MAXIMAL LINEABILITY OF THE CLASS OF DARBOUX NOT CONNECTIVITY MAPS ON $\mathbb{R}$

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Abstract. We provide an elegant argument showing, in ZFC, that the class $\mathrm{PES}\setminus\mathrm{Conn}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ that are perfectly everywhere surjective (so Darboux) but not connectivity is $2^{\mathfrak{c}}$-lineable, that is, that there exists a linear subspace of $\mathbb{R}^{\mathbb{R}}$ of dimension $2^{\mathfrak{c}}$ that is contained in $(\mathrm{PES}\setminus\mathrm{Conn})\cup\{0\}$. This solves a problem from a 2020 paper of G.M. Albkwre, K.C. Ciesielski, and J. Wojciechowski. The construction utilizes a transcendental basis of $\mathbb{R}$.

1. Introduction and preliminaries

Over the last two decades a lot of mathematicians have been interested in finding the largest possible vector spaces that are contained in various families of real functions, see e.g. survey [4], monograph [3], and the literature cited there. (More recent work in this direction include [2,5,8].) Specifically, given a cardinal number κ, a subset M of a vector space X is said to be *κ-lineable (in X)* provided there exists a linear space $Y\subset M\cup\{0\}$ of dimension κ. This notion was first studied by Vladimir Gurariy [9], even though he did not use the term lineability. He showed that the set of continuous nowhere differentiable functions on [0,1], together with the constant 0 function, contains an infinite-dimensional vector space, that is, it is ω-lineable. In what follows $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

The goal of this note is to show that the class $\mathrm{PES}\setminus\mathrm{Conn}$ is $2^{\mathfrak{c}}$-lineable, where Conn stands for the class of all *connectivity functions* in $\mathbb{R}^{\mathbb{R}}$ (i.e., from $\mathbb{R}$ to $\mathbb{R}$), that is, having connected graphs (as subspaces of $\mathbb{R}^2$), while PES is the family of all *perfectly everywhere surjective* maps $f\colon\mathbb{R}\to\mathbb{R}$, that is, such that $f[P]=\mathbb{R}$ for every non-empty perfect set $P\subset\mathbb{R}$. Notice that every $f\in\mathrm{PES}$ is also Darboux. The $\mathfrak{c}^+$-lineability of $\mathrm{PES}\setminus\mathrm{Conn}$, under the assumption that $\mathfrak{c}$ is a regular cardinal number, has been proved in a 2020 paper [1] by G.M. Albkwre, K.C. Ciesielski, and J. Wojciechowski. In that paper the authors also asked [1, problem 4.1(iii)] if $2^{\mathfrak{c}}$-lineability of $\mathrm{PES}\setminus\mathrm{Conn}$ can be proved in ZFC. Below we give an affirmative answer to this question.

For an $f\in\mathbb{R}^{\mathbb{R}}$ its *support* is defined as

$$\mathrm{supp}(f)\coloneqq\{x\in\mathbb{R}\colon f(x)\neq 0\}.$$

Note that we do not take the closure of the set above.

For a family $\mathcal{F}\subseteq\mathbb{R}^{\mathbb{R}}$ of non-zero functions with pairwise disjoint supports let $V_{\mathcal{F}}$ be the collections of all maps $f_s\coloneqq\sum_{f\in\mathcal{F}}s(f)\cdot f$, where $s\colon\mathcal{F}\to\{0,1\}$. Notice that each f_s is well defined and that $V_{\mathcal{F}}$ has cardinality 2^{κ}, where $\kappa=|\mathcal{F}|$, is the

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cardinality of $\mathcal{F}$. Let $W_{\mathcal{F}}$ be the linear subspace of $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$ spanned by $V_{\mathcal{F}}$. So, the following remark is obvious.

Remark 1.1. *If $|\mathcal{F}| = \mathfrak{c}$, then $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$.*

Recall (see e.g. [6]) that $B \subset \mathbb{R}$ is a *Bernstein set* provided $P \cap B \neq \emptyset \neq P \setminus B$ for every non-empty perfect set $P \subset \mathbb{R}$.

For an $S \subset \mathbb{R}$ let $\mathbb{Q}(S)$ denote the subfield of $\mathbb{R}$ generated by S (i.e., the smallest smallest subfield of $\mathbb{R}$ containing S) and let $\bar{\mathbb{Q}}(S)$ be the algebraic closure of $\mathbb{Q}(S)$ in $\mathbb{R}$. Recall that S is *algebraically independent* provided $s \notin \bar{\mathbb{Q}}(S \setminus \{s\})$ for every $s \in S$; and that S is a *transcendental basis* provided it is a maximal algebraically independent subset of $\mathbb{R}$. It is well known that there exists a transcendental basis T that is also a Bernstein set—it can be constructed by an easy transfinite induction.[1] (Compare [6, theorem 7.3.4].)

