a, x, c, b\}$,

$$\begin{aligned} L(f,Q) - L(f,P) &= f(x)(c-x) + f(c)(b-c) - f(x)(b-x) \\ &= (f(c) - f(x))(b-c), \end{aligned}$$

$$\begin{aligned} L(g,Q) - L(g,P) &= g(c)(c-x) + g(b)(b-c) - g(b)(b-x) \\ &= (g(c) - g(b))(c-x), \end{aligned}$$

and

$$\frac{g(c)-g(b)}{b-c} = \frac{f(c)-f(x)}{c-x}. \tag{8}$$

By (7) and (8), we have $\frac{f(c)-f(a)}{c-a} = \frac{f(c)-f(x)}{c-x}$. In particular, since f is continuous at b, for every $x \in (a, b)$,

$$\frac{f(x)-f(a)}{x-a} = \frac{f(c)-f(a)}{c-a} \to_{c\to b} \frac{f(b)-f(a)}{b-a},$$

that is, f is a line with slope $m := \frac{f(b)-f(a)}{b-a}$. Substituting this in (7) with $c = x$ we get $\frac{g(x)-g(b)}{b-x} = m$ for every $x \in (a, b)$ and, by continuity of g at a, also for $x = a$. Therefore, g is a line with slope $-m$. Since, as before, we have $f(a) = g(b)$, $\langle f, g\rangle$ constitutes an $\times$-pair. ■

3. THE CASE OF RIEMANN INTEGRABLE FUNCTIONS. But what happens within the class of all Riemann integrable functions? Is the characterization from Theorem 1 valid when we weaken the assumption "continuous" to "Riemann integrable?" If you think that such naïve generalization cannot be true, you are right. At least up to a point, see Theorem 2 below.

Basically, the characterization from Theorem 1 does not hold for the class $\mathcal{R}$ of all Riemann integrable maps (on $I = [a, b]$), since $\mathcal{R}$ is much bigger than the class of continuous functions. Specifically, recall that a $D \subset \mathbb{R}$ is *null* (or of *Lebesgue measure zero*) provided that for every $\varepsilon > 0$ there is a family $\{(c_i, d_i) \colon i \in \mathbb{N}\}$ of open intervals such that $D \subset \bigcup_{i\in\mathbb{N}}(c_i, d_i)$ and $\sum_{i\in\mathbb{N}}(d_i - c_i) < \varepsilon$. In particular, it is well known and easy to see that

$$\text{no null set contain an interval and a union of two such sets is still null.} \tag{9}$$

A well-known Lebesgue characterization of $\mathcal{R}$, whose nice short proof can be found in [**1**], is as follows:

$$f \in \mathcal{R} \text{ if, and only if, } f \text{ is bounded and the set } D(f) \text{ of points of discontinuity of } f \text{ is null.}$$

Recall also that *the functions f and g are equal almost everywhere*, abbreviated as $f = g$ *a.e.*, provided the set $[f \neq g] := \{x \in I \colon f(x) \neq g(x)\}$ is null.

Now, it is easy to see that if the functions $f, g \colon I \to \mathbb{R}$ are such that

$$\text{for some } \bigtimes\text{-pair } \langle \bar{f}, \bar{g}\rangle \text{ we have } f \geq \bar{f},\ g \geq \bar{g},\ f = \bar{f} \text{ a.e., and } g = \bar{g} \text{ a.e.,} \tag{10}$$

then the pair $\langle f, g\rangle$ also satisfies condition (a_1): they are not equal (even a.e.) by (9), while the other part of (a_1) holds since $\inf f([c, d]) = \min \bar{f}([c, d])$ and $\inf g([c, d]) = \min \bar{g}([c, d])$ whenever $a \leq c < d \leq b$. At the same time, if the sets $[f \neq \bar{f}]$ and $[g \neq \bar{g}]$ are nonempty, then $\langle f, g\rangle$ is *not* an $\bigtimes$-pair, that is, the characterization from Theorem 1 does not hold. This is in spite of the fact that the maps f and g as in (10) can be Riemann integrable, e.g., when the sets $[f \neq \bar{f}]$ and $[g \neq \bar{g}]$ are finite.

