MAXIMAL LINEABILITY OF THE CLASSES OF FUNCTIONS IN THE ALGEBRA OF DARBOUX-LIKE MAPS THAT ARE DARBOUX BUT NOT CONNECTIVITY

GBREL M. ALBKWRE AND KRZYSZTOF CHRIS CIESIELSKI

Abstract. Consider the set $\mathbb{R}^\mathbb{R}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ as a vector space over $\mathbb{R}$. A family $\mathcal{G} \subset \mathbb{R}^\mathbb{R}$ is *κ-lineable* provided there exists a κ-dimensional linear subspace of $\mathbb{R}^\mathbb{R}$ contained in $\mathcal{G} \cup \{0\}$. The goal of this work is to show $2^\mathfrak{c}$-lineability of all non-empty classes of functions in the algebra $\mathcal{A}(\mathbb{D})$ of Darboux-like maps that are contained in the family $\mathcal{D} \setminus \mathrm{Conn}$ of Darboux functions that have disconnected graphs. The arguments for these classes stand apart from the proofs of $2^\mathfrak{c}$-lineability of other classes in the algebra $\mathcal{A}(\mathbb{D})$: the proofs we present below are considerably more delicate and heavily rely on the existence of algebraically independent subsets of $\mathbb{R}$ having different structures. The presented results generalize recently published proof of $2^\mathfrak{c}$-lineability of the class $\mathcal{D} \setminus \mathrm{Conn}$.

1. Introduction

The theory of lineability of different subfamilies of vector spaces has become a noticeable trend in mathematical research of early 21st century, as can be seen in a 2014 survey [7], a 2016 monograph [6], and the literature cited therein. A more recent work in this direction includes [5, 8, 19]. The roots of this subject can be traced to a 1966 paper [23] of Vladimir Gurariĭ where it is shown that the set of continuous nowhere differentiable functions on $[0, 1]$, together with the constant 0 function, contains an infinite dimensional vector space, that is, it is ω-lineable. Of course, for the families $\mathcal{G} \subset \mathbb{R}^\mathbb{R}$ the best possible κ-lineability is for $\kappa = 2^\mathfrak{c}$, the dimension of $\mathbb{R}^\mathbb{R}$.

The term Darboux-like functions refers to several classes of generalized continuous maps and their study goes back at least to 1875 paper [20] of Jean-Gaston Darboux who first systematically investigated functions from $\mathbb{R}$ to $\mathbb{R}$ that have the intermediate value property, that is, that map every connected subset of $\mathbb{R}$ (i.e., an interval) onto a connected set. Nowadays we refer to such maps as *Darboux functions* and the class of all such functions is denoted by a symbol $\mathcal{D}$. Another family of functions that are among Darboux-like is that of *connectivity functions* Conn, that is, maps whose graphs are connected subsets of $\mathbb{R}^2$. This notion can be traced to a 1956 problem [28] stated by John Forbes Nash. (See also 1950's papers [26, 30].) It is easy to see that $\mathrm{Conn} \subset \mathcal{D}$.

The other three classes of Darboux-like maps that we will investigate here are defined as follows.

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PR of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with perfect road*, that is, such that for every $x \in \mathbb{R}$ there exists a perfect $P \subset \mathbb{R}$ having x as a bilateral limit point (i.e., with x being a limit point of $(-\infty, x) \cap P$ and of $(x, \infty) \cap P$) such that $f \restriction P$ is continuous at x. This class was introduced in a 1936 paper [27] of Isaie Maximoff, where he proved that $\mathcal{D} \cap \mathcal{B}_1 = \mathrm{PR} \cap \mathcal{B}_1$, where $\mathcal{B}_1$ is the class of Baire class 1 functions.

CIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Cantor Intermediate Value Property*, that is, such that for all $p, q \in \mathbb{R}$ with $f(p) < f(q)$ and for every perfect set $K \subset (f(p), f(q))$ there exists a perfect set P between p and q such that $f[P] \subset K$. This class was first introduced in a 1982 paper [22] of Richard G. Gibson and Fred William Roush.

