DIFFERENT NOTIONS OF SIERPIŃSKI-ZYGMUND FUNCTIONS

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ABSTRACT. A function $f: \mathbb{R} \to \mathbb{R}$ is Sierpiński-Zygmund, $f \in \mathrm{SZ(C)}$, provided its restriction $f \restriction M$ is discontinuous for any $M \subset \mathbb{R}$ of cardinality continuum. Often, it is slightly easier to construct a function $f: \mathbb{R} \to \mathbb{R}$, denoted as $f \in \mathrm{SZ(Bor)}$, with a seemingly stronger property that $f \restriction M$ is not Borel for any $M \subset \mathbb{R}$ of cardinality continuum. It has been recently noticed that the properness of the inclusion $\mathrm{SZ(Bor)} \subseteq \mathrm{SZ(C)}$ is independent of ZFC. In this paper we explore the classes $\mathrm{SZ}(\Phi)$ for arbitrary families Φ of partial functions from $\mathbb{R}$ to $\mathbb{R}$. We investigate additivity and lineability coefficients of the class $\mathbb{S} := \mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$. In particular we show that if $\mathfrak{c} = \kappa^+$ and $\mathbb{S} \neq \emptyset$, then the additivity of $\mathbb{S}$ is κ, that $\mathbb{S}$ is $\mathfrak{c}^+$-lineable, and it is consistent with ZFC that $\mathbb{S}$ is $\mathfrak{c}^{++}$-lineable. We also construct several examples of functions from $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ that belong also to other important classes of real functions.

1. BACKGROUND

The restriction theorems constitute a very important research subject in analysis, see e.g. [9] or [20]. A typical example of such result is a 1912 theorem of N. Lusin, according to which every (Lebesgue) measurable function $f: \mathbb{R} \to \mathbb{R}$ has a continuous restriction $f \restriction E$ to a closed $E \subset \mathbb{R}$, where the measure of $\mathbb{R} \setminus E$ is arbitrary small. Another, less known result in this direction is a 1984 theorem of M. Laczkovich [36] (see also [20, Section 3.2]) stating that for every continuous $f: \mathbb{R} \to \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ such that $f \restriction P$ is differentiable.

As these two examples indicate, the majority of restriction results assume that the original function $f: \mathbb{R} \to \mathbb{R}$ has some "nice" property. An exception here is a 1922 theorem of H. Blumberg [8] (see also [13]) stating that for *every* function $f: \mathbb{R} \to \mathbb{R}$ there exists a dense set $D \subset \mathbb{R}$ such that the restriction $f \restriction D$ is continuous. The set D constructed by Blumberg is countable. Thus, a natural question arises whether each $f: \mathbb{R} \to \mathbb{R}$ has a continuous restriction to an uncountable set. Answering this question, W. Sierpiński and A. Zygmund constructed, in a 1923 paper [43], an $f: \mathbb{R} \to \mathbb{R}$ such that the restriction $f \restriction M$ is discontinuous for any $M \subset \mathbb{R}$ of cardinality continuum $\mathfrak{c}$. Nowadays, any such map is called a *Sierpiński-Zygmund* (or just SZ-) *function*.

A construction of an SZ-function f is actually quite simple once one recalls the following result of K. Kuratowski, see e.g., [32, p. 16].

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Proposition 1.1. *For every continuous function g from an $S \subset \mathbb{R}$ to $\mathbb{R}$ there exist a G_δ-set $G \subset \mathbb{R}$ containing S and a continuous extension $\bar{g}\colon G \to \mathbb{R}$ of g.*

Now, if $\{x_\xi \colon \xi < \mathfrak{c}\}$ and $\{g_\xi \colon \xi < \mathfrak{c}\}$ are the enumerations, respectively, of $\mathbb{R}$ and the family of all continuous maps from G_δ subsets of $\mathbb{R}$ to $\mathbb{R}$ and one chooses, by induction on $\xi < \mathfrak{c}$, the values

$$f(x_\xi) \in \mathbb{R} \setminus \{g_\zeta(x_\xi) \colon \zeta < \xi \ \& \ x_\xi \in \mathrm{dom}(g_\zeta)\}, \tag{1}$$

then this defines a map $f \colon \mathbb{R} \to \mathbb{R}$ which, by Proposition 1.1, is in SZ(C).

The condition "$x_\xi \in \mathrm{dom}(g_\zeta)$" in (1) is essential, since functions g_ζ are only partial: the set G in Proposition 1.1 cannot be, in general, equal to $\mathbb{R}$. Of course, function $\bar{g}$ from Proposition 1.1 can be further extended to a Borel map $\hat{g} \colon \mathbb{R} \to \mathbb{R}$. But, in fact, there is another result of K. Kuratowski, see e.g., [32, p. 73], that is useful for us:

Proposition 1.2. *For every Borel function g from an $S \subset \mathbb{R}$ to $\mathbb{R}$ there exists a Borel extension $\bar{g} \colon \mathbb{R} \to \mathbb{R}$ of g.*

Now, if $\{g_\xi \colon \xi < \mathfrak{c}\}$ is an enumeration of all Borel functions $g \colon \mathbb{R} \to \mathbb{R}$, then the condition "$x_\xi \in \mathrm{dom}(g_\zeta)$" in (1) can be removed. Moreover, by Proposition 1.2, the resulted $f \colon \mathbb{R} \to \mathbb{R}$ is in SZ(Bor). This last approach, which produces a function in SZ(Bor), can be found in [34], [24], [6], [26], or [4]. On the other hand, the original approach from [43], which produces an $f \in$ SZ(C), is used in [22], [3], [15], and [16].

Obviously, SZ(Bor) $\subseteq$ SZ(C) and, as proved in [5, Theorem 4.4] (see also Corollary 2.5 below), the equality between these classes is both consistent with and independent from the usual axioms ZFC of set theory. In spite of the differences between classes SZ(Bor) and SZ(C) that we plan to explore in this paper, their definitions are clearly very similar. To emphasize this, in the reminder of this paper we will consider the generalized class SZ($\mathcal{G}$) of Sierpiński-Zygmund functions, which will encompass both SZ(Bor) and SZ(C).

The main results of this article concern the algebraic properties the family SZ(C) $\setminus$ SZ(Bor) investigated in terms of two measures that are commonly used for this purpose in the literatures, see e.g. [25, 30, 37]. The first of them, so called *additivity*, connected to the linear space generated by SZ(C) $\setminus$ SZ(Bor), is considered in Section 3. The second one, called *lineability* and concerning the maximal size of a linear space contained in $\{0\} \cup$ SZ(C) $\setminus$ SZ(Bor), is studied in Section 4.

In the last section we construct examples of functions from SZ(C) $\setminus$ SZ(Bor) that belong also to other important classes of real functions: additive functions, Hamel functions, and almost continuous functions of Stallings.

In what follows, for a set X its cardinality is denoted as $|X|$. Also, for a cardinal number κ, we define $[X]^\kappa := \{A \subset X \colon |A| = \kappa\}$, and $[X]^{\leq\kappa} := \{A \subset X \colon |A| \leq \kappa\}$, etc. If f is a function, then $\mathrm{dom}(f)$ denotes its domain. For the sets X and Y we use symbol Y^X to denote the family of all functions from X to Y. An ordinal number α is identified with the set of all ordinals $\xi < \alpha$. A cardinal number κ is identified with the first ordinal of size κ. The cardinality of the set of all reals is denoted by $\mathfrak{c}$.

2. SZ($\mathcal{G}$) functions

Throughout this section the symbol X will always stand for a non-empty separable metric space. In particular, this means that $|X| \leq \mathfrak{c}$.

Let $\mathcal{G}$ be a family of partial maps from X to $\mathbb{R}$. Then $\mathrm{SZ}_X(\mathcal{G})$ is defined as the family all functions $f\colon X \to \mathbb{R}$ such that $f{\restriction}M \notin \mathcal{G}$ for every $M \in [X]^{\mathfrak{c}}$, that is,

$$\mathrm{SZ}_X(\mathcal{G}) := \left\{ f \in \mathbb{R}^X \colon f{\restriction}M \notin \mathcal{G} \text{ for every } M \in [X]^{\mathfrak{c}} \right\}. \tag{2}$$

We write $\mathrm{SZ}(\mathcal{G})$ for $\mathrm{SZ}_{\mathbb{R}}(\mathcal{G})$. This definition agrees with our earlier definition of the families SZ(Bor) and SZ(C), where symbols Bor and C denote the classes of all partial function from $\mathbb{R}$ to $\mathbb{R}$ that are, respectively, Borel and continuous. In what follows we will also consider, for any n in $\mathbb{N} := \{1, 2, 3, \ldots\}$, the classes D^n of all partial functions from $\mathbb{R}$ to $\mathbb{R}$ that are n-times differentiable.

Of course, if $|X| < \mathfrak{c}$, then $\mathrm{SZ}_X(\mathcal{G}) = \mathbb{R}^X$ for any family $\mathcal{G}$. So, we will be mainly interested in $\mathrm{SZ}_X(\mathcal{G})$ when $|X| = \mathfrak{c}$.

Assume $\mathcal{G}$ is a family of partial functions on X. Then $\mathrm{cf}(\mathcal{G})$, the *cofinality of* $\mathcal{G}$, is the minimal cardinality of a subfamily $\mathcal{G}_0 \subset \mathcal{G}$ such that every $g \in \mathcal{G}$ is covered (i.e., $g \subset g_0$) by some $g_0 \in \mathcal{G}_0$. Any such family $\mathcal{G}_0$ is called a *basis* for $\mathcal{G}$. We say that a family $\mathcal{G}$ is *hereditary* if it is closed onto subfunctions: if $g \in \mathcal{G}$ and $g_0 \subset g$, then $g_0 \in \mathcal{G}$. A family $\mathcal{G}$ is called *nice* if it is hereditary, contain all functions defined on singletons, and have cofinality less than or equal to $\mathfrak{c}$. Note that the following families of partial functions from $\mathbb{R}$ to $\mathbb{R}$ are nice:

- Bor of all Borel functions;
- C of all continuous functions;
- D^n of all n-times differentiable functions, $n \in \mathbb{N}$;
- C^n of all maps in D^n, $n \in \mathbb{N}$, with continuous nth derivative.

The fact that classes D^n and C^n are nice follows from an appropriate version of Proposition 1.1, that every partial D^n function has an extension to a D^n function defined on a Borel set, see a 2018 paper [19, theorem 5.4] of Ciesielski and Seoane-Sepúlveda. (Surprisingly, it seems that for $n > 1$ this result was previously unknown.)

The original construction of Sierpiński and Zygmund can easily be generalized as follows.

Theorem 2.1. *Let $\mathcal{G}$ be a non-empty family of real valued partial functions on X and assume that* $\mathrm{cf}(\mathcal{G}) \leq \mathfrak{c}$. *Then* $\mathrm{SZ}_X(\mathcal{G}) \neq \emptyset$.

Proof. If $|X| < \mathfrak{c}$, then $\mathrm{SZ}_X(\mathcal{G}) = \mathbb{R}^X \neq \emptyset$. So, we can assume that $|X| = \mathfrak{c}$. Let $\mathcal{G}_0 = \{g_\xi \colon \xi < \mathfrak{c}\}$ be a basis for $\mathcal{G}$ and $X = \{x_\xi \colon \xi < \mathfrak{c}\}$. For every $\xi < \mathfrak{c}$ choose $f(x_\xi)$ as in (1). Then $f \in \mathrm{SZ}_X(\mathcal{G})$. □

It is easy to observe that if $\mathcal{G} \subset \mathcal{F}$, then $\mathrm{SZ}_X(\mathcal{F}) \subset \mathrm{SZ}_X(\mathcal{G})$ and one can ask when this inclusion is proper. This question is related to the following notions.