Of course, property (10) is not that far from the condition (b_1) of Theorem 1. Surprisingly, (10) is also the condition (b_2) that could be used in the characterization we seek for the Riemann integrable functions:

Theorem 2. *For every Riemann integrable $f, g \colon I \to \mathbb{R}$ the following conditions are equivalent:*

(a_2) $L(f, P) = L(g, P)$ *for every partition P of I and f and g are not equal a.e.*
(b_2) *There is an $\bigtimes$-pair $\langle \bar{f}, \bar{g}\rangle$ such that $\bar{f} \leq f$, $\bar{g} \leq g$, $\bar{f} = f$ a.e., and $\bar{g} = g$ a.e.*

Informally, if $f, g \colon I \to \mathbb{R}$ are Riemann integrable such that $L(f, P) = L(g, P)$ for every partition P of I, then

$$f \neq g \text{ in a.e. sense if, and only if, } \langle f, g\rangle \text{ is an } \bigtimes\text{-pair in a.e. sense.}$$

Thus, in a sense, the naïve generalization of Theorem 1 for $f, g \in \mathcal{R}$ is achieved.

We have also the following characterization for the Riemann integrable maps where the "a.e." requirement in condition (b_2) is omitted. For this, however, we need to further strengthen the assumption (a_2).

Corollary 3. *For every Riemann integrable $f, g\colon I \to \mathbb{R}$ the following conditions are equivalent:*

(a_3**)** *$L(f,P) = L(g,P)$ and $U(f,P) = U(g,P)$ for every partition P of I and f and g are not equal a.e.*
(b_1**)** *$\langle f, g\rangle$ is an $\times$-pair.*

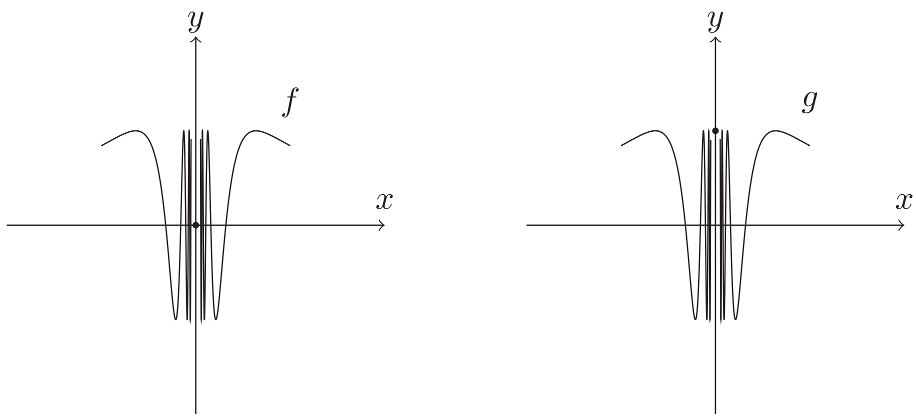

Figure 4. Maps showing that in Corollary 3 clause a.e. cannot be dropped.

Notice that in Corollary 3 the clause "f and g are not equal a.e." cannot be simply reduced to "$f \neq g$." Indeed, this is justified by the functions $f, g\colon [-1,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases}$$

See Figure 4.

The remainder of this article is dedicated to the proofs of the above results. Specifically, we first deduce Corollary 3 from Theorem 2. Then, we prove Theorem 2.

Proof of Corollary 3. This follows from the fact that $-L(f,P) = U(-f,P)$ for every f and partition P. Using this, or by inspecting the argument in the proof of Proposition, we see that (b_1) implies (a_3), that is, any $\times$-pair satisfies (a_3).

To prove the other implication, assume (a_3) and notice that, by Theorem 2, there exists an $\times$-pair $\langle \bar{f}, \bar{g}\rangle$ such that $f \geq \bar{f}$, $g \geq \bar{g}$, and functions $\phi := f - \bar{f} \geq 0$ and $\psi := g - \bar{g} \geq 0$ are equal 0 a.e.