2. The construction

Let id be the identity map from $\mathbb{R}$ to $\mathbb{R}$ and for $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ let $\mathrm{id} \cdot \mathcal{F} := \{\mathrm{id} \cdot f : f \in \mathcal{F}\}$.

Theorem 2.1. *There exists a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of cardinality $\mathfrak{c}$ with pairwise disjoint supports such that $W_{\mathrm{id}\cdot\mathcal{F}} \subset (\mathrm{PES} \setminus \mathrm{Conn}) \cup \{0\}$. In particular, the class $\mathrm{PES} \setminus \mathrm{Conn}$ is $2^{\mathfrak{c}}$-lineable.*

Proof. Let $\{b_\xi : \xi < \mathfrak{c}\}$ be an enumeration of a transcendental basis T that is also a Bernstein set and let $\{B_r^\eta : r \in \mathbb{R}, \eta < \mathfrak{c}\}$ be a partition of T into Bernstein sets such that for every $r \in \mathbb{R}$ and $\eta < \mathfrak{c}$:

(a) $r \in \bar{\mathbb{Q}}(\{b_\zeta : \zeta < \xi\})$ for every $\xi < \mathfrak{c}$ with $b_\xi \in B_r^\eta$.

Such a partition without the property (a) can be found by an easy transfinite induction. Then, the property (a) can be additionally imposed by removing from each initially constructed set B_r^η, with $\eta < \mathfrak{c}$ and $r \neq 0$, a set of cardinality less than $\mathfrak{c}$ and adding each such removed part to B_0^η.

For every $r \in \mathbb{R} \setminus \{0\}$ and $\eta < \mathfrak{c}$ put

$$D_r^\eta := r \cdot B_r^\eta,$$

and notice that

(b) the sets in $\{D_r^\eta : r \in \mathbb{R} \setminus \{0\} \ \& \ \eta < \mathfrak{c}\}$ are Bernstein and pairwise disjoint.

To see pairwise disjointness, choose distinct $\langle r, \eta \rangle, \langle r', \eta' \rangle \in (\mathbb{R} \setminus \{0\}) \times \mathfrak{c}$ and numbers $r \cdot b \in D_r^\eta$ and $r' \cdot b' \in D_{r'}^{\eta'}$. We need to show that $r \cdot b \neq r' \cdot b'$. Indeed, $b \neq b'$ as they belong to disjoint sets B_r^η and $B_{r'}^{\eta'}$, respectively. So, there are distinct $\xi, \xi' < \mathfrak{c}$ such that $b = b_\xi$ and $b' = b_{\xi'}$. We can assume that $\xi' > \xi$. Now, if $r \cdot b = r' \cdot b'$ then, by (a), $b_{\xi'} = b' = (r')^{-1} r \cdot b_\xi \in \bar{\mathbb{Q}}(\{b_\zeta : \zeta < \xi'\})$, contradicting algebraic independence of T. To finish the proof of (b) it is enough to notice that $D_r^\eta = r \cdot B_r^\eta$ intersects every non-empty perfect set P, as $r^{-1} P$ is perfect and $B_r^\eta \cap (r^{-1} P) \neq \emptyset$.

For every $r \in \mathbb{R} \setminus \{0\}$ and $\eta < \mathfrak{c}$ define

$$f_r^\eta(x) := \begin{cases} r/x & \text{if } x \in D_r^\eta \\ 0 & \text{if } x \in \mathbb{R} \setminus D_r^\eta, \end{cases}$$

[1] Let $\{P_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all non-empty perfect subsets of $\mathbb{R}$, for every ordinal $\xi < \mathfrak{c}$ choose $b_\xi \in \bar{\mathbb{Q}}(\{b_\zeta : \zeta < \xi\})$. Then $\{b_\xi : \xi < \mathfrak{c}\}$ is algebraically independent, so there is a transcendental basis T extending it, see e.g. [10]. It is Bernstein, as it and its complement (containing a shift of T by 1) intersect every non-empty perfect set.

and put

$$f_\eta := \sum_{r\in\mathbb{R}\setminus\{0\}} f_r^\eta.$$

The functions f_r^η and f_η are well defined since $0 \notin D_r^\eta$ and, by (b), the supports of the maps f_r^η are pairwise disjoint. We claim that the family $\mathcal{F} := \{f_\eta : \eta < \mathfrak{c}\}$ is as needed.

Indeed, clearly $\mathrm{id}\cdot\mathcal{F} = \{\mathrm{id}\cdot f : f \in \mathcal{F}\}$ consists of $\mathfrak{c}$-many distinct functions with pairwise disjoint supports. So, by Remark 1.1, $W_{\mathrm{id}\cdot\mathcal{F}}$ has needed dimension $2^\mathfrak{c}$.