Definition 2.2. *Let $\mathcal{G}$ be a family of partial functions on X. A function $f\colon X \to \mathbb{R}$ is called:*

- *$(\kappa, \mathcal{G})$-decomposable if there exists a partition $\{X_\alpha\}_{\alpha<\kappa}$ of X such that the restriction of f to any X_α belongs to $\mathcal{G}$;*
- *$(< \lambda, \mathcal{G})$-decomposable when f is $(\kappa, \mathcal{G})$-decomposable for some $\kappa < \lambda$.*

Notice that the notions of $(\omega, \mathcal{G})$- and $(< \mathfrak{c}, \mathcal{G})$-decomposability coincide under CH.

The following proposition characterizes the $(<\mathfrak{c},\mathcal{G})$-decomposable functions under the assumption that $\mathfrak{c}$ is a regular cardinal. This result generalizes a characterization of countable continuity formulated (for $X=\mathbb{R}$, under CH, and without a proof) by Darji in [23, Theorem 10]. See also [5, Proposition 4.3].

Proposition 2.3. *Let $\mathcal{G}$ be a nice family of partial functions on X and let $A\in[X]^{\mathfrak{c}}$.*

(1) *If $f\colon A\to\mathbb{R}$ satisfies the following condition*
 $(*)$ *for every $U\in[A]^{\mathfrak{c}}$ there exists a $V\in[U]^{\mathfrak{c}}$ such that $f\restriction V\in\mathcal{G}$ (i.e. $f\restriction U\notin \mathrm{SZ}_U(\mathcal{G})$ for every $U\in[A]^{\mathfrak{c}}$),*
 then f is $(<\mathfrak{c},\mathcal{G})$-decomposable.

(2) *If $\mathfrak{c}$ is a regular cardinal, then every $(<\mathfrak{c},\mathcal{G})$-decomposable $f\colon A\to\mathbb{R}$ satisfies the condition $(*)$.*

Proof. (1) Suppose that $f\colon A\to\mathbb{R}$ is not $(<\mathfrak{c},\mathcal{G})$-decomposable. We need to show that $(*)$ is false. So, let $\mathcal{G}_0=\{g_\alpha\colon \alpha<\mathfrak{c}\}$ be a basis of $\mathcal{G}$ and for each $\alpha<\mathfrak{c}$ choose an $x_\alpha\in A\setminus\{x_\beta\colon\beta<\alpha\}$ such that $\langle x_\alpha,f(x_\alpha)\rangle\notin\bigcup_{\beta<\alpha}g_\beta$. This is possible as $f\not\subset\bigcup_{\beta<\alpha}\{\langle x_\beta,f(x_\beta)\rangle\}\cup\bigcup_{\beta<\alpha}g_\beta$, since f is not $(<\mathfrak{c},\mathcal{G})$-decomposable. Then $U:=\{x_\alpha\colon\alpha<\mathfrak{c}\}\in[A]^{\mathfrak{c}}$ justifies the negation of $(*)$, since $|(f\restriction U)\cap g|<\mathfrak{c}$ for each $g\in\mathcal{G}_0$.

(2) Next assume that $\mathfrak{c}$ is regular and $f\colon A\to\mathbb{R}$ is $(<\mathfrak{c},\mathcal{G})$-decomposable. Then, for some $\kappa<\mathfrak{c}$, A can be represented as $A=\bigcup_{\alpha<\kappa}X_\alpha$ with $f\restriction X_\alpha\in\mathcal{G}$ for each $\alpha<\kappa$. To see $(*)$, fix a $U\in[A]^{\mathfrak{c}}$. Since $\mathfrak{c}$ is regular, there exists an $\alpha<\kappa$ for which $V:=U\cap X_\alpha$ is of size $\mathfrak{c}$. Then V justifies $(*)$. Indeed, $f\restriction V\in\mathcal{G}$ since $\mathcal{G}$ is hereditary and contains $f\restriction X_\alpha$. $\square$

Now assume that $\mathcal{F}$ and $\mathcal{G}$ are families of partial functions on X such that $\bigcup\mathcal{F}\subset\bigcup\mathcal{G}$ (i.e., so that every $f\in\mathcal{F}$ is covered by functions from $\mathcal{G}$). Notice, that this assumption is satisfied when $\mathcal{G}$ contains all singletons. By $\mathrm{dec}(\mathcal{F},\mathcal{G})$ we denote the minimal cardinal κ such that every $f\in\mathcal{F}$ is $(\kappa,\mathcal{G})$-decomposable, that is, such that for every $f\in\mathcal{F}$ there is a partition $\{X_\alpha\colon\alpha<\kappa\}$ of X for which $f\restriction X_\alpha\in\mathcal{G}$ for every $\alpha<\kappa$. This cardinal has been defined by J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki in [11]. (See also [12].)

Theorem 2.4. *Assume that $|X|=\mathfrak{c}$ and let $\mathcal{F}$ and $\mathcal{G}$ be the families of partial functions on X such that $\mathcal{G}$ is nice and $\mathcal{F}$ is hereditary.*

(1) *Assume that there exists an $f_0\in\mathcal{F}$ such that $\mathrm{dec}(\{f_0\},\mathcal{G})=\mathfrak{c}$. Then, there exists an $f\in\mathbb{R}^X$ in $\mathrm{SZ}_X(\mathcal{G})\setminus\mathrm{SZ}_X(\mathcal{F})$. In particular, such an f exists, when $\mathfrak{c}$ is a successor cardinal and $\mathrm{dec}(\mathcal{F},\mathcal{G})=\mathfrak{c}$.*

(2) *If $\mathfrak{c}$ is a regular cardinal and $\mathrm{dec}(\mathcal{F},\mathcal{G})<\mathfrak{c}$, then $\mathrm{SZ}_X(\mathcal{G})\setminus\mathrm{SZ}_X(\mathcal{F})=\emptyset$.*

Proof. (1) By our assumption, there exists an $f_0\colon A\to\mathbb{R}$ in $\mathcal{F}$ which is not $(<\mathfrak{c},\mathcal{G})$-decomposable. Then $A\in[X]^{\mathfrak{c}}$. So, by Proposition 2.3, there exists a $U\in[A]^{\mathfrak{c}}$ such that $f_0\restriction V\notin\mathcal{G}$ for any $V\in[U]^{\mathfrak{c}}$, that is, with the property that $f_0\restriction U\in\mathrm{SZ}_U(\mathcal{G})$. Also, by Theorem 2.1, there is a map $f_1\colon X\setminus U\to\mathbb{R}$ in $\mathrm{SZ}_{X\setminus U}(\mathcal{G})$. Then $f=(f_0\restriction U)\cup f_1$ is as needed. Indeed, to see that $f\in\mathrm{SZ}_X(\mathcal{G})$ fix an $M\in[X]^{\mathfrak{c}}$ and, by way of contradiction, assume that $f\restriction M\in\mathcal{G}$. Then $f\restriction(M\cap U), f\restriction(M\setminus U)\in\mathcal{G}$, as $\mathcal{G}$ is hereditary. Also, at least one of the sets $M\cap U$ and $M\setminus U$ must have cardinality $\mathfrak{c}$. But $|M\cap U|=\mathfrak{c}$ contradicts the fact that $f\restriction U=f_0\restriction U\in\mathrm{SZ}_U(\mathcal{G})$, while $|M\setminus U|=\mathfrak{c}$ contradicts $f\restriction(M\setminus U)=f_1\restriction(M\setminus U)\in\mathrm{SZ}_{X\setminus U}(\mathcal{G})$. So, indeed $f\in\mathrm{SZ}_X(\mathcal{G})$. Also, $f\notin\mathrm{SZ}_X(\mathcal{F})$ since $\mathcal{F}$ is hereditary so that $f\restriction U=f_0\restriction U\in\mathcal{F}$.

(2) To see this, take an $f\colon X \to \mathbb{R}$ not in $\mathrm{SZ}_X(\mathcal{F})$. We need to show that $f \notin \mathrm{SZ}_X(\mathcal{G})$. Indeed, $f \notin \mathrm{SZ}_X(\mathcal{F})$ implies that there is an $A \in [X]^{\mathfrak{c}}$ with $f{\restriction}A \in \mathcal{F}$. Since $\mathrm{dec}(\mathcal{F},\mathcal{G}) = \kappa < \mathfrak{c}$, our $f{\restriction}A$ can be decomposed into κ maps $\{f{\restriction}B_\xi \colon \xi < \kappa\}$ from $\mathcal{G}$. Since $\mathfrak{c}$ is a regular cardinal and $\kappa < \mathfrak{c}$, there is $\xi < \mathfrak{c}$ with $|B_\xi| = \mathfrak{c}$. This and $f{\restriction}B_\xi \in \mathcal{G}$ imply that $f \notin \mathrm{SZ}_X(\mathcal{G})$, as needed. □

This implies the following result about classes SZ(Bor) and SZ(C).

Corollary 2.5. *For any uncountable Polish space X the equality $\mathrm{SZ}_X(\mathrm{Bor}) = \mathrm{SZ}_X(\mathrm{C})$ is independent of ZFC.*

Proof. Let Bor_0 denote the family of all Borel functions $f\colon X \to \mathbb{R}$. Cichoń, Morayne, Pawlikowski and Solecki proved that

$$\mathrm{cov}\,(\mathcal{M}) \le \mathrm{dec}(\mathrm{Bor}_0, \mathrm{C}) \le \mathfrak{d}, \tag{3}$$

where $\mathrm{cov}\,(\mathcal{M})$, the *covering of category*, is the smallest cardinality of a covering of $\mathbb{R}$ by meager sets, and $\mathfrak{d}$, the *dominating number*, is the smallest cardinality of a dominating family $D \subset \omega^\omega$. (See [11, Theorem 5.7] and [10, Theorem 4.3]. Compare also [12, Theorem 4.1].) Note that $\mathrm{dec}(\mathrm{Bor}_0, \mathrm{C}) = \mathrm{dec}(\mathrm{Bor}, \mathrm{C})$. Indeed, since $\mathrm{Bor}_0 \subset \mathrm{Bor}$, $\mathrm{dec}(\mathrm{Bor}_0, \mathrm{C}) \le \mathrm{dec}(\mathrm{Bor}, \mathrm{C})$. The opposite inequality follows easily from the fact that each partially Borel function can be extended to a Borel function defined on whole X. (Compare Proposition 1.2.)

Next, observe that $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) \neq \emptyset$ is consistent with ZFC. Specifically, this holds whenever $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, which is implied by the Continuum Hypothesis and, more general, by Martin Axiom. This is the case, since $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$ implies that there there exists a Borel function $f_0 \in \mathbb{R}^{\mathbb{R}}$ for which $\mathrm{dec}(\{f_0\}, \mathrm{C}) = \mathfrak{c}$, see [11, Theorem 5.7]. So, by part (1) of Theorem 2.4, $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) \neq \emptyset$.