Also, by the above remark, the functions $-f$ and $-g$ satisfy (a_2) from Theorem 2. Thus, there exists an $\times$-pair $\langle \bar{f}_1, \bar{g}_1\rangle$ such that $-f \geq \bar{f}_1$, $-g \geq \bar{g}_1$, and functions $\phi_1 := -f - \bar{f}_1 \geq 0$ and $\psi_1 := -g - \bar{g}_1 \geq 0$ are equal 0 a.e. But this implies that $\bar{f}_1 = -f = -\bar{f}$ a.e and, as the maps $\bar{f}_1$ and $\bar{f}$ are continuous, that $\bar{f}_1 = -\bar{f}$. In particular, $\phi = f - \bar{f} = f + \bar{f}_1 = -\phi_1$. Hence $\phi = \phi_1 = 0$, as both of these functions are nonnegative.

This implies that $f = \bar{f}$. Similarly, $g = \bar{g}$. Hence $\langle f, g\rangle = \langle \bar{f}, \bar{g}\rangle$ is an $\times$-pair, that is, (b_1) is satisfied. ■

Proof of Theorem 2. The fact that the condition (b_2), which is identical to (10), implies (a_2) was given above. So, in what follows we will assume that the Riemann integrable maps $f, g\colon I \to \mathbb{R}$ satisfy (a_2) and show that this implies (b_2).

For a bounded function $h\colon I \to \mathbb{R}$, define $\bar{h}\colon I \to \mathbb{R}$ via the formula

$$\bar{h}(x) := \lim_{\varepsilon \to 0^+} \inf h([x-\varepsilon, x+\varepsilon]), \tag{11}$$

where $h([x-\varepsilon, x+\varepsilon]) := \{h(t)\colon t \in I \cap [x-\varepsilon, x+\varepsilon]\}$. Clearly $\bar{h} \le h$. Also, let

$$\mu_h := \inf h(I).$$

It is easy to see that for every $x \in I$

$$\bar{h}(x) = \mu_h \iff \lim_{n\to\infty} h(x_n) = \mu_h \text{ for some } \langle x_n \rangle_{n\in\mathbb{N}} \text{ in } I \text{ with } x_n \to x. \tag{12}$$

In particular, the set $\bar{h}^{-1}(\mu_h) := \left\{x \in I\colon \bar{h}(x) = \mu_h\right\}$ is compact and nonempty.

Let $\bar{f}$ and $\bar{g}$ be defined from f and g by (11). We will show that these functions satisfy (b_2). Indeed, that $\bar{f} \le f$ and $\bar{g} \le g$ follow from (11). To see that $f = \bar{f}$ a.e., notice that, by (11), $\bar{f}(x) = f(x)$ at every point x of continuity of f. Hence, by the theorem of Lebesgue mentioned above, $f = \bar{f}$ a.e. Similarly, $g = \bar{g}$ a.e. Hence, to finish the proof, we just need to show that $\langle \bar{f}, \bar{g} \rangle$ constitutes an $\bigtimes$-pair. This will be done in the steps similar to those used in the proof of Theorem 1.

By (12), the following equalities

$$\mu_f = \inf f(I) = \min \bar{f}(I) \quad \text{and} \quad \mu_g = \inf g(I) = \min \bar{g}(I)$$

hold. Also, using (a_2) for $P = \{a, b\}$, we see that $\mu_f = \mu_g$. Therefore, in what follows we will use the symbol μ to denote $\mu_f = \mu_g$.

The format and proof of the following fact is very close to that of Fact 2.1. Thus, we will only sketch its proof, emphasizing the differences.

Fact 3.1. $\bar{f}^{-1}(\mu) \cap \bar{g}^{-1}(\mu) = \emptyset$.

Proof. As in Fact 2.1 we assume, by way of contradiction, that there exists an $x \in \bar{f}^{-1}(\mu) \cap \bar{g}^{-1}(\mu)$. An argument identical to that given in the proof of Fact 2.1 shows that for every $t \in (a, b)$, $t \ne x$, and every $s \in (a, t)$ close enough to t we have $\inf_{u\in[s,t]} f(u) = \inf_{u\in[s,t]} g(u)$. Therefore, for every point $t \in (a, b)$ at which both f and g are continuous, we have

$$f(t) = \lim_{s\to t^-} \left(\inf_{u\in[s,t]} f(u) \right) = \lim_{s\to t^-} \left(\inf_{u\in[s,t]} g(u) \right) = g(t).$$

This means that $f(t) = g(t)$ for every $t \in I$ not in the null set $D(f) \cup D(g) \cup \{a, b\}$. Thus, $f = g$ a.e., which contradicts (a_2). ■

Notice that, by Fact 3.1, we have $\inf \bar{f}^{-1}(\mu) \ne \inf \bar{g}^{-1}(\mu)$.