It remains to show that every non-zero $f \in W_{\mathrm{id}\cdot\mathcal{F}}$ is in $\mathrm{PES}\setminus\mathrm{Conn}$. To see this, notice that there exist $n \in \{1,2,3,\ldots\}$, $a_1,\ldots,a_n \in \mathbb{R}$, and $f_{s_1},\ldots,f_{s_n} \in V_{\mathrm{id}\cdot\mathcal{F}}$ such that

$$f = \sum_{i=1}^n a_i f_{s_i} = \sum_{\eta<\mathfrak{c}} \left(\sum_{i=1}^n a_i s_i(\eta)\right)\cdot(\mathrm{id}\cdot f_\eta), \tag{1}$$

where $f_{s_i} := \sum_{\eta<\mathfrak{c}} s_i(\eta)\cdot\mathrm{id}\cdot f_\eta$ for appropriate $s_i : \mathfrak{c} \to \{0,1\}$. Also, for every $r \in \mathbb{R}\setminus\{0\}$ and $\eta < \mathfrak{c}$ we have $\mathrm{id}\cdot f_r^\eta = r\chi_{D_r^\eta}$, where χ_D is the characteristic function of D, as $(\mathrm{id}\cdot f_\eta)(x) = x f_r^\eta(x) = r$ for every $x \in D_r^\eta$. In particular,

$$\mathrm{id}\cdot f_\eta = \sum_{r\in\mathbb{R}\setminus\{0\}} \mathrm{id}\cdot f_r^\eta = \sum_{r\in\mathbb{R}\setminus\{0\}} r\chi_{D_r^\eta}. \tag{2}$$

Also, there is an $\eta < \mathfrak{c}$ so that the number $c_\eta := \sum_{i=1}^n a_i s_i(\eta)$ is non-zero, since f is non-zero. Moreover, since sets $D_\eta := \bigcup_{r\in\mathbb{R}\setminus\{0\}} D_r^\eta$ and $B^0 := \bigcup_{\eta<\mathfrak{c}} B_0^\eta$ are Bernstein and disjoint, (1) and (2) imply that for every non-empty perfect $P \subset \mathbb{R}$ we have

$$\mathbb{R} = c_\eta\cdot\mathbb{R} = c_\eta\cdot\big((\mathrm{id}\cdot f_\eta)\restriction(D_\eta\cup B^0)\big)[P] = \big(f\restriction(D_\eta\cup B^0)\big)[P] \subset f[P],$$

proving that $f \in \mathrm{PES}$.

To finish the proof, it is enough to show that $f \notin \mathrm{Conn}$. To see this, notice that every c_η belongs to the finite set $C := \left\{\sum_{i=1}^n a_i t(i) : t \in \{0,1\}^{\{1,\ldots,n\}}\right\}$. Since B^0 is infinite, we can choose an $a \in B^0$ such that $a \notin \bar{\mathbb{Q}}(C\cup(\mathbb{R}\setminus B^0))$. We claim that

$$f(x) \neq ax \text{ for every } x \in \mathbb{R}\setminus\{0\}. \tag{3}$$

Indeed, $f(x) = ax \neq 0$ implies that $x = rb$ for some $\eta < \mathfrak{c}$, $r \neq 0$, and $b \in B_r^\eta$. But, by (1) and (2), this means that $arb = ax = f(x) = c_\eta(\mathrm{id}\cdot f_r^\eta)(x) = c_\eta\cdot r$ and so $a = \frac{c_\eta}{b} \in \bar{\mathbb{Q}}(C\cup(\mathbb{R}\setminus B^0))$, contradicting the choice of a.

Finally, since $f \in \mathrm{PES}$, its graph is dense, so there exist $q > p > 0$ such that $f(p) > ap$ and $f(q) < aq$. But this, together with (3) implies that the three-segment set $(\{p\}\times(-\infty,ap]) \cup \{(x,ax) : x \in [p,q]\} \cup (\{q\}\times[aq,\infty))$ separates the graph of f. □

As mentioned above, the class D of all *Darboux functions* (i.e, all functions $f : \mathbb{R}\to\mathbb{R}$ mapping mapping every interval into an interval) obviously contains PES. In particular, Theorem 2.1 immediately implies that

Corollary 2.2. *The class* $D\setminus\mathrm{Conn}$ *is* $2^\mathfrak{c}$*-lineable.*

The results presented in this paper constitute a starting point of an extensive study of the maximal lineabilities for all classes in the algebra of Darboux-like maps on $\mathbb{R}$. This study is expected to lead to several other papers and a Ph.D. dissertation of Mr. Gbrel Albkwre. For the study of the additivity coefficients for the same classes of functions see [7].

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