Finally, the equality $\mathrm{SZ}(\mathrm{Bor}) = \mathrm{SZ}(\mathrm{C})$ holds in any model of ZFC in which $\mathfrak{c}$ is a regular cardinal and $\mathfrak{d} < \mathfrak{c}$, for example in the iterated perfect set model (or any model of ZFC in which the Covering Property Axiom CPA holds, see [17] or [18, Theorem 3.1]). Indeed, by (3), in such case we have $\mathrm{dec}(\mathrm{Bor}, \mathrm{C}) \le \mathfrak{d} < \mathfrak{c}$. So, by part (2) of Theorem 2.4, $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) = \emptyset$. In particular, we have $\mathrm{SZ}(\mathrm{C}) \subset \mathrm{SZ}(\mathrm{Bor})$. Since inclusion $\mathrm{SZ}(\mathrm{Bor}) \subset \mathrm{SZ}(\mathrm{C})$ always holds, as $\mathrm{C} \subset \mathrm{Bor}$, we get desired $\mathrm{SZ}(\mathrm{Bor}) = \mathrm{SZ}(\mathrm{C})$. □

The similar conclusions can be also established for the other pairs of classes of nice functions mentioned above:

Corollary 2.6. *For every $n \in \mathbb{N}$:*

(i) *the strict inclusion $\mathrm{SZ}(\mathrm{D}^n) \subsetneq \mathrm{SZ}(\mathrm{C}^n)$ is independent of ZFC;*
(ii) *the strict inclusion $\mathrm{SZ}(\mathrm{C}^{n-1}) \subsetneq \mathrm{SZ}(\mathrm{D}^n)$ is provable in ZFC.*

Proof. As in the proof of Corollary 2.5, by Theorem 2.4 these results can be reduced to the decomposition results. Specifically, it is proved in [18] (see also [17, Example 4.5.5]) that, for every $n \in \mathbb{N}$, $\mathrm{dec}(\mathrm{C}^{n-1}, \mathrm{D}^n) = \mathfrak{c}$ is provable in ZFC (in fact, there exists a function $f \in \mathrm{C}^{n-1}$ with $\mathrm{dec}(\{f\}, \mathrm{D}^n) = \mathfrak{c}$), while $\mathrm{dec}(\mathrm{D}^n, \mathrm{C}^n) = \mathfrak{c}$ is independent of ZFC: it fails[1] under CPA, when we also have $\mathfrak{c} = \omega_2$; it is implied by CH. □

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[1]The proof in [18] (and [17]) that, under CPA, $\mathrm{dec}(\mathrm{D}^n, \mathrm{C}^n) < \mathfrak{c}$ has a gap—it relies on a false lemma. This has been corrected in [19].

In the remainder of this paper we restrict our study to the family $\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})$, that is, the family $\mathrm{SZ}_X(\mathcal{G})\setminus\mathrm{SZ}_X(\mathcal{F})$ from Theorem 2.4 with $X=\mathbb{R}$, $\mathcal{G}=\mathrm{C}$—the family of all partial continuous functions, and $\mathcal{F}=\mathrm{Bor}$—the family of all partial Borel functions.

3. Additivity coefficient

Recall that the *additivity cardinal coefficient* $\mathrm{A}(\mathcal{F})$ of an $\mathcal{F}\subset\mathbb{R}^{\mathbb{R}}$ is defined as the minimal cardinality $|F|$ of a family $F\subset\mathbb{R}^{\mathbb{R}}$ that cannot be shifted into $\mathcal{F}$ by any single $\varphi\in\mathbb{R}^{\mathbb{R}}$:

$$\mathrm{A}(\mathcal{F}):=\min\big(\{|F|\colon F\subset\mathbb{R}^{\mathbb{R}} \text{ and } \varphi+F\not\subset\mathcal{F} \text{ for every } \varphi\in\mathbb{R}^{\mathbb{R}}\}\cup\{(2^{\mathfrak{c}})^+\}\big).$$

This notion was introduced in the early 1990's by Natkaniec [37, 38] and thoroughly studied in a 1996 paper [30] of Jordan. (See also [12, 31].) The basic properties of this operator are listed in the following proposition, that comes from [30].

Proposition 3.1. *For every $\mathcal{F},\mathcal{G}\subset\mathbb{R}^{\mathbb{R}}$ the following holds.*

(1) $1\le\mathrm{A}(\mathcal{F})\le(2^{\mathfrak{c}})^+$.
(2) *If $\mathcal{F}\subset\mathcal{G}$, then $\mathrm{A}(\mathcal{F})\le\mathrm{A}(\mathcal{G})$.*
(3) *$\mathrm{A}(\mathcal{F})=1$ if, and only if, $\mathcal{F}=\emptyset$.*
(4) *$\mathrm{A}(\mathcal{F})=(2^{\mathfrak{c}})^+$ if, and only if, $\mathcal{F}=\mathbb{R}^{\mathbb{R}}$.*
(5) *If $\mathcal{F}\ne\emptyset$, then $\mathrm{A}(\mathcal{F})=2$ if, and only if, $\mathcal{F}-\mathcal{F}\ne\mathbb{R}^{\mathbb{R}}$.*

In [15, Theorem 2.14] it is shown that $\mathrm{A}(\mathrm{SZ}(\mathrm{C}))=d_{\mathfrak{c}}$, where the cardinal

$$d_\kappa=:\min\left\{|F|\colon F\subset\kappa^\kappa \;\&\; \forall_{h\in\kappa^\kappa}\,\exists_{f\in F}\,|f\cap h|=\kappa\right\}.$$

is defined for any infinite cardinal κ. It is known that $d_{\mathfrak{c}}>\mathfrak{c}$, and $d_{\mathfrak{c}}$ can be different in different models of ZFC, see [15, Corollaries 2.10 and 2.12]. It is worth to notice that the equality $\mathrm{A}(\mathrm{SZ}(\mathrm{C}))=d_{\mathfrak{c}}$ is true in a more general setting:

Proposition 3.2. *Assume that $\mathcal{G}$ is a family of partial functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathcal{G}$ contains a constant zero function and $\mathrm{cf}(\mathcal{G})\le\mathfrak{c}$. Then $\mathrm{A}(\mathrm{SZ}(\mathcal{G}))=d_c$. In particular, $\mathrm{A}(\mathrm{SZ}(\mathrm{Bor}))=d_c>\mathfrak{c}$.*

Proof. The proof of this fact is identical to the proof from [15, Theorem 2.14] that $\mathrm{A}(\mathrm{SZ}(\mathrm{C}))=d_{\mathfrak{c}}$. We will not repeat it here, since in what follows we will use the proposition only to justify that $\mathrm{A}(\mathrm{SZ}(\mathrm{Bor}))>\mathfrak{c}$, which can be easier proved by a simple transfinite induction. $\square$

The goal of this section is to study the number $\mathrm{A}(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor}))$. Since, by Proposition 3.1(3), its value is 1 when $\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})=\emptyset$, in the rest of this section we assume that $\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})\ne\emptyset$.

The main result of this section is the following theorem.

Theorem 3.3. *Assume that $\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})\ne\emptyset$. Then,*

(a) $\omega\le\mathrm{A}(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor}))\le\mathfrak{c}$.
(b) *If $\mathfrak{c}=\kappa^+$ for some cardinal κ, then $\mathrm{A}(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor}))=\kappa$.*
(c) *If $\mathfrak{c}$ is a regular limit cardinal (that is, it is a weakly inaccessible cardinal, see e.g. [35]), then $\mathrm{A}(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor}))=\mathfrak{c}$.*

In particular

$$\big(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})\big)+\big(\mathrm{SZ}(\mathrm{C})\setminus\mathrm{SZ}(\mathrm{Bor})\big)=\mathbb{R}^{\mathbb{R}}.$$

The additional statement follows from (a) and Proposition 3.1(5). The upper bounds in Theorem 3.3 follow from the next two lemmas.

Lemma 3.4. $\mathrm{A}(\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})) \leq \mathfrak{c}$.

Proof. To see this, let $F := \mathrm{Bor} \cap \mathbb{R}^{\mathbb{R}}$ and fix a $g \in \mathbb{R}^{\mathbb{R}}$. Since $|F| = \mathfrak{c}$, it is enough to show that $g + F \not\subset \mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$.

Indeed, $g = g + 0 \in g + F$. If $g \in \mathrm{SZ}(\mathrm{Bor})$, then we are done. So, assume that $g \notin \mathrm{SZ}(\mathrm{Bor})$. Then, there is an $h \in \mathrm{Bor}$ contained in g with $X := \mathrm{dom}(h)$ of cardinality $\mathfrak{c}$. By Proposition 1.2 there is an extension $\bar{h} \in F$ of h. However, $g - \bar{h} \notin \mathrm{SZ}(\mathrm{C})$, since it is equal 0 on the set X of cardinality $\mathfrak{c}$. □

Lemma 3.5. *If* $\mathfrak{c} = \kappa^+$*, then* $\mathrm{A}(\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})) \leq \kappa$.

Proof. First notice that, under $\mathfrak{c} = \kappa^+$,

- there exists an $F = \{f_\zeta \in \mathbb{R}^{\mathbb{R}} \colon \zeta < \kappa\}$ such that $|h \setminus \bigcup F| \leq \kappa$ for every $h \in \mathrm{Bor} \cap \mathbb{R}^{\mathbb{R}}$.

Indeed, if $\mathbb{R} = \{r_\xi \colon \xi < \mathfrak{c}\}$ and $\mathrm{Bor} \cap \mathbb{R}^{\mathbb{R}} = \{h_\xi \colon \xi < \mathfrak{c}\}$, then for every $\xi < \mathfrak{c}$ choose the values $f_\zeta(r_\xi)$, $\zeta < \kappa$, so that $\{f_\zeta(r_\xi) \colon \zeta < \kappa\} \supset \{h_\eta(r_\xi) \colon \eta < \xi\}$. This works.

Next, let $\bar{F} := F \cup \{\Theta\}$, where Θ is the constant zero function, and fix a $g \in \mathbb{R}^{\mathbb{R}}$. It is enough to show that $g + \bar{F} \not\subset \mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$.

To see this, we can assume that $g = g + \Theta \notin \mathrm{SZ}(\mathrm{Bor})$. Then, by Proposition 1.2, there is an $h \in \mathrm{Bor} \cap \mathbb{R}^{\mathbb{R}}$ and $X \in [\mathbb{R}]^{\mathfrak{c}}$ with $g = -h$ on X. Since $(h \restriction X) \setminus \bigcup \bar{F} \subset h \setminus \bigcup F$ has cardinality at most $\kappa < \kappa^+ = \mathfrak{c}$, there exists an $f \in F$ and $Y \in [X]^{\mathfrak{c}}$ so that $f = h$ on Y. Therefore, $g + f = -h + h = \Theta$ on Y, so that $g + f \notin \mathrm{SZ}(\mathrm{C})$, as needed. □

The next lemma will be used to prove the lower bounds in Theorem 3.3. It is quite technical and, perhaps, a little long but its proof is provided in full detail.

Lemma 3.6. *Assume that* $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) \neq \emptyset$ *and let* $\kappa < \mathfrak{c}$ *be an infinite cardinal. If either* $\kappa = \omega$ *or* $\mathfrak{c}$ *is a regular cardinal, then* $\mathrm{A}(\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})) \geq \kappa$.

Proof. Choose an $F \subset \mathbb{R}^{\mathbb{R}}$ with $|F| < \kappa$. We need to find a $\bar{g} \in \mathbb{R}^{\mathbb{R}}$ such that $\bar{g} + F \subset \mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$.

Since $\mathrm{A}(\mathrm{SZ}(\mathrm{Bor})) > \mathfrak{c}$ (see Proposition 3.2), there exists a $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + F \subset \mathrm{SZ}(\mathrm{Bor})$. We claim that is enough to show that

(F) there exists a family $\{Y_\phi \in [\mathbb{R}]^{\mathfrak{c}} \colon \phi \in F\}$ of pairwise disjoint sets such that for every $\phi \in F$ there is a $g^\phi \colon Y_\phi \to \mathbb{R}$ so that $g^\phi + (\phi \restriction Y_\phi) \in \mathrm{Bor}$ and $g^\phi + (f \restriction Y_\phi) \in \mathrm{SZ}_{Y_\phi}(\mathrm{C})$ for every $f \in F$.

Indeed, in such a case the function

$$\bar{g}(x) := \begin{cases} g^\phi(x) & \text{when } x \in Y_\phi \text{ for some } \phi \in F; \\ g(x) & \text{otherwise} \end{cases}$$

is as needed.

To prove (F) we consider two cases.

Case $\kappa = \omega$. Let $h \in \mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$. Then, $h \notin \mathrm{SZ(Bor)}$ implies that there is a set $X \in [\mathbb{R}]^{\mathfrak{c}}$ so that $h \restriction X \in \mathrm{Bor}$, while $h \in \mathrm{SZ(C)}$ implies that

($*$) $h \restriction Y$ is discontinuous for every $Y \in [X]^{\mathfrak{c}}$.