Fact 3.2. *If* $\inf \bar{f}^{-1}(\mu) < \inf \bar{g}^{-1}(\mu)$ *and* (a_2) *holds, then* $\bar{f}^{-1}(\mu) = \{a\}$.

Proof. Once again, the proof is a variation of the one for Fact 2.2. The difficulties are caused by the lack of an extreme value theorem for f and g.

First, we note that $\inf \bar{f}^{-1}(\mu) = a$. To see this, assume, by way of contradiction, that the number $x := \inf \bar{f}^{-1}(\mu)$ is greater than a. Then $x < \inf \bar{g}^{-1}(\mu) \le b$. Let

$d \in (x, \inf \bar{g}^{-1}(\mu))$ and notice that $\inf g([a,d]) > \mu$, since otherwise, by (12), there is a $t \in [a,d]$ with $\bar{g}(t) = \mu$. Thus, there exists an $\varepsilon > 0$ with $\inf g([a,d]) \geq \mu + \varepsilon$. In particular, if $c \in (a,x)$, then for every $t \in (c,d)$ and $P = \{a,t,b\}$ we have

$$\begin{aligned} L(g,P) &\geq (t-a)(\mu+\varepsilon) + (b-t)\mu \\ &= \mu(b-a) + \varepsilon(t-a) \geq \mu(b-a) + \varepsilon(c-a). \end{aligned} \tag{13}$$

At the same time, clearly

$$L(f,P) \leq f(t)(b-a). \tag{14}$$

Choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in I converging to x such that $\lim_{n\to\infty} f(x_n) = \mu$. In particular, as $\varepsilon(c-a) > 0$, there exists an $n \in \mathbb{N}$ such that $x_n \in (c,d)$ and $f(x_n)(b-a) < \mu(b-a) + \varepsilon(c-a)$. But this, together with (13) and (14) used with $t := x_n$, implies that

$$L(f,P) \leq f(x_n)(b-a) < \mu(b-a) + \varepsilon(c-a) \leq L(g,P),$$

a contradiction.

The above argument, together with the definition (11), shows that $a \in \bar{f}^{-1}(\mu)$. To finish the proof, assume, by way of contradiction, that there also exists an $x \in (a,b]$ with $\bar{f}(x) = \mu$. Since we have $a = \inf \bar{f}^{-1}(\mu) < \inf \bar{g}^{-1}(\mu)$, we can choose a $d \in (a, \min\{x, \inf \bar{g}^{-1}(\mu)\})$ and an $\varepsilon > 0$ such that $\inf g([a,d]) \geq \mu + \varepsilon$. Then, for $P = \{a,d,b\}$,

$$\begin{aligned} L(g,P) \geq (d-a)(\mu+\varepsilon) + (b-d)\mu = \mu(b-a) + \varepsilon(d-a) \\ > \mu(b-a) = L(f,P), \end{aligned}$$

a contradiction. ■

In the remainder of this proof we will assume that $\inf \bar{f}^{-1}(\mu) < \inf \bar{g}^{-1}(\mu)$, the other case being symmetric. Thus, by Fact 3.2, we have $\bar{f}^{-1}(\mu) = \{a\}$. Once again, we can also deduce that $\bar{g}^{-1}(\mu) = \{b\}$. Therefore, in what follows we will assume that

$$\bar{f}^{-1}(\mu) = \{a\} \quad \text{and} \quad \bar{g}^{-1}(\mu) = \{b\}. \tag{15}$$

Now, we are ready for the final piece of the puzzle.

Fact 3.3. *If (15) and (a_2) hold, then $\bar{f}$ is increasing and $\bar{g}$ is decreasing.*

Proof. We will prove this only for $\bar{f}$, the case of $\bar{g}$ being symmetric. Here the difficulty comes from an lack of the intermediate value theorem.