SCIVP of all functions $f \in \mathbb{R}^{\mathbb{R}}$ *with Strong Cantor Intermediate Value Property*, that is, such that for all $p, q \in \mathbb{R}$ with $f(p) < f(q)$ and for every perfect $K \subset (f(p), f(q))$ there exists a perfect set P between p and q such that $f[P] \subset K$ and $f \restriction P$ is continuous. This notion was introduced in a 1992 paper [29] of Harvey Rosen, R. Gibson, and F. Roush.

Clearly SCIVP $\subset$ CIVP $\subset$ PR.

Beside these five classes, usually the following three classes of functions, not studied in this article, are listed as Darboux-like: PC of peripherally continuous functions, AC of almost continuous functions (in the sense of Stallings), and Ext of extendable functions. All eight of these classes, that is, the elements of $\mathbb{D} = \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathcal{D}, \mathrm{PC}, \mathrm{SCIVP}, \mathrm{CIVP}, \mathrm{PR}\}$, were extensively studied in the late 20th century, see surveys [9, 14, 21] and the literature cited therein. In particular, it is known that $\mathrm{Ext} \subsetneq \mathrm{AC} \subsetneq \mathrm{Conn} \subsetneq \mathcal{D} \subsetneq \mathrm{PC}$, $\mathrm{Ext} \subsetneq \mathrm{SCIVP} \subsetneq \mathrm{CIVP} \subsetneq \mathrm{PR} \subsetneq \mathrm{PC}$, and that these are all inclusions among these classes. In particular, the algebra $\mathcal{A}(\mathbb{D})$ of subsets of $\mathrm{PC} \subset \mathbb{R}^{\mathbb{R}}$ generated by these eight classes has 17 non-empty minimal subclasses (atoms), four of which are the subject of our study: the intersection of $\mathcal{D} \setminus \mathrm{Conn}$ with the classes $\mathbb{R}^{\mathbb{R}} \setminus \mathrm{PR}$, $\mathrm{PR} \setminus \mathrm{CIVP}$, $\mathrm{CIVP} \setminus \mathrm{SCIVP}$, and SCIVP. More precisely, using notation $\neg\mathcal{G} := \mathbb{R}^{\mathbb{R}} \setminus \mathcal{G}$ for $\mathcal{G} \in \mathbb{D}$, the four classes in $\mathcal{A}(\mathbb{D})$ we are interested in can be written as

$$\begin{array}{lllll} \mathcal{D} \cap & \neg\,\mathrm{Conn} \cap \neg\,\mathrm{PR} & \qquad \mathcal{D} \cap \mathrm{SCIVP} \cap \neg\,\mathrm{Conn} \\ \mathcal{D} \cap \mathrm{PR} \cap \neg\,\mathrm{Conn} \cap \neg\,\mathrm{CIVP} & & \qquad \mathcal{D} \cap \mathrm{CIVP} \cap \neg\,\mathrm{Conn} \cap \neg\,\mathrm{SCIVP} \end{array} \tag{1}$$

The systematic study of the properties of the atoms in $\mathcal{A}(\mathbb{D})$ was initiated in the recent paper [16], though its traces can be found earlier, see e.g. [18] or [17]. The lineabilities of the classes in $\mathbb{D}$ have been previously studied—they are all maximally- (i.e., $2^{\mathfrak{c}}$-) lineable, see [7] and [13]. The systematic study of the lineabilities of the atoms of $\mathcal{A}(\mathbb{D})$ is the subject of a Ph.D. dissertation of the first author, written under the supervision of the second author, and this paper constitutes the final part of this project. (Though, some unanswered questions remain.) The lineabilities of the other atoms of $\mathcal{A}(\mathbb{D})$ are discussed in the papers [1, 2, 4]. (Compare also [3].) The summary of all these results can be found in [1].

2. Preliminaries

Our terminology is standard and follows [10]. In particular, the symbols $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of all rational and all real numbers, respectively. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. The cardinality of a set X is denoted by $|X|$ and the symbol

$\mathfrak{c}$ denotes $|\mathbb{R}|$. For an $A \subset \mathbb{R}$ we use symbol χ_A to denote the characteristic function of A.