Let $\{X_f \in [X]^{\mathfrak{c}} \colon f \in F\}$ be a partition of X and fix a $\phi \in F$. We will find a $Y_\phi \in [X_\phi]^{\mathfrak{c}}$ and a $g^\phi \colon Y_\phi \to \mathbb{R}$ satisfying (F). Let $\{f_0, \ldots, f_n\}$ be an enumeration of F with $f_0 = \phi$. We will prove, by induction on $k \le n$, that

(I_k) there exist $X^k_\phi \in [X_\phi]^{\mathfrak{c}}$ and finite $A_k \subset \mathbb{N}$ such that $(ah - \phi + f_i) \restriction X^k_\phi \in \mathrm{SZ}_{X^k_\phi}(\mathrm{C})$ for every $a \in \mathbb{N} \setminus A_k$ and $i \le k$.

Indeed, by ($*$), for $k = 0$ this holds with $X^0_\phi := X_\phi$ and $A_0 := \emptyset$. So, assume that (I_k) holds for some $k < n$. We need to prove (I_{k+1}).

If $(ah - \phi + f_{k+1}) \restriction X^k_\phi \in \mathrm{SZ}_{X^k_\phi}(\mathrm{C})$ for every $a \in \mathbb{N} \setminus A_k$, then $X^{k+1}_\phi := X^k_\phi$ and $A_{k+1} := A_k$ clearly satisfy (I_{k+1}). So, assume that this is not the case. Then, there exist an $a_k \in \mathbb{N} \setminus A_k$ and an $X^{k+1}_\phi \in [X^k_\phi]^{\mathfrak{c}}$ such that $(a_k h - \phi + f_{k+1}) \restriction X^{k+1}_\phi \in \mathrm{C}$. We claim that this X^{k+1}_ϕ and $A_{k+1} := A_k \cup \{a_k\}$ satisfy (I_{k+1}). Indeed, otherwise there exist an $a \in \mathbb{N} \setminus A_{k+1}$ and a $Y \in [X^{k+1}_\phi]^{\mathfrak{c}}$ such that $(ah - \phi + f_{k+1}) \restriction Y \in \mathrm{C}$. Also, $(a_k h - \phi + f_{k+1}) \restriction Y \in \mathrm{C}$, as $Y \subset X^{k+1}_\phi$. Therefore, their difference $(a_k - a)h \restriction Y$ is continuous, what contradicts ($*$) as $a_k - a \neq 0$. This completes the inductive argument.

Next, let $Y_\phi := X^n_\phi$ and $g^\phi := (ah - \phi) \restriction Y_\phi$ for the first $a \in \mathbb{N} \setminus A_n$. Then (F) is satisfied by (I_n). This completes the proof in the case when $\kappa = \omega$.

Case $\kappa > \omega$ and $\mathfrak{c}$ is a regular cardinal. We can assume that F is an additive group. Choose a family $\mathcal{B} \in \left[\mathbb{R}^{\mathbb{R}}\right]^{\kappa}$ of linearly independent functions such that $\mathrm{LIN}(\mathcal{B}) \subset \mathrm{SZ}(C) \cup \{0\}$ and, for some $X \in [\mathbb{R}]^{\mathfrak{c}}$, we have $f \restriction X \in \mathrm{Bor}(X)$ for all $f \in \mathcal{B}$. (It exists by [5, Corollary 4.6], see the begin of the proof of Theorem 4.3.) Since $|h \cap h'| < \mathfrak{c}$ for every distinct $h, h' \in \mathcal{B}$ and $\mathfrak{c}$ is regular, decreasing X (by a set of cardinality $< \mathfrak{c}$) if necessary, we can also assume that

$$h(x) \neq h'(x) \text{ for every } x \in X \text{ and distinct } h, h' \in \mathcal{B}. \tag{4}$$

Let $\{X_f \colon f \in F\} \subset [X]^{\mathfrak{c}}$ be a decomposition of X.

Fix an $\phi \in F$. We will find a $Y_\phi \in [X_\phi]^{\mathfrak{c}}$ and a $g^\phi \colon Y_\phi \to \mathbb{R}$ satisfying (F). For this, let $\{g_\gamma \colon \gamma < \mathfrak{c}\}$ be an enumeration of the family of all continuous maps $g \colon G \to \mathbb{R}$ with G being a G_δ set in $\mathbb{R}$ and for every $\gamma < \mathfrak{c}$ define

$$\bar{g}_\gamma := g_\gamma \setminus \bigcup_{\xi < \gamma} g_\xi.$$

Thus, the sets (partial continuous functions) $\{\bar{g}_\gamma \colon \gamma < \mathfrak{c}\}$ are pairwise disjoint and $\bigcup_{\xi \le \gamma} \bar{g}_\xi = \bigcup_{\xi \le \gamma} g_\xi$ for every $\gamma < \mathfrak{c}$.

For every $\gamma < \mathfrak{c}$, $h \in \mathcal{B}$, and $f \in F$ let

$$D^f_{h,\gamma} := \mathrm{dom}(\bar{g}_\gamma \cap (h - f)).$$

Notice that, for every $h \in \mathcal{B}$ and $f \in F$, the sets in the family $\left\{D^f_{h,\gamma} \colon \gamma < \mathfrak{c}\right\}$ are pairwise disjoint, since so are the functions $\{\bar{g}_\gamma \colon \gamma < \mathfrak{c}\}$. In addition, for every $\gamma, \gamma' < \mathfrak{c}$ and distinct $h, h' \in \mathcal{B}$ the set $D := D^f_{h,\gamma} \cap D^f_{h',\gamma'}$ has cardinality $< \mathfrak{c}$. Indeed, if $x \in D$, then we have $h(x) - \bar{g}_\gamma(x) = f(x) = h'(x) - \bar{g}_{\gamma'}(x)$. Thus $h - h' = \bar{g}_{\gamma'} - \bar{g}_\gamma$ on D. Since $h - h'$ is a restriction of a function in SZ(C) and $(\bar{g}_{\gamma'} - \bar{g}_\gamma) \restriction D$ is continuous, we see that indeed $|D| < \mathfrak{c}$.

Next, for every $\gamma < \mathfrak{c}$, $h \in \mathcal{B}$, and $f \in F$ let

$$\bar{D}^f_{h,\gamma} := D^f_{h,\gamma} \setminus \bigcup_{\beta \le \gamma} \bigcup_{h' \in \mathcal{B}\setminus\{h\}} D^f_{h',\beta}$$

and notice that $\left|D^f_{h,\gamma} \setminus \bar{D}^f_{h,\gamma}\right| < \mathfrak{c}$. Also, $\bar{D}^f_{h,\gamma} \cap \bar{D}^f_{h',\beta} = \emptyset$, whenever the pairs $\langle h,\gamma\rangle, \langle h',\beta\rangle \in \mathcal{B}\times\mathfrak{c}$ are distinct: for $h \neq h'$ this follows from the above definition, while for $h = h'$ from pairwise disjointness of $\{\bar{D}^f_{h,\gamma} \colon \gamma < \mathfrak{c}\}$, which follows from the same property of the family $\{D^f_{h,\gamma} \colon \gamma < \mathfrak{c}\}$.

Now, for every $f \in F$ let $D^f := \bigcup\{\bar{D}^f_{h,\gamma} \colon h \in \mathcal{B}\ \&\ \gamma < \mathfrak{c}\} \cap X_\phi$ and define $f^* \colon D^f \to \mathbb{R}$ so that $f^* := f + \bar{g}_\gamma$ on every set $\bar{D}^f_{h,\gamma}$. This is well defined, since $\bar{D}^f_{h,\gamma}$ are pairwise disjoint.

We claim that

(∗) there exists an $h \in \mathcal{B}$ such that the set $T_h := (h \restriction X_\phi) \setminus \bigcup\{f^* \colon f \in F\}$ has cardinality $\mathfrak{c}$.

By way of contradiction, assume that this is not the case, that is, that $|T_h| < \mathfrak{c}$ for every $h \in \mathcal{B}$. Then, by regularity of $\mathfrak{c}$, the set $Z := \bigcup_{h\in\mathcal{B}} \mathrm{dom}(T_h)$ has cardinality less than $\mathfrak{c}$. Choose an $x \in X_\phi \setminus Z$ and notice that, by (4), the set $\{h(x) \colon h \in \mathcal{B}\}$ is of cardinality $|\mathcal{B}| = \kappa$ while it is contained in the set $\{f^*(x) \colon f \in F\}$ of cardinality $\le |F| < \kappa$, a contradiction.

Finally, let $h \in \mathcal{B}$ be as in (∗) and $Y_\phi := \mathrm{dom}(T_h)$. Define $g^\phi \colon Y_\phi \to \mathbb{R}$ as $g^\phi := h - \phi \restriction Y_\phi$. We claim that it satisfies (F).

Clearly $g^\phi + (\phi \restriction Y_\phi) = h \restriction Y_\phi \in \mathrm{Bor}$. To finish the proof, fix an $\hat{f} \in F$ and let $f := \phi - \hat{f} \in F$. We need to show that $g^\phi + (\hat{f} \restriction Y_\phi) \in \mathrm{SZ}_{Y_\phi}(\mathrm{C})$. To see this, first notice that $g^\phi + \hat{f} = h - \phi + \hat{f} = h - f$ on Y_ϕ. By way of contradiction, assume that $h - f$ is continuous on some subset of Y_ϕ of cardinality $\mathfrak{c}$. So, there exists a $\gamma < \mathfrak{c}$ such that the set $A := Y_\phi \cap \mathrm{dom}((h-f)\cap g_\gamma)$ has cardinality $\mathfrak{c}$. We can assume that γ is the smallest with this property. Then also the set $\bar{A} := Y_\phi \cap \mathrm{dom}((h-f)\cap \bar{g}_\gamma)$ has cardinality $\mathfrak{c}$. Notice, that for every $x \in \bar{A}$ we have $(h-f)(x) = \bar{g}_\gamma(x)$, that is, $x \in \mathrm{dom}(\bar{g}_\gamma \cap (h-f)) = D^f_{h,\gamma}$. So, $\bar{A} \subset D^f_{h,\gamma}$ and, since $|D^f_{h,\gamma} \setminus \bar{D}^f_{h,\gamma}| < \mathfrak{c}$, we may assume that $\bar{A} \subset \bar{D}^f_{h,\gamma}$. Therefore, $\bar{A} \subset \bar{D}^f_{h,\gamma} \cap Y_\phi$, and on $\bar{A}$ we have $h = f + \bar{g}_\gamma = f^*$. But this contradicts the fact that $\bar{A} \subset Y_\phi = \mathrm{dom}(T_h)$. This finishes the proof. □

Proof of Theorem 3.3. The upper bound in part (b) follows from Lemma 3.5, while in parts (a) and (c) from Lemma 3.4. The lower bounds follow from Lemma 3.6. □

Problem 3.7. *Assume that* $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) \neq \emptyset$. *What can be said about the number* $\mathrm{A}(\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}))$ *when* $\mathfrak{c}$ *is a singular cardinal number?*

Problem 3.8. *Can we prove Theorem 3.3 for more general classes in place of* Bor *and* C*?*

4. Lineability

The goal of this section is to investigate the lineability of $(\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}))$, that is, finding the largest possible subfamily of $\{0\} \cup (\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}))$ that forms a linear subspace of the space $\mathbb{R}^\mathbb{R}$ (over the field $\mathbb{R}$). We say that a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ is *κ-lineable* if $\mathcal{F} \cup \{0\}$ contains a linear subspace of dimension κ, see [1, 6, 7, 25].