By way of contradiction, assume that this $\bar{f}$ is not increasing. Then, there are $s < t$ in I with $\bar{f}(s) > \bar{f}(t)$. Let $y := \inf f([s,b])$ and $v := \sup\{x \in [a,s] \colon \bar{f}(x) \leq y\}$. Notice that $v < s$, as otherwise, by (11), we would have $\bar{f}(s) \leq y \leq \bar{f}(t)$, contradicting the choice of s and t. Also, by (11), $\bar{f}(v) \leq y$ and $f(u) \geq \bar{f}(u) > y$ holds for all $u \in (v,s]$. Consider the following two cases.

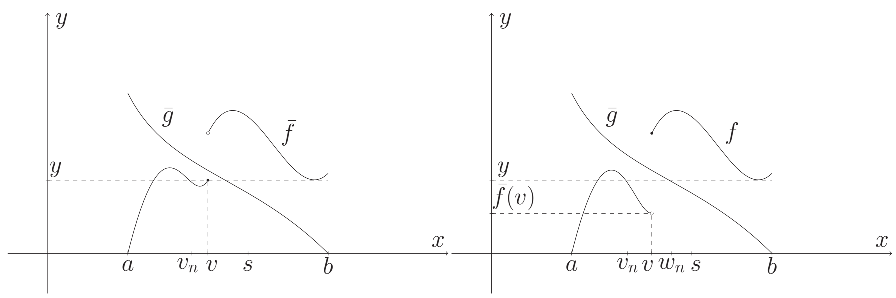

Figure 5. Illustrations for Fact 3.3: left for $\bar{f}(v) = y$ and right for $\bar{f}(v) < y$.

Case $\bar{f}(v) = y$**:** There is a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in $[a, v] \subset I$ with $v_n \to v$ and $\lim_{n\to\infty} f(v_n) = \bar{f}(v) = y$. Let $P_n = \{a, v_n, s, b\}$ and $Q_n = \{a, v_n, b\}$, see Figure 5, and notice that

$$L(f, P_n) - L(f, Q_n) = \inf f([v_n, s])(s - v_n) + y(b - s) - \inf f([v_n, b])(b - v_n).$$

Recalling the definition of $\bar{f}$ at v, see (11), we infer from $\lim_{n\to\infty} f(v_n) = \bar{f}(v) = y$ and $v_n \to v$ that

$$\inf f([v_n, s]) \to_n y \quad \text{and} \quad \inf f([v_n, b]) \to_n y.$$

Thus, $L(f, P_n) - L(f, Q_n) \to_n 0$.

Also

$$\begin{aligned} L(g, P_n) &= \inf g([a, v_n])(v_n - a) + \inf g([v_n, s])(s - v_n) + \mu(b - s), \\ L(g, Q_n) &= \inf g([a, v_n])(v_n - a) + \mu(s - v_n) + \mu(b - s). \end{aligned}$$

Hence,

$$\begin{aligned} L(g, P_n) - L(g, Q_n) &= \big[\inf g([v_n, s]) - \mu\big](s - v_n) \\ &\geq \big[\inf g([a, s]) - \mu\big](s - v_n) \to_n \big[\inf g([a, s]) - \mu\big](s - v) > 0. \end{aligned}$$

Thus, there exists an $n \in \mathbb{N}$ such that $L(f, P_n) - L(f, Q_n) \neq L(g, P_n) - L(g, Q_n)$, contradicting (a_2).

Case $\bar{f}(v) < y$**:** In this case, there is a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in $[a, v]$ such that $v_n \to v$ and $\lim_{n\to\infty} f(v_n) = \bar{f}(v)$. Choose a sequence $\langle w_n \rangle_{n\in\mathbb{N}}$ in $(v, s]$ converging to v and notice that, by the choice of v, $\inf f([w_n, b]) = y$ for all $n \in \mathbb{N}$. This time let $P_n = \{a, v_n, w_n, b\}$ and $Q_n = \{a, v_n, b\}$, see Figure 5, and notice that

$$\begin{aligned} L(g, P_n) &= \inf g([a, v_n])(v_n - a) + \inf g([v_n, w_n])(w_n - v_n) + \mu(b - w_n) \\ L(g, Q_n) &= \inf g([a, v_n])(v_n - a) + \mu(w_n - v_n) + \mu(b - w_n). \end{aligned}$$