For an $f \in \mathbb{R}^{\mathbb{R}}$ its *support* is defined as

$$\operatorname{supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Note that we do not take the closure of the set above. Also, for $G \subset \mathbb{R}$ and a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we define

$$\operatorname{supp}(\mathcal{F}) := \bigcup_{f \in \mathcal{F}} \operatorname{supp}(f) \quad \text{ and } \quad \mathcal{F} \restriction G := \{f \cdot \chi_G : f \in \mathcal{F}\}.$$

For an $A \subset \mathbb{R}$ the symbol $\mathbb{Q}(A)$ denotes the subfield of $\mathbb{R}$ generated by A, that is, $\mathbb{Q}(A)$ is the intersection of all subfields of $\mathbb{R}$ that contain A. By $\bar{\mathbb{Q}}(A)$ we denote the algebraic closure of $\mathbb{Q}(A)$ in $\mathbb{R}$, that is, $\bar{\mathbb{Q}}(A)$ is the set of $x \in \mathbb{R}$ that are algebraic over $\mathbb{Q}(A)$. We say that an $S \subset \mathbb{R}$ is: *algebraically independent* when it is algebraically independent over $\mathbb{Q}$; it is a *transcendental basis* provided it is a maximal algebraically independent subset of $\mathbb{R}$. For every algebraically independent set $S \subset \mathbb{R}$ there exists a transcendental basis T with $S \subset T$, see e.g. [24]. If T is transcendental basis, then every $x \in \mathbb{R}$ is algebraic over $\mathbb{Q}(T)$, that is $\bar{\mathbb{Q}}(T) = \mathbb{R}$.

For an $A \subset \mathbb{R}$, a *transcendence degree* of $\mathbb{R}$ over $\bar{\mathbb{Q}}(A)$ is the cardinality of any transcendental basis of $\mathbb{R}$ over $\bar{\mathbb{Q}}(A)$. Notice that if $S \subset \mathbb{R}$ is algebraically independent such that $\bar{\mathbb{Q}}(S) = \bar{\mathbb{Q}}(A)$ and $T \supset S$ is a transcendental basis, then the transcendence degree of $\mathbb{R}$ over $\bar{\mathbb{Q}}(A)$ equals the cardinality of the set $T \setminus S$.

For a Polish space X (we will use this notion only for only $X = \mathbb{R}$ and $X = 2^{\omega}$) we say that $B \subset X$ is a *Bernstein set (in X)* provided both B and $X \setminus B$ intersect every perfect subset of X.

2.1. Canonical linear space $W_{\mathcal{F}}$.

For a family $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ of functions having pairwise disjoint supports let

$$V_{\mathcal{F}} := \left\{\textstyle\sum_{f \in \mathcal{F}} s(f) \cdot f : s \in \{0,1\}^{\mathcal{F}}\right\} = \left\{\textstyle\sum_{g \in \mathcal{G}} g : \mathcal{G} \subset \mathcal{F}\right\}$$

and notice that $V_{\mathcal{F}} \subset \mathbb{R}^{\mathbb{R}}$ by the disjoint support assumption. The space $W_{\mathcal{F}}$ is defined as the linear subspace of $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$ spanned by $V_{\mathcal{F}}$. The space $W_{\mathcal{F}}$ was first used in [11]; however, it is closely related to a bigger natural vector space $\mathcal{L}_{\mathcal{F}} := \left\{\sum_{f \in \mathcal{F}} s(f) \cdot f : s \in \mathbb{R}^{\mathcal{F}}\right\}$ considered in [2] as well as in the earlier papers cited therein. All linear spaces we use in this paper to justify $2^{\mathfrak{c}}$-lineability are in the form $W_{\mathcal{F}}$.

Notice that if $\mathcal{F}$ is infinite, then $W_{\mathcal{F}}$ has dimension $2^{|\mathcal{F}|}$. This is obvious when $2^{|\mathcal{F}|} > |\mathbb{R}| = \mathfrak{c}$, the only case we are interested in this paper, while a simple argument for this in the case when $2^{|\mathcal{F}|} = \mathfrak{c}$ can be found in [2]. In particular, the following remark is obvious.

Remark 2.1. *If $|\mathcal{F}| = \mathfrak{c}$, then $W_{\mathcal{F}}$ has dimension $2^{\mathfrak{c}}$.*

Notice also the following simple fact.

Remark 2.2. *Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ be a family of functions having pairwise disjoint supports. If $g \in W_{\mathcal{F}}$, then there is a finite set $A_g \subset \mathbb{R}$ such that*

- *for every $f \in \mathcal{F}$ there exists an $a_f \in \mathbb{Q}(A_g)$ so that $g = a_f \cdot f$ on* $\operatorname{supp}(f)$.

Proof. Let $g = \sum_{i<n} a_i\varphi_i \in W_{\mathcal{F}}$ with $\varphi_i = \sum_{f\in\mathcal{F}} s^i(f)f \in V_{\mathcal{F}}$ for every $i < n$. Then, $A_g := \{a_i : i < n\}$ is as needed. Indeed, for every $f \in \mathcal{F}$ and $x \in \operatorname{supp}(f)$ we have

$$g(x) = \sum_{i<n} a_i\varphi_i(x) = \sum_{i<n} a_i s^i(f)f(x) = a_f f(x),$$

where $a_f := \sum_{i<n} a_i s^i(f) \in \mathbb{Q}(A_g)$. □

2.2. **Subclases** ES **and** PES **of** $\mathcal{D}$**.** Consider also the following classes of Darboux functions studied in the lineability context[1] (see e.g. [6, 12, 13]):

- ES of all *everywhere surjective* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that $f[(a,b)] = \mathbb{R}$ for all $a < b$ or, equivalently, such that $f^{-1}(r)$ is dense in $\mathbb{R}$ for every $r \in \mathbb{R}$;
- PES of all *perfectly everywhere surjective* functions $f \in \mathbb{R}^{\mathbb{R}}$, that is, such that for every perfect $P \subset \mathbb{R}$ we have $f[P] = \mathbb{R}$ or, equivalently, $f^{-1}(r) \cap P \neq \emptyset$ for every $r \in \mathbb{R}$.

Clearly $\mathrm{PES} \subsetneq \mathrm{ES} \subsetneq \mathcal{D}$. It is also easy to see that $\mathrm{PES} \subset \neg\,\mathrm{PR}$.

All non-zero functions in $\mathbb{R}^{\mathbb{R}}$ we will consider in this paper will have dense graphs. In particular, the following simple remark will be useful for us.

Remark 2.3. *If $f \in \mathbb{R}^{\mathbb{R}}$ has a dense graph in $\mathbb{R}^2$, then $f \in \mathcal{D}$ if, and only if, $f \in \mathrm{ES}$.*

2.3. **Families $\mathcal{H}$ for which $W_{\mathcal{H}}$ ensures lineability of the classes in $\mathbb{D}$.**

Proposition 2.4. *Let $\mathcal{F}, \mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$ be families of functions with pairwise disjoint supports such that all maps in $\mathcal{F}$ have graphs dense in $\mathbb{R}^2$. Assume that $B \subset \mathbb{R}$ is such that $\operatorname{supp}(\mathcal{F}) \subset B$ and that $\mathcal{F} \restriction B = \mathcal{H} \restriction B$.*

(i) *The graph of every non-zero $g \in W_{\mathcal{H}}$ is dense in $\mathbb{R}^2$.*
(ii) *If $\mathcal{F} \subset \mathrm{CIVP}$, then $W_{\mathcal{H}} \subset \mathrm{CIVP}$.*
(iii) *If $\mathcal{F} \subset \mathrm{SCIVP}$, then $W_{\mathcal{H}} \subset \mathrm{SCIVP}$.*
(iv) *If $\mathcal{F} \subset \mathcal{D}$ and $B \setminus \operatorname{supp}(\mathcal{F})$ is dense in $\mathbb{R}$, then $W_{\mathcal{H}} \subset \mathrm{ES} \cup \{0\}$.*
(v) *If $\mathcal{F} \subset \mathrm{PES}$ and $B \setminus \operatorname{supp}(\mathcal{F})$ intersects every perfect subset of $\mathbb{R}$, then $W_{\mathcal{H}} \subset \mathrm{PES} \cup \{0\}$.*

Proof. Choose a non-zero $g \in W_{\mathcal{H}}$. By Remark 2.2, there exists an $h \in \mathcal{H}$ and non-zero $c \in \mathbb{R}$ such that $g = c \cdot h$ on $\operatorname{supp}(h)$. By our assumption, there exists an $f \in \mathcal{F}$ such that $f \restriction B = h \restriction B$. In particular, $g = c \cdot f$ on $\operatorname{supp}(f)$.

To see (i) notice that $f \restriction \operatorname{supp}(f)$, as well as its multiplication by c, has graph dense in $\mathbb{R}^2$.

To see (ii) notice that the assumption that $f \in \mathrm{CIVP}$ implies that $c \cdot f \in \mathrm{CIVP}$ and in the definition of the class CIVP we can restrict our attention to perfect sets $K \subset \mathbb{R} \setminus \{0\}$ in which case the condition $c \cdot f[P] \subset K$ is achieved only for $P \subset \operatorname{supp}(f)$, so that $g[P] = c \cdot f[P] \subset K$, as needed. The argument for (iii) is essentially the same.

To see (iv) first notice that, by Remark 2.3, $\mathcal{F} \subset \mathrm{ES}$. In particular, for every $r \in \mathbb{R} \setminus \{0\}$ we have $g^{-1}(r) \supset (c \cdot f)^{-1}(r)$ and this last set is dense, since $c \cdot f \in \mathrm{ES}$. Finally, $g^{-1}(0)$ is dense, since it contains $B \setminus \operatorname{supp}(\mathcal{F})$. The argument for (v) is essentially the same. □

Notice that, in particular, all non-zero functions in the considered spaces $W_{\mathcal{F}}$ (which will have dense graphs) will be in ES.

[1] The class ES was studied earlier by the second author under the name *strongly Darboux maps*, see e.g. [10, sec 7.2]. Compare also [15].

2.4. **Families $\mathcal{H}$ for which $W_{\mathcal{H}}$ ensures lineability of $\neg\,$Conn.** Let $\mathrm{id}_\star \in \mathbb{R}^{\mathbb{R}}$ be defined as $\mathrm{id}_\star(x) = x$ for $x \neq 0$ and $\mathrm{id}_\star(0) = 1$. Notice that this ensures that the function $1/\mathrm{id}_\star$ is well defined at all points, including $x = 0$.

The following lemma is extracted from the proof of [11, theorem 2.1].

Lemma 2.5. *Let $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$ be a family of functions with pairwise disjoint supports and graphs dense in $\mathbb{R}^2$. If $(g/\mathrm{id}_\star)[\mathbb{R}] \neq \mathbb{R}$ for every $g \in W_{\mathcal{H}}$, then $W_{\mathcal{H}} \subset \neg\,\mathrm{Conn} \cup \{0\}$.*

Proof. Let $g \in W_{\mathcal{H}}$ be non-zero. Then, by Proposition 2.4(i), g has a dense graph. To see that $g \in \neg\,$Conn choose an $a \in \mathbb{R} \setminus (g/\mathrm{id}_\star)[\mathbb{R}]$. This means that $(g/\mathrm{id}_\star)(x) \neq a$ for every $x \in \mathbb{R}$. In particular, $g(x) \neq a\,\mathrm{id}_\star(x) = ax$ for every $x \neq 0$. Since g has a dense graph, there exist $q > p > 0$ such that $g(p) > ap$ and $g(q) < aq$. But this implies that the three-segment set $(\{p\} \times (-\infty, ap]) \cup \{(x, ax) : x \in [p, q]\} \cup (\{q\} \times [aq, \infty))$ separates the graph of g. So, indeed $g \in \neg\,$Conn. □

In Lemma 2.5 we assume that no function in $(1/\mathrm{id}_\star) \cdot W_{\mathcal{H}}$ is surjective. But how to ensure this together with the needed property that $W_{\mathcal{H}} \subset \mathrm{ES} \cup \{0\}$? To see this first notice that $(1/\mathrm{id}_\star) \cdot W_{\mathcal{H}} = W_{\mathcal{G}}$ where $\mathcal{G} = (1/\mathrm{id}_\star) \cdot \mathcal{H}$. To ensure that no function in $W_{\mathcal{G}}$ is surjective for such $\mathcal{G}$, we will use the following simple lemma extracted from [11].

Lemma 2.6. *Let $S_0 \subset \mathbb{R}$ be such that $\mathbb{R}$ has infinite transcendence degree over $\bar{\mathbb{Q}}(S_0)$. If $\mathcal{G} \subset \mathbb{Q}(S_0)^{\mathbb{R}}$ is a family of functions with pairwise disjoint supports, then no function in $W_{\mathcal{G}}$ is surjective.*

Proof. Let S be a transcendental basis of $\bar{\mathbb{Q}}(S_0)$ (over $\mathbb{Q}$) and T be a transcendental basis with $S \subset T$. Fix a $g \in W_{\mathcal{G}}$ and choose a finite $S_g \subset T$ such that $A_g \subset \mathbb{Q}(S_g)$, where A_g as in the Remark 2.2. Then, the range of g is contained in $\bar{\mathbb{Q}}(S_g \cup S)$, while $\bar{\mathbb{Q}}(S_g \cup S) \neq \bar{\mathbb{Q}}(T) = \mathbb{R}$ by our assumption that $\mathbb{R}$ has infinite transcendence degree over $\bar{\mathbb{Q}}(S_0) = \bar{\mathbb{Q}}(S)$. So, indeed g is not surjective. □

Notice that the conclusion of Lemma 2.6 does not hold if in its statement we replace space $W_{\mathcal{G}}$ with $\mathcal{L}_{\mathcal{G}}$. This is the main reason we work here with the spaces $W_{\mathcal{F}}$ rather than with $\mathcal{L}_{\mathcal{G}}$.

2.5. **Families $\mathcal{H}$ with $W_{\mathcal{H}}$ ensuring lineability of $\mathrm{ES} \cap \neg\,\mathrm{Conn}$.** $2^{\mathfrak{c}}$-lineability of ES is established via well known and perhaps the easiest construction presented in this paper. Specifically, for a family $\Delta := \{D^r_\xi : r \in \mathbb{R} \;\&\; \xi < \mathfrak{c}\}$ of pairwise disjoint dense subsets of $\mathbb{R}$ define the functions

(φ) $\varphi_\xi := \sum_{r \in \mathbb{R}} r \cdot \chi_{D^r_\xi}$

and let $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$.

Clearly functions in $\mathcal{F}(\varphi)$ are ES and have pairwise disjoint supports. So, $W_{\mathcal{F}(\varphi)}$ is well defined. To ensure that $W_{\mathcal{F}(\varphi)}$ is contained in $\neg\,\mathrm{Conn} \cup \{0\}$ we will consider the sets D^r_ξ of the form $\mathrm{id}_\star(r) \cdot S^r_\xi$ for some dense sets $S^r_\xi \subset \mathbb{R}$.

Proposition 2.7. *Let $\mathcal{S} = \{S^r_\xi \subset \mathbb{R} \setminus \{0\} : r \in \mathbb{R} \;\&\; \xi < \mathfrak{c}\}$ be a family of dense sets and put $S := \bigcup \mathcal{S}$. For $r \in \mathbb{R}$ and $\xi < \mathfrak{c}$ let $D^r_\xi := \mathrm{id}_\star(r) \cdot S^r_\xi$. Then every function φ_ξ is* ES *and $\varphi_\xi/\mathrm{id}_\star$ has range contained in $\mathbb{Q}(S)$.*

In particular, if $\mathbb{R}$ has infinite transcendence degree over $\bar{\mathbb{Q}}(S)$, the sets in

$$\Delta := \{\mathrm{id}_\star(r) \cdot S^r_\xi : r \in \mathbb{R} \;\&\; \xi < \mathfrak{c}\} \tag{2}$$

are pairwise disjoint, and $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$*, then the space* $W_{\mathcal{F}(\varphi)}$ *is well defined and* $W_{\mathcal{F}(\varphi)}$ *justifies* $2^{\mathfrak{c}}$*-lineability of* $\mathrm{ES} \setminus \mathrm{Conn}$.

Proof. Clearly $\varphi_\xi \in \mathrm{ES}$, since each set D^r_ξ is dense. To see $(\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(S)$ notice that

$$(\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] = \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \frac{r \cdot \chi_{D^r_\xi}}{\mathrm{id}_\star}[\mathbb{R}] = \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{r}{\mathrm{id}_\star(x)} : x \in D^r_\xi \right\}$$

$$= \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{r}{\mathrm{id}_\star(\mathrm{id}_\star(r)x)} : x \in S^r_\xi \right\} = \{0\} \cup \bigcup_{r \in \mathbb{R} \setminus \{0\}} \left\{ \frac{1}{x} : x \in S^r_\xi \right\}.$$

So, the first part is proved.

To see the second part, notice that the family $W_{\mathcal{F}(\varphi)}$ is well defined and that, by Remark 2.1, it has dimension $2^{\mathfrak{c}}$. The fact that $W_{\mathcal{F}(\varphi)} \subset \mathrm{ES} \cup \{0\}$ is justified by Proposition 2.4(iv) (used with $B = \mathbb{R}$ and $\mathcal{H} = \mathcal{F}(\varphi)$) while $W_{\mathcal{F}(\varphi)} \subset \neg\,\mathrm{Conn} \cup \{0\}$ from Lemmas 2.6 and 2.5. □

To successfully use Proposition 2.7 to show $2^{\mathfrak{c}}$-lineability of $\mathrm{ES} \setminus \mathrm{Conn}$ we still need to find the families $\mathcal{S}$ satisfying its assumptions. This is a relatively easy task if we ignore the requirement that the sets in the family Δ from (2) need to be pairwise disjoint. In fact, such a family with considerably stronger properties (including $\mathfrak{c}$-density of each set S^r_ξ) is constructed in Lemma 4.1. The refining of such family to one ensuring also pairwise disjointness of the sets in Δ can then be found using the following result.

Lemma 2.8. *Let* $\mathcal{S}$ *be a family of pairwise disjoint sets such that* $\bigcup \mathcal{S}$ *is algebraically independent and for every* $S \in \mathcal{S}$ *let* $r_S \in \mathbb{R} \setminus \{0\}$*. Then for every* $S \in \mathcal{S}$ *there exists a set* $N_S \subset \bigcup \mathcal{S}$ *of cardinality less than* $\mathfrak{c}$ *such that the sets in* $\Delta := \{r_S \cdot (S \setminus N_S) : S \in \mathcal{S}\}$ *are pairwise disjoint. Moreover, if* $r_S \in \mathbb{Q}$*, then* $N_S = \varnothing$.

Proof. The proof of this lemma is implicitly included in the proof of [11, theorem 2.1]. (See (a) and (b) in that proof.) Specifically, if $T = \{t_\xi : \xi < \mathfrak{c}\}$ is a transcendental basis extending $\bigcup \mathcal{S}$ and $\eta_S < \mathfrak{c}$ is the smallest such that $r_S \in \bar{\mathbb{Q}}(\{t_\xi : \xi < \eta_S\})$, then the sets $N_S := \bar{\mathbb{Q}}(\{t_\xi : \xi < \eta_S\}) \cap \bigcup \mathcal{S}$ are as needed. For more details see [11]. □

3. $2^{\mathfrak{c}}$-lineability of $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \neg\,\mathrm{PR}$ and of $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \mathrm{SCIVP}$

We start with examining the two classes from the top row of (1).

3.1. The class $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \neg\,\mathrm{PR}$.

The $2^{\mathfrak{c}}$-lineability of this class is an easy corollary of the main result from [11].

Theorem 3.1. *There exists a family* $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ *of* $\mathfrak{c}$*-many maps with pairwise disjoint supports such that* $W_{\mathcal{F}} \subset (\mathrm{PES} \cap \neg\,\mathrm{Conn}) \cup \{0\}$*. In particular,* $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \neg\,\mathrm{PR}$ *is* $2^{\mathfrak{c}}$*-lineabile.*

Proof. The existence of an $\mathcal{F}$ satisfying the first part of the theorem was proved in [11, theorem 2.1]. (See also Subsection 3.3.) To finish the proof just recall that $\mathrm{PES} \subset \neg\,\mathrm{PR}$. □

3.2. **The class** $\mathrm{ES}\cap\neg\,\mathrm{Conn}\cap\mathrm{SCIVP}$**.** In the proof of $2^{\mathfrak{c}}$-lineability of this class we will use the following well known fact, where $\mathcal{B}=\{(p,q): p<q\ \&\ p,q\in\mathbb{Q}\}$ is the standard countable basis for $\mathbb{R}$.

Lemma 3.2. *There exists a family* $\mathcal