Proposition 4.1. *For any cardinal number κ the following conditions are equivalent.*

(1) $\mathrm{SZ(C)}$ *is κ-lineable;*
(2) $\mathrm{SZ(Bor)}$ *is κ-lineable;*
(3) *there exists a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ.*

Proof. The equivalence of conditions (2) and (3) is proved in [26]. Since $\mathrm{SZ(Bor)} \subset \mathrm{SZ(C)}$, the condition (2) implies (1). Finally, (1) implies (3) because if $f, g \in \mathrm{SZ(C)}$ are linearly independent, then they are $\mathfrak{c}$-almost disjoint. (The same argument is used in [26] in the proof of implication (2)⇒(3).) □

The following fact is proved in [5, Corollary 4.6] by so-called *exponential like function method*, which has been used in many earlier papers, see e.g. [2, 27, 28].

Proposition 4.2. *If the family* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ *is non-empty, then it is $\mathfrak{c}$-lineable.*

We will show below that this result is not optimal.

Assume that X is a set of size $\mathfrak{c}$. We say that a cardinal κ has the *double star property* $\binom{\star}{\star}$ if there exists a sequence $\{f_\xi \in \omega^X : \xi < \kappa\}$ such that for every $n \in \mathbb{N}$

(1) the sets in $\{f_\xi^{-1}(n) \in [X]^{\mathfrak{c}} : \xi < \kappa\}$ are pairwise $\mathfrak{c}$-almost disjoint;
(2) for all one-to-one sequences $\langle i_1, \ldots, i_n\rangle \in \mathbb{N}^n$ and $\langle \xi_1, \ldots, \xi_n\rangle \in \kappa^n$

$$\bigcap_{k=1}^{n} f_{\xi_k}^{-1}(i_k) \in [X]^{\mathfrak{c}}.$$

Theorem 4.3. *Assume that* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)} \neq \emptyset$*. If a cardinal $\kappa \geq \mathfrak{c}$ has the property $\binom{\star}{\star}$, then the family* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ *is κ-lineable.*

Proof. Let $h \in \mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$. Then, $h \notin \mathrm{SZ(Bor)}$ implies that there is a set $X \in [\mathbb{R}]^{\mathfrak{c}}$ so that $h{\restriction}X \in \mathrm{Bor}$, while $h \in \mathrm{SZ(C)}$ implies that

($*$) $h \restriction Y$ is discontinuous for every $Y \in [X]^{\mathfrak{c}}$.

We may assume that $|\mathbb{R} \setminus X| = \mathfrak{c}$. For every $n \in \mathbb{N}$ let $h_n(x) := e^{nh(x)}$. Then the set $\{h_n : n \in \mathbb{N}\}$ is linearly independent and any $h \in \mathrm{LIN}(\{h_n : n \in \mathbb{N}\})$ satisfies the property ($*$). (See [24, Lemma 5.9]. Compare also [5, Corollary 4.6].)

Notice that, by the condition (1) of the property $\binom{\star}{\star}$, there exists a family satisfying (3) of Proposition 4.1. Hence, there exists also a linearly independent family $\{g_\xi \in \mathbb{R}^{\mathbb{R}} : \xi < \kappa\}$ with span in $\mathrm{SZ(Bor)} \cup \{0\}$. Let $\{f_\xi \in \omega^X : \xi < \kappa\}$ be a family witnessing the property $\binom{\star}{\star}$ and for every $\xi < \kappa$ let $\bar{f}_\xi : \mathbb{R} \to \omega$ be an extension of f_ξ such that $\bar{f}_\xi(x) = 0$ for every $x \in \mathbb{R} \setminus X$.

For every $\xi < \kappa$ let

$$\tilde{g}_\xi := \left(g_\xi \restriction \bar{f}_\xi^{-1}(0)\right) \cup \bigcup_{n \in \mathbb{N}} \left(h_n \restriction f_\xi^{-1}(n)\right)$$

that is,

$$\tilde{g}_\xi(x) := \begin{cases} h_n(x) & \text{when } x \in f_\xi^{-1}(n) \text{ for some } n \in \mathbb{N}; \\ g_\xi(x) & \text{otherwise.} \end{cases}$$

First notice that the functions $\{\tilde{g}_\xi : \xi < \kappa\}$ are linearly independent. To see this, assume that $a_1\tilde{g}_{\xi_1} + \cdots + a_n\tilde{g}_{\xi_n} = 0$ for some $a_1, \ldots, a_n \in \mathbb{R}$ and $\xi_1 < \cdots < \xi_n$. We need to show that $a_1 = \cdots = a_n = 0$. Indeed, let $g := a_1 g_{\xi_1} + \cdots + a_n g_{\xi_n}$. Then $g \in \mathrm{SZ(Bor)} \cup \{0\}$ must be constant 0, since otherwise $g \in \mathrm{SZ(Bor)}$, in spite the fact that on the set $\mathbb{R}\setminus X$ of cardinality $\mathfrak{c}$ we have $g = a_1\tilde{g}_{\xi_1} + \cdots + a_n\tilde{g}_{\xi_n} = 0$, that is, $g \restriction \mathbb{R}\setminus X \in \mathrm{Bor}$. Hence, $a_1 g_{\xi_1} + \cdots + a_n g_{\xi_n} = 0$ and, by linear independence of $\{g_\xi \in \mathbb{R}^{\mathbb{R}} \colon \xi < \kappa\}$, this implies that $a_1 = \cdots = a_n = 0$.

By the above fact, $\mathrm{LIN}\left(\{\tilde{g}_\xi \colon \xi < \kappa\}\right)$ has dimension κ. So, to finish the proof, it is enough to verify that

$$\mathrm{LIN}\left(\{\tilde{g}_\xi \colon \xi < \kappa\}\right) \subset \{0\} \cup \mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}.$$

To see this, fix an $f \in \mathrm{LIN}\left(\{\tilde{g}_\alpha : \alpha < \kappa\}\right)\setminus\{0\}$. Then $f = a_1\tilde{g}_{\xi_1} + \cdots + a_n\tilde{g}_{\xi_n}$ for some $a_1, \ldots, a_n \in \mathbb{R}\setminus\{0\}$ and $\xi_1 < \cdots < \xi_n$. We need for show that $f \in \mathrm{SZ(C)}\setminus\mathrm{SZ(Bor)}$.

To argue for $f \notin \mathrm{SZ(Bor)}$, observe that $f(x) = a_1 h_1(x) + \cdots + a_n h_n(x)$ for any $x \in Y := \bigcap_{k=1}^n f_{\xi_k}^{-1}(k) \in [\mathbb{R}]^{\mathfrak{c}}$. Hence, $f{\restriction}Y$ is a Borel function on $Y \in [\mathbb{R}]^{\mathfrak{c}}$ and so $f \notin \mathrm{SZ(Bor)}$.

To see that $f \in \mathrm{SZ(C)}$ first notice that $\{\bigcap_{k=1}^n f_{\xi_k}^{-1}(i_k) \colon \langle i_1, \ldots, i_n\rangle \in \omega^n\}$ is a countable partition of $\mathbb{R}$. Therefore, it is enough to prove that f is SZ(C) on any of these sets. So, fix $\langle i_1, \ldots, i_n\rangle \in \omega^n$ and let $Z := \bigcap_{k=1}^n f_{\xi_k}^{-1}(i_k)$.

To argue for $f \restriction Z \in \mathrm{SZ}_Z(\mathrm{C})$ define the sets $A := \{k \in \{1, \ldots, n\} \colon i_k \neq 0\}$ and $B := \{1, \ldots, n\} \setminus A$. For every $x \in Z$, put

$$\tilde{f}(x) := \sum_{k\in A} a_k h_{i_k}(x) \quad \text{and} \quad \hat{f} := \sum_{k \in B} a_k g_{\xi_k}(x).$$

We can assume that $i_k \neq i_j$ for any distinct $j, k \in A$, since otherwise, by (1), $f_{\xi_k}^{-1}(i_k) \cap f_{\xi_j}^{-1}(i_j) \supset Z$ has cardinality less than $\mathfrak{c}$ and so $f{\restriction}Z \in \mathrm{SZ}_Z(\mathrm{C})$.

We have three cases.

- $A = \emptyset$**:** Then, by the choice of maps g_ξ, we have $f{\restriction}Z = \hat{f}{\restriction}Z \in \mathrm{SZ}_Z(\mathrm{Bor})$.
- $B = \emptyset$**:** Then, since the functions h_n, $n \in \mathbb{N}$, are linearly independent and so $f{\restriction}Z = \tilde{f}{\restriction}Z = (\sum_{i=1}^n a_i h_{i_k}){\restriction}Z \in \mathrm{SZ}_Z(\mathrm{C})$.
- $A \neq \emptyset \neq B$**:** Then, $Z \subset X$, $\tilde{f}{\restriction}Z \in \mathrm{Bor}$, and $\hat{f}{\restriction}Z \in \mathrm{SZ}_Z(\mathrm{Bor})$. Therefore, clearly we have $f{\restriction}Z = \tilde{f}{\restriction}Z + \hat{f}{\restriction}Z \in \mathrm{SZ}_Z(\mathrm{Bor})$.

Thus, $f \restriction Z \in \mathrm{SZ}_Z(\mathrm{C})$, as needed. □

Theorem 4.3 and the following lemma give an independent proof of Proposition 4.2.

Lemma 4.4. *The cardinal* $\mathfrak{c}$ *has the property* $\binom{\star}{\star}$.

Proof. Let X be the family of all $g \in \omega^{\mathfrak{c}}$ such that $\mathrm{supp}(g) := \{\xi < \mathfrak{c} \colon g(\xi) > 0\}$ is countable infinite and g is one-to-one on $\mathrm{supp}(g)$. Notice that X is of size $\mathfrak{c}$. For every $\xi < \mathfrak{c}$ define $f_\xi \colon X \to \omega$ by the formula: $f_\xi(g) = n$ if, and only if, $g(\xi) = n$. Then the family $\{f_\xi \colon \xi < \mathfrak{c}\}$ witnesses the property $\binom{\star}{\star}$. □

4.1. **$\mathfrak{c}^+$-lineability.** The goal of this subsection is to show that in Proposition 4.2 we can actually have $\mathfrak{c}^+$-lineability when $\mathfrak{c}$ is a regular cardinal. This will follow immediately from the next lemma.

Lemma 4.5. *If* $\mathfrak{c}$ *is a regular cardinal, then the cardinal* $\mathfrak{c}^+$ *has the property* $\binom{\star}{\star}$.

Proof. To see this, fix an $X \in [\mathbb{R}]^{\mathfrak{c}}$. We will construct, by transfinite induction $\alpha \leq \mathfrak{c}^+$, the sequences

$$F_\alpha := \langle f_\xi \in \omega^X : \xi < \alpha\rangle$$

subject to the following inductive conditions, where $\mathbb{F}_\alpha$ is the family of all one-to-one maps s from a $D \in [\alpha]^{<\omega}$ into $\mathbb{N}$.

(I_α): For every $n \in \mathbb{N}$ the sets in the family $\{f_\xi^{-1}(n) \in [X]^{\mathfrak{c}} \colon \xi < \alpha\}$ are pairwise $\mathfrak{c}$-almost disjoint.

(J_α): $T_s := \bigcap_{\nu \in \mathrm{dom}(s)} f_\nu^{-1}(s(\nu)) \in [X]^{\mathfrak{c}}$ for every $s \in \mathbb{F}_\alpha$.

(M_α): $F_\xi \subset F_\eta$ for $\xi \le \eta < \alpha$.

Notice that any set $\bigcap_{k=1}^n f_{\xi_k}^{-1}(i_k)$ from the part (2) of definition of $\binom{\star}{\star}$ equals to T_s, where $s := \{\langle \xi_1, i_1\rangle, \ldots, \langle \xi_n, i_n\rangle\} \in \mathbb{F}_\kappa$. Thus, the properties (I_κ) and (J_κ) represent (1) and (2) from $\binom{\star}{\star}$. (For $s = \emptyset$, we understand T_s as equal X.)

Our induction starts with $\alpha = \mathfrak{c}$, for which the desired sequence exists by Lemma 4.4. It is easy to see that if $\alpha \le \mathfrak{c}^+$ is a limit ordinal then a sequence $F_\alpha = \bigcup_{\xi<\alpha} F_\xi$, satisfies $(I_{\bar\alpha})\&(J_{\bar\alpha})\&(M_{\bar\alpha})$ for every $\bar\alpha < \alpha$; therefore, it satisfies also $(I_\alpha)\&(J_\alpha)\&(M_\alpha)$.

To finish the inductive proof, assume that for some $\alpha < \mathfrak{c}^+$, with $\alpha \ge \mathfrak{c}$, we already have a sequence $\langle f_\xi \in \omega^X \colon \xi < \alpha\rangle$ that satisfies $(I_\alpha)\&(J_\alpha)$. We need to construct an $f_\alpha \in \omega^X$ for which the extended sequence $\langle f_\xi \in \omega^X \colon \xi < \alpha+1\rangle$ still satisfies $(I_{\alpha+1})\&(J_{\alpha+1})$.

Let $\langle\langle s_\xi, n_\xi\rangle \colon \xi < \mathfrak{c}\rangle$ be a sequence of all pairs $\langle s, n\rangle \in \mathbb{F}_\alpha \times \mathbb{N}$ so that n does not belong to the range of s and such that each such pair appears in it $\mathfrak{c}$-many times. Let $\{f_{\delta_\xi} \colon \xi < \mathfrak{c}\}$ be an enumeration of $\{f_\xi \colon \xi < \alpha\}$. By induction define a sequence $\langle t_\xi \colon \xi < \mathfrak{c}\rangle$ so that

$$t_\xi \in Z_\xi := \bigcap_{\nu \in \mathrm{dom}(s_\xi)} f_\nu^{-1}(s_\xi(\nu)) \setminus \left(\{t_\zeta \colon \zeta < \xi\} \cup \bigcup_{\zeta<\xi} f_{\delta_\zeta}^{-1}(n_\xi) \right).$$

To see that such choice is possible, notice that there exists an $\eta \in \mathfrak{c} \setminus \mathrm{dom}(s_\xi)$ such that $f_\eta \ne f_{\delta_\zeta}$ for all $\zeta < \xi$. Then, by (J_α), the set $f_\eta^{-1}(n_\xi) \cap \bigcap_{\nu \in \mathrm{dom}(s_\xi)} f_\nu^{-1}(s_\xi(\nu))$ has cardinality $\mathfrak{c}$ (as n_ξ does not belong to the range of s_ξ). Therefore, also the set $f_\eta^{-1}(n_\xi) \cap Z_\xi$ has cardinality $\mathfrak{c}$, since $\mathfrak{c}$ is a regular cardinal and, by (I_α), each set $f_\eta^{-1}(n_\xi) \cap f_{\delta_\zeta}^{-1}(n_\xi)$ has cardinality $< \mathfrak{c}$.

Define

$$f_\alpha(x) := \begin{cases} n_\xi & \text{when } x = t_\xi \text{ for some } \xi < \mathfrak{c}; \\ 0 & \text{otherwise} \end{cases}$$

and notice that it is as needed.

Indeed, $(I_{\alpha+1})$ holds, since for every $\xi < \alpha$ there is a $\zeta < \mathfrak{c}$ such that $f_{\delta_\zeta} = f_\xi$ and, by our definition, for every $n \in \mathbb{N}$ the set $f_\alpha^{-1}(n) \cap f_\xi^{-1}(n) = f_\alpha^{-1}(n) \cap f_{\delta_\zeta}^{-1}(n)$ is contained in $\{t_\gamma \colon \gamma \le \zeta\}$.

To see $(J_{\alpha+1})$, it is enough to show that for every set T_s from (J_α) and every $n \in \mathbb{N}$ not in the range of s the set $T_s \cap f_\alpha^{-1}(n)$ has cardinality $\mathfrak{c}$. But this is the case, since there is $\mathfrak{c}$-many $\xi < \mathfrak{c}$ such that $\langle s_\xi, n_\xi\rangle = \langle s, n\rangle$ and $t_\xi \in T_s \cap f_\alpha^{-1}(n)$ for any such ξ. □

The above lemma and Theorem 4.3 immediately imply the following corollary.

Corollary 4.6. *If* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)} \ne \emptyset$ *and* $\mathfrak{c}$ *is a regular cardinal, then the family* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ *is* $\mathfrak{c}^+$*-lineable.*

4.2. **Forcing axioms and $\mathfrak{c}^{++}$-lineability.** In this subsection we will show that it is consistent with ZFC that the family $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ is $\mathfrak{c}^{++}$-lineable. Notice that such result cannot be obtained in ZFC, since it contradicts $2^{\mathfrak{c}} = \mathfrak{c}^+$. Moreover, there are models of ZFC in which $\mathfrak{c}^{++} = 2^{\mathfrak{c}}$ and without $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of size $2^{\mathfrak{c}}$, see [21, Theorem 3.3]. In such models the family SZ(C) is not $2^{\mathfrak{c}}$-lineable, see Proposition 4.1. Thus $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ has the same property.

For every cardinal κ and partially ordered set $\mathbb{P}$ consider the following statement, a specific form of Generalized Martins Axiom to which we will refer as *κ-Martin's Axiom for* $\mathbb{P}$. (See [14]. Compare also [15].)

$\mathrm{MA}_\kappa(\mathbb{P})$**:** For any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, if $|\mathcal{D}| < \kappa$, then there exists a $\mathcal{D}$-generic filter G in $\mathbb{P}$, that is, such that $D \cap G \neq \emptyset$ for every $D \in \mathcal{D}$.

In a manner similar to that used in [15], for every $X \in [\mathbb{R}]^{\mathfrak{c}}$ and a family $\mathcal{F} \subset \mathbb{R}^X$ satisfying properties (1) and (2) from the definition of $\binom{\star}{\star}$ consider the following notions of forcings: $\mathbb{P}_X := \left\{p \in \omega^D : D \in [X]^{\leq\omega}\right\}$ ordered by the reversed inclusion and

$$\mathbb{P}_{\mathcal{F}} := \mathbb{P}_X \times [\mathcal{F}]^{\leq\omega}$$

ordered as

$$\begin{aligned} \langle p, E\rangle \leq \langle q, F\rangle \quad \iff \quad & p \supseteq q \text{ and } E \supseteq F \text{ and} \\ & p(x) \neq f(x) \text{ for every } f \in F \text{ and } x \in \mathrm{dom}(p) \setminus \mathrm{dom}(q). \end{aligned}$$

It is not difficult to see that each forcing $\mathbb{P}_{\mathcal{F}}$ is ω-closed and, under CH, also ω_2-cc. In particular, the standard iterated forcing technique, identical to that used to prove [14, theorem 3.7], gives the following result.

Proposition 4.7. *Let M be a model of ZFC+GCH in which $X \in [\mathbb{R}]^{\mathfrak{c}}$ and let $\lambda \geq \kappa \geq \omega_2$ be the cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then, there exists a generic model N of ZFC+CH extending M, having the same cardinals and real numbers than M, in which $2^{\mathfrak{c}} = \lambda$ and $\mathrm{MA}_\kappa(\mathbb{P}_{\mathcal{F}})$ holds for every appropriate family $\mathcal{F}$.*

With this result, we are ready to prove the main theorem of this subsection.

Theorem 4.8. *For every $X \in [\mathbb{R}]^{\mathfrak{c}}$ and $\lambda \geq \omega_3$ with $\mathrm{cf}(\lambda) > \omega_1$ it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$ and $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ is $\mathfrak{c}^{++}$-lineable.*

Proof. It is enough to prove that $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ is $\mathfrak{c}^{++}$-lineable in the model N from Proposition 4.7 used with $\kappa = \omega_3$.

To see this, first notice that in this model we have $\mathfrak{c} = \omega_1$ and $\kappa = \omega_3 = \mathfrak{c}^{++}$. Also, $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)} \neq \emptyset$, since we have CH. Therefore, by Theorem 4.3, it is enough to show that, under $\kappa > \mathfrak{c}$, $\mathrm{MA}_\kappa(\mathbb{P}_{\mathcal{F}})$ implies that κ has the property $\binom{\star}{\star}$.

For this, similarly as in Lemma 4.4, we will construct, by transfinite induction $\alpha \leq \kappa$, the sequences

$$F_\alpha := \langle f_\xi \in \omega^X : \xi < \alpha\rangle$$

satisfying the following inductive conditions, where $\mathbb{F}_\alpha$ is the family of all one-to-one maps s from a $D \in [\alpha]^{<\omega}$ into $\mathbb{N}$.

(I_α)**:** For every $n \in \mathbb{N}$ the sets in the family $\{f_\xi^{-1}(n) \in [X]^{\mathfrak{c}} : \xi < \alpha\}$ are pairwise $\mathfrak{c}$-almost disjoint.

(J_α)**:** $T_s := \bigcap_{\nu\in\mathrm{dom}(s)} f_\nu^{-1}(s(\nu)) \in [X]^{\mathfrak{c}}$ for every $s \in \mathbb{F}_\alpha$.

(M_α)**:** $F_\xi \subset F_\eta$ for $\xi \leq \eta < \alpha$.

By Lemma 4.4, we can start our induction with $\alpha = \mathfrak{c}^+$. Once again, it is easy to see that if $\alpha \leq \kappa$ is a limit ordinal then a sequence $F_\alpha = \bigcup_{\xi<\alpha} F_\xi$ satisfies $(I_{\bar\alpha})\&(J_{\bar\alpha})\&(M_{\bar\alpha})$ for every $\bar\alpha < \alpha$; therefore, it satisfies also $(I_\alpha)\&(J_\alpha)\&(M_\alpha)$.

To finish the inductive proof, assume that for some $\alpha < \kappa$, with $\alpha \geq \mathfrak{c}^+$, we already have a sequence $\langle f_\xi \in \omega^X : \xi < \alpha \rangle$ that satisfies $(I_\alpha)\&(J_\alpha)$. We need to construct an $f_\alpha \in \omega^X$ for which the extended sequence $\langle f_\xi \in \omega^X : \xi < \alpha + 1 \rangle$ still satisfies $(I_{\alpha+1})\&(J_{\alpha+1})$. We will use $\mathrm{MA}_\kappa(\mathbb{P}_\mathcal{F})$ with $\mathcal{F} = \{f_\xi : \xi \leq \alpha\}$[2] to construct such f_α.

For this, let $\{x_\xi : \xi < \mathfrak{c}\}$ be an enumeration of X and notice that for every $\zeta < \alpha$, $n \in \mathbb{N}$, $\gamma < \mathfrak{c}$, and $s \in \mathbb{F}_\alpha$ so that n is not in the range $\mathrm{rng}(s)$ of s, the following sets

$$D_\zeta := \{\langle p, E \rangle \in \mathbb{P}_\mathcal{F} : f_\zeta \in E\},$$

$$D^s_{\gamma,n} := \{\langle p, E \rangle \in \mathbb{P}_\mathcal{F} : \text{ there is a } \xi > \gamma \text{ with } \langle x_\xi, n \rangle \in p \text{ and } x_\xi \in T_s\},$$

are dense in $\mathbb{P}_\mathcal{F}$.

Indeed, each D_ζ is dense, since for every $\langle p, E \rangle \in \mathbb{P}_\mathcal{F}$ the condition $\langle p, E \cup \{f_\zeta\} \rangle$ is in D_ζ and extends $\langle p, E \rangle$. To see the density of $D^s_{\gamma,n}$, fix a $\langle p, E \rangle \in \mathbb{P}_\mathcal{F}$. It is enough to find a $\xi > \gamma$ such that $\langle p \cup \{\langle x_\xi, n \rangle\}, E \rangle$ is in $D^s_{\gamma,n}$ and extends $\langle p, E \rangle$.

The choice of x_ξ is similar to that of t_ξ in the proof of Lemma 4.4. Specifically, choose an $\eta < \alpha$ such that $\eta \notin \mathrm{dom}(s)$ and $f_\eta \notin E$. Then, by (J_α), we have $T_s \cap f_\eta^{-1}(n) \in [X]^\mathfrak{c}$. At the same time, by (I_α), we have $|f_\eta^{-1}(n) \cap f_\zeta^{-1}(n)| < \mathfrak{c}$ for every $f_\zeta \in E$. Thus, there exists a $\xi > \gamma$ such that $x_\xi \in T_s \cap f_\eta^{-1}(n) \setminus \mathrm{dom}(p)$ and

$$x_\xi \notin f_\eta^{-1}(n) \cap f_\zeta^{-1}(n) \text{ for every } f_\zeta \in E. \tag{5}$$

We claim that such x_ξ is as needed. Indeed, we have $q := p \cup \{\langle x_\xi, n \rangle\} \in \mathbb{P}_X$ since $x_\xi \in X \setminus \mathrm{dom}(p)$. So, $\langle q, E \rangle \in D^s_{\gamma,n}$, as $\xi > \gamma$. Also, to see that $\langle q, E \rangle \leq \langle p, E \rangle$ we need to show that $n = q(x_\xi) \neq f(x_\xi)$ for every $f \in E$. But this is ensured by (5).

By the above, all sets in the family

$$\mathcal{D} := \big\{D_\zeta : \zeta < \alpha\big\} \cup \big\{D^s_{\gamma,n} : \gamma < \mathfrak{c} \ \&\ n \in \mathbb{N} \ \&\ s \in \mathbb{F}_\alpha \ \&\ n \notin \mathrm{rng}(s)\big\}$$

are dense in $\mathbb{P}_\mathcal{F}$. Since $|\mathcal{D}| \leq |\alpha| < \kappa$, by $\mathrm{MA}_\kappa(\mathbb{P}_\mathcal{F})$ there exists a $\mathcal{D}$-generic filter G in $\mathbb{P}_\mathcal{F}$. Let $\hat{f} = \{p : \langle p, E \rangle \in G\}$. Since G is a filter, it is easy to see that $\hat{f}$ is a function. Let $f_\alpha \in \mathbb{R}^X$ be an extension of $\hat{f}$ such that $f_\alpha(x) = 0$ for every $x \in X \setminus \mathrm{dom}(\hat{f})$. Then f_α is as needed.

Indeed, $(I_{\alpha+1})$ holds, since for every $\zeta < \alpha$ there is a $\langle p, E \rangle \in G \cap D_\zeta$. Specifically, by the definition of order in $\mathbb{P}_\mathcal{F}$, for every $n \in \mathbb{N}$, if $f_\alpha(x) = f_\zeta(x) = n$, then $x \in \mathrm{dom}(p)$. (Indeed, if $x \notin \mathrm{dom}(\hat{f})$, then $f_\alpha(x) = 0 \neq n$. If $x \in \mathrm{dom}(\hat{f})$, then there is $\langle q, F \rangle \in G$ extending $\langle p, E \rangle$ with $x \in \mathrm{dom}(q)$. So, $f_\alpha(x) = q(x) \neq f_\zeta(x)$, unless $x \in \mathrm{dom}(p)$.) Thus, $|f_\alpha^{-1}(n) \cap f_\zeta^{-1}(n)| < \mathfrak{c}$, as required.

To see $(J_{\alpha+1})$, it is enough to show that for every set T_s from (J_α) and every $n \in \mathbb{N}$ not in the range of s the set $T_s \cap f_\alpha^{-1}(n)$ has cardinality $\mathfrak{c}$. Since $\mathfrak{c} = \omega_1$ is a regular cardinal, to see it, it is enough to show that for every $\gamma < \mathfrak{c}$ there is a $\xi > \gamma$ such that $x_\xi \in T_s \cap f_\alpha^{-1}(n)$. But this follows immediately from the fact that there is a $\langle p, E \rangle \in G \cap D^s_{\gamma,n}$, since then there is a $\xi > \gamma$ with $x_\xi \in T_s$ and $f_\alpha(x_\xi) = p(x_\xi) = n$. □

Finally notice that if the family $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ is κ-lineable, then there exists a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^\mathfrak{c}$ of cardinality κ, see Proposition 4.1. It would be good to know, if these two conditions are equivalent.

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[2]Note that this cannot be done in ZFC, since the number of sets T_s in (J_α) that we need consider (and also sets in I_α)) is $> \mathfrak{c}$.

Problem 4.9. (1) *Is the double star property $\binom{\star}{\star}$ equivalent to the existence of a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ?*

(2) *Assume that* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)} \neq \emptyset$ *and κ is a cardinal such that there exists a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ. Does this imply that the family* $\mathrm{SZ(C)} \setminus \mathrm{SZ(Bor)}$ *is κ-lineable?*

5. Examples

5.1. **Additive $\mathrm{SZ}(\mathcal{G})$ functions.** It is well-known that there are additive SZ(C) functions, that is, functions which satisfies the Cauchy equation:

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Examples of such functions can be found in [3], [39], and [40]. Exactly in the same way one can construct an example of additive SZ(Bor) function (see [34]). In the next theorem we generalize these results to the case when $\mathcal{G}$ is a nice family of partial functions on $\mathbb{R}$.

Theorem 5.1. *Suppose $\mathcal{G}$ is a nice family of partial functions on $\mathbb{R}$. Then there exists an additive function $f \in \mathrm{SZ}(\mathcal{G})$.*

Proof. Let $\mathbb{R} = \{x_\alpha \colon \alpha < \mathfrak{c}\}$ and $\{g_\alpha \colon \alpha < \mathfrak{c}\}$ be a basis for $\mathcal{G}$. We will construct a sequence $\langle f_\beta \colon \beta < \mathfrak{c}\rangle$ such that each f_β is an additive function defined on a linear subspace V_β of $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. The construction is by transfinite induction on $\alpha \leq \mathfrak{c}$ so that each of its initial segments $\langle f_\beta \colon \beta < \alpha\rangle$ satisfies the following inductive conditions.

(1) $f_\beta \subset f_\gamma$ for all $\beta \leq \gamma < \alpha$;
(2) $|V_\beta| \leq \max(\omega, \beta)$ for all $\beta < \alpha$;
(3) $x_\beta \in V_\beta$ for all $\beta < \alpha$;
(4) $\mathrm{dom}(f_\gamma \cap g_\beta) \subset V_\beta$ for all $\beta \leq \gamma < \alpha$.

First notice that if for $\alpha = \mathfrak{c}$ such a sequence is constructed, then $f := \bigcup_{\beta<\mathfrak{c}} f_\beta$ is our desired function. Indeed, clearly f is an additive function defined, by (3), on $\mathbb{R}$. It is in $\mathrm{SZ}(\mathcal{G})$ by (2) and (4).

To make an inductive step of our construction, fix an $\alpha < \mathfrak{c}$ so that $\langle f_\beta \colon \beta < \alpha\rangle$ satisfies (1)–(4). Let $f^\alpha := \bigcup_{\beta<\alpha} f_\beta$ and $V^\alpha = \bigcup_{\beta<\alpha} V_\beta$. We will find an extension f_α of f^α so that the sequence $\langle f_\beta \colon \beta < \alpha + 1\rangle$ still satisfies the inductive assumptions.

If $x_\alpha \in V^\alpha$, then we can just put $f_\alpha := f^\alpha$ and $V_\alpha := V^\alpha$. So assume that $x_\alpha \notin V^\alpha$ and let

$$V_\alpha := V^\alpha + \mathbb{Q}x_\alpha. \tag{6}$$

Choose a $y_\alpha \in \mathbb{R}$ so that

$$y_\alpha \notin \mathrm{rng}(f^\alpha) + \bigcup_{\beta \leq \alpha} \mathbb{Q} g_\beta[\mathrm{dom}(g_\beta) \cap V_\alpha]. \tag{7}$$

Such a choice is possible since the set $\mathrm{rng}(f^\alpha) + \bigcup_{\beta\leq\alpha} \mathbb{Q} g_\beta[\mathrm{dom}(g_\beta) \cap V_\alpha]$ has cardinality less than $\mathfrak{c}$. Define $f_\alpha \colon V_\alpha \to \mathbb{R}$ as the unique additive extension of $f^\alpha \cup \{\langle x_\alpha, y_\alpha\rangle\}$. Then, clearly, conditions (1), (2), and (3) are satisfied.

To verify (4) we will use (7). So, let $\beta < \alpha$ and $x \in V_\alpha \cap \mathrm{dom}(g_\beta)$ be such that $f_\alpha(x) = g_\beta(x)$. We need to show that $x \in V_\beta$. This clearly holds by an inductive assumption when $x \in V^\alpha$. So, by way of contradiction, assume that

$x \notin V^\alpha$. Then there are $z \in V^\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$ such that $x = z + qx_\alpha$. Hence, $f^\alpha(z) + qy_\alpha = f_\alpha(x) = g_\beta(x)$ and so

$$y_\alpha = q^{-1}g_\beta(x) - q^{-1}f^\alpha(z) \in \mathrm{rng}(f^\alpha) + \bigcup_{\beta \le \alpha} \mathbb{Q}g_\beta[\mathrm{dom}(g_\beta) \cap V_\alpha],$$

contrary to (7). $\square$

Next, we will prove a similar result for the maps in $\mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F})$.

Theorem 5.2. *Suppose $\mathcal{G}$ and $\mathcal{F}$ are families of partial functions on $\mathbb{R}$, such that $\mathcal{F}$ is hereditary while $\mathcal{G}$ is nice and closed onto compositions (outer and inner) with linear operations $\ell(x) = ax + b$. If there exists an $\varphi \in \mathcal{F}$ such that* $\mathrm{dec}(\{\varphi\}, \mathcal{G}) = \mathfrak{c}$*, then there is an additive function $f \in \mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F})$. In particular, there exists such an f when* $\mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F}) \neq \emptyset$ *and $\mathfrak{c}$ is a successor cardinal.*

Proof. Let $\varphi \in \mathcal{F}$ be such that $\mathrm{dec}(\{\varphi\}, \mathcal{G}) = \mathfrak{c}$. Notice that, under the assumption that $\mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F}) \neq \emptyset$ and $\mathfrak{c}$ is a successor cardinal, such function φ exists since $\mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F}) \neq \emptyset$ and Proposition 2.3(2) imply that $\mathrm{dec}(\mathcal{F}, \mathcal{G}) = \mathfrak{c}$, while this, and our assumption on $\mathfrak{c}$, ensures existence of the desired $\varphi \in \mathcal{F}$.

Let $\mathbb{R} = \{x_\alpha \colon \alpha < \mathfrak{c}\}$ and $\{g_\alpha \colon \alpha < \mathfrak{c}\}$ be a basis for $\mathcal{G}$. We will construct a sequence $\langle f_\beta \colon \beta < \mathfrak{c}\rangle$ of additive functions and a one-to-one mapping $\mathfrak{c} \ni \beta \mapsto t_\beta \in \mathrm{dom}(\varphi)$. The construction is by transfinite induction on $\alpha \le \mathfrak{c}$ so that each of its initial segments $\langle f_\beta \colon \beta < \alpha\rangle$ satisfies the following inductive conditions.

(1) $f_\beta \subset f_\gamma$ for all $\beta \le \gamma < \alpha$;
(2) $|V_\beta| \le \max(\omega, \beta)$ for all $\beta < \alpha$;
(3) $x_\beta \in V_\beta$ for all $\beta < \alpha$;
(4) $\mathrm{dom}(f_\gamma \cap g_\beta) \subset V_\beta$ for all $\beta \le \gamma < \alpha$;
(5) $\langle t_\beta, \varphi(t_\beta)\rangle \in f_\beta$.

First notice that if for $\alpha = \mathfrak{c}$ such a sequence is constructed, then $f := \bigcup_{\beta<\mathfrak{c}} f_\beta$ is our desired function. Indeed, clearly f is an additive function defined, by (3), on $\mathbb{R}$. It is in $\mathrm{SZ}(\mathcal{G})$ by (2) and (4). It is not in $\mathrm{SZ}(\mathcal{F})$ since (5) implies that $|f \cap \varphi| = \mathfrak{c}$.

To make an inductive step of our construction, fix an $\alpha < \mathfrak{c}$ so that $\langle f_\beta \colon \beta < \alpha\rangle$ satisfies (1)–(5). Let $f^\alpha := \bigcup_{\beta<\alpha} f_\beta$ and $V^\alpha = \bigcup_{\beta<\alpha} V_\beta$. We will find an extension f_α of f^α so that the sequence $\langle f_\beta \colon \beta < \alpha+1\rangle$ still satisfies the inductive conditions.

First, we will find an extension $\hat{f}_\alpha$ of f^α that ensures satisfaction of (1)–(4). This is done the same way as in the proof of Theorem 5.1. Next, choose a point $t_\alpha \in \Big(\mathrm{dom}(\varphi) \setminus \mathrm{dom}(\hat{f}_\alpha) \cup \{t_\beta \colon \beta < \alpha\}\Big)$ such that

$$\varphi(t_\alpha) \notin \bigcup \{qg_\beta(v) + b \colon v \in \mathrm{dom}(g_\beta) \cap \mathrm{dom}(\hat{f}_\alpha) \ \&\ b \in \mathrm{rng}(\hat{f}_\alpha)\}. \tag{8}$$

Such t_α exists by the assumption that $\mathrm{dec}(\{\varphi\}, \mathcal{G}) = \mathfrak{c}$, since $\mathcal{G}$ contains all singletons and each of the maps $qg_\beta + b$ is in $\mathcal{G}$.

We define f_α on $V_\alpha := \mathrm{dom}(\hat{f}_\alpha) + \mathbb{Q}t_\alpha$ as the unique additive extension of the map $\hat{f}^\alpha \cup \{\langle t_\alpha, \varphi(t_\alpha)\rangle\}$. Then, clearly conditions (1), (2), (3), and (5) are satisfied. The proof that the property (4) is satisfies is, once again, essentially identical to one in Theorem 5.1. $\square$

As a consequence of the above theorem we immediately obtain the following corollary.

Corollary 5.3. *If* $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor}) \neq \emptyset$ *and* $\mathfrak{c}$ *is a successor cardinal, then there exists an additive function* $f\colon \mathbb{R} \to \mathbb{R}$ *which belongs to the class* $\mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$.

Recall also (compare Corollary 2.6) that there exists an $f \in \mathrm{C}^{n-1}$ such that $\mathrm{dec}(\{f\}, \mathrm{D}^n) = \mathfrak{c}$, see [17, Example 4.5.5]. So, by Theorem 5.2, we have also

Corollary 5.4. *For every* $n \in \mathbb{N}$ *there exists an additive function* $f\colon \mathbb{R} \to \mathbb{R}$ *which belongs to the class* $\mathrm{SZ}(\mathrm{D}^n) \setminus \mathrm{SZ}(\mathrm{C}^{n-1})$.

5.2. **Hamel functions.** Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called *Hamel function* provided its graph is a Hamel basis of the linear space $\mathbb{R}^2$ over $\mathbb{Q}$, see [41]. An example of Hamel function in the class SZ(C) was constructed by Plotka [42].

Theorem 5.5. *Suppose* $\mathcal{G}$ *and* $\mathcal{F}$ *are families of partial functions on* $\mathbb{R}$*, such that* $\mathcal{F}$ *is hereditary while* $\mathcal{G}$ *is nice. If there exists a* $\varphi \in \mathcal{F}$ *such that* $\mathrm{dec}(\{\varphi\}, \mathcal{G}) = \mathfrak{c}$*, then there is a Hamel function* $f \in \mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F})$*. In particular, there exists such an* f *when* $\mathrm{SZ}(\mathcal{G}) \setminus \mathrm{SZ}(\mathcal{F}) \neq \emptyset$ *and* $\mathfrak{c}$ *is a successor cardinal.*

Proof. Fix $\varphi \in \mathcal{F}$ such that $\mathrm{dec}(\{\varphi\}, \mathcal{G}) = \mathfrak{c}$.

Let $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of $\mathbb{R}$ with $x_0 = 0$ and let $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be a basis for $\mathcal{G}$. We will construct a sequence $\langle f_\beta\colon \beta < \mathfrak{c}\rangle$ and a one-to-one mapping $\mathfrak{c} \ni \beta \mapsto t_\beta \in \mathrm{dom}(\varphi)$. The construction is by transfinite induction on $\alpha \le \mathfrak{c}$ so that each of its initial segments $\langle f_\beta\colon \beta < \alpha\rangle$ satisfies the following inductive conditions for every $\beta \le \gamma < \alpha$:

(1) f_β is a function which graph is a linearly independent subset of $\mathbb{R}^2$;
(2) $f_\beta \subset f_\gamma$;
(3) $|f_\beta| \le \max(\omega, \beta)$;
(4) $x_\beta \in \mathrm{dom}(f_\beta)$ and $\langle 0, x_\beta\rangle \in \mathrm{LIN}\,(f_\beta)$;
(5) $f_\gamma \cap g_\beta \subset f_\beta$;
(6) $\langle t_\beta, \varphi(t_\beta)\rangle \in f_\beta$.

First notice that if for $\alpha = \mathfrak{c}$ such a sequence is constructed, then $f := \bigcup_{\beta<\mathfrak{c}} f_\beta$ is our desired function. Indeed, clearly f is a function which graph a linearly independent subset of $\mathbb{R}^2$. By (4), $\mathrm{dom}(f) = \mathbb{R}$ and $\mathrm{LIN}\,(f) = \mathbb{R}^2$, so that f is a Hamel function. By (6), $|f \cap \varphi| = \mathfrak{c}$, so $f \notin \mathrm{SZ}(\mathcal{F})$. Finally, the statements (3) and (5) yield $f \in \mathrm{SZ}(\mathcal{G})$.

To make an inductive step of our construction, fix an $\alpha < \mathfrak{c}$ so that $\langle f_\beta\colon \beta < \alpha\rangle$ satisfies (1)–(6). Let $f^\alpha := \bigcup_{\beta<\alpha} f_\beta$ and and observe that its graph is a linearly independent subset of $\mathbb{R}^2$. We will find, in few steps, an extension f_α of f^α so that the sequence $\langle f_\beta\colon \beta < \alpha+1\rangle$ still satisfies the inductive conditions.

Step 1. If $x_\alpha \in \mathrm{dom}(f^\alpha)$, then put $f'_\alpha := f^\alpha$. If $\alpha = 0$, define $f'_\alpha := \{\langle 0, 1\rangle\}$. Otherwise observe that

$$Y_\alpha = \mathrm{LIN}\,(\mathrm{rng}(f^\alpha)) \cup \{g_\xi(x_\alpha)\colon \xi \le \alpha \;\&\; x_\alpha \in \mathrm{dom}(g_\xi)\}$$

is of size $< \mathfrak{c}$, choose $y \in \mathbb{R} \setminus Y_\alpha$, and put

$$f'_\alpha := f^\alpha \cup \{\langle x_\alpha, y\rangle\}.$$

Step 2. If $\langle 0, x_\alpha\rangle \in \mathrm{LIN}\,(f'_\alpha)$, put $f''_\alpha = f'_\alpha$. Otherwise choose $y \notin \mathrm{LIN}\,(\mathrm{dom}(f'_\alpha))$ and

$$z \in \mathbb{R} \setminus \{g_\xi(x), x_\alpha - g_\xi(x)\colon \xi \le \alpha \;\&\; x \in \mathrm{dom}(g_\xi)\}.$$

and put

$$f''_\alpha = f'_\alpha \cup \{\langle y, z\rangle, \langle -y, x_\alpha - z\rangle\}.$$

Notice that f''_α satisfies the conditions (1)–(5).

STEP 3. Finally notice that φ cannot be covered by α-many functions from $\mathcal{G}$. Therefore, there exists a $t_\alpha \in \mathrm{dom}(\varphi) \setminus \mathrm{LIN}\,(\mathrm{dom}(f''_\alpha) \cup \{t_\beta \colon \beta < \alpha\})$ such that $\varphi(t) \notin \{g_\xi(t) \colon \xi \leq \alpha \ \&\ t \in \mathrm{dom}(g_\xi)\}$. Then define

$$f_\alpha = f''_\alpha \cup \{\langle t, \varphi(t)\rangle\}$$

and observe that f_α satisfies all conditions (1)-(6). □

Remark 5.6. *In the same way (or even simpler) one can prove that if $\mathcal{G}$ is a nice family of partial functions on $\mathbb{R}$, then there exists a Hamel function $f \in \mathrm{SZ}(\mathcal{G})$.*

Also, since $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$ implies that there exists a $\varphi \in \mathrm{Bor}$ with $\mathrm{dec}(\{\varphi\}, \mathrm{C}) = \mathfrak{c}$, see [11, Theorem 5.7], we get

Corollary 5.7. *If $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, then there exists an $f \in \mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$ which is a Hamel function.*

5.3. **Almost continuous functions.** A function $f \colon \mathbb{R} \to \mathbb{R}$ is *almost continuous* in the sense of Stallings, denoted $f \in \mathrm{AC}$, if every open subset of $\mathbb{R}^2$ containing f contains also a continuous function from $\mathbb{R}$ to $\mathbb{R}$, see [44]. It is known that existence of a function $f \in \mathrm{AC} \cap \mathrm{SZ}(\mathrm{C})$ cannot be proved in ZFC, see [3]. In the same paper the authors construct an example $f \in \mathrm{AC} \cap \mathrm{SZ}(\mathrm{C})$ under assumption that $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$. If, in the construction of $f \in \mathrm{AC} \cap \mathrm{SZ}(\mathrm{C})$ from [3] we replace the family C with the family Bor, we obtain the following result.

Theorem 5.8. *Assume that $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$. Then there exists an $f \in \mathrm{AC} \cap \mathrm{SZ}(\mathrm{Bor})$. Moreover, the graph of f is dense in $\mathbb{R}^2$.*

We have also another variety on the same topic.

Theorem 5.9. *If $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, then there exists a $g \in \mathrm{AC} \cap \mathrm{SZ}(\mathrm{C}) \setminus \mathrm{SZ}(\mathrm{Bor})$.*

Proof. Fix an $f \in \mathrm{SZ}(\mathrm{Bor}) \cap \mathrm{AC}$ which is dense in $\mathbb{R}^2$. It is known that every modification of such f on a nowhere dense set is still almost continuous; that is, if $C \subset \mathbb{R}$ is nowhere dense and $g \colon \mathbb{R} \to \mathbb{R}$ is such that $f = g$ on $\mathbb{R} \setminus C$, then $g \in \mathrm{AC}$, see [33] or [29]. Let C be the ternary Cantor set. Since $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, there exists a Borel function $\varphi \in \mathbb{R}^C$ with $\mathrm{dec}(\{\varphi\}, \mathrm{C}) = \mathfrak{c}$, see [11, theorem 5.7]. Therefore, by part (1) of Theorem 2.4, there exists a $X \in [C]^{\mathfrak{c}}$ with $\varphi{\restriction}X \in \mathrm{SZ}_X(\mathrm{C})$. Then $g := f{\restriction}(\mathbb{R} \setminus X) \cup \varphi{\restriction}X$ is as needed. □

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