So, since g is bounded,

$$L(g, P_n) - L(g, Q_n) = \big[\inf g([v_n, w_n]) - \mu\big](w_n - v_n) \to_n 0.$$

Also

$$L(f, P_n) = \mu(v_n - a) + \inf f([v_n, w_n])(w_n - v_n) + y(b - w_n)$$
$$L(f, Q_n) = \mu(v_n - a) + \inf f([v_n, b])(w_n - v_n) + \inf f([v_n, b])(b - w_n).$$

So, as $\left[\inf f([v_n, w_n]) - \inf f([v_n, b])\right](w_n - v_n) \to_n 0$,

$$\lim_{n\to\infty}(L(g, P_n) - L(g, Q_n)) = \lim_{n\to\infty}(y - \inf f([v_n, b]))(b - w_n) = (y - \bar{f}(v))(b - v) > 0.$$

Therefore, once again, $L(f, P_n) - L(f, Q_n) \neq L(g, P_n) - L(g, Q_n)$ for some $n \in \mathbb{N}$, contradicting (a_2). ■

Now, by Fact 3.3 and (11), $\bar{f}$ is continuous at b and $\bar{g}$ is continuous at a. Consequently, the functions $\bar{f}$ and $\bar{g}$ satisfy the assumptions of Lemma. Hence (a_2) indeed implies that $\langle \bar{f}, \bar{g} \rangle$ constitutes an $\times$-pair.

REFERENCES

1. Brown, A. B. (1936). A proof of the Lebesgue condition for Riemann integrability. *Amer. Math. Monthly*. 43(7): 396–398. doi.org/10.2307/2301737
2. Darboux, G. (1875). Mémoire sur les fonctions discontinues. *Annales scientifiques de l'É.N.S. 2e série.* 4: 57–112. www.numdam.org/article/ASENS_1875_2_4__57_0.pdf
3. Hughes-Hallett, D., McCullum, W. G., et al. (2017). *Calculus: Single and Multivariable*, 7th ed. Hoboken, NJ: Wiley.
4. Laczkovich, M., Sós, V. T. (2015). *Real Analysis—Foundations and Functions of One Variable*, 5th ed. Undergraduate Texts in Mathematics. New York, NY: Springer. Translated from the third (2005) Hungarian edition by the authors.
5. Riemann, B. (1868). Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe. (On the representability of a function by a trigonometric series; i.e., when can a function be represented by a trigonometric series.) *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Proceedings of the Royal Philosophical Society at Göttingen)* 13: 87–132. Available at: books.google.com/books?id=PDVFAAAAcAAJ&pg=RA1-PA87#v=onepage&q&f=false

KRZYSZTOF CHRIS CIESIELSKI received his master and Ph.D. degrees in pure mathematics from Warsaw University, Poland, in 1981 and 1985, respectively. He works at West Virginia University since 1989. Since 2006 he holds a position of adjunct professor in the Department of Radiology at the University Pennsylvania. He is author of three books and over 140 journal research articles. Ciesielski's research interests include both pure mathematics (real analysis, topology, set theory) and applied mathematics (image processing, especially image segmentation). He is an editor of "Real Analysis Exchange," "Journal of Applied Analysis," and "Journal of Mathematical Imaging and Vision."
Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA and
Department of Radiology, MIPG, University of Pennsylvania, Philadelphia, PA 19104-6021, USA
kcies@math.wvu.edu

YICHEN LIU received his bachelor degree in engineering with first class honor from Xi'an Jiaotong-Liverpool University, China, in 2011, and Ph.D. degree in mathematical sciences from the University of Liverpool, UK, in 2015. After that, he spent for two years at Beijing International Center for Mathematical Research in Peking University as a postdoc fellow. He works at Xi'an Jiaotong-Liverpool University as a lecturer since 2018. He is author of 10 journal research articles and his research focuses on the intersection among the fields: nonlinear analysis, partial differential equations, calculus of variations, and control and optimization theory.
Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, Suzhou Industrial Park, Suzhou, Jiangsu China 215123
Yichen.Liu01@xjtlu.edu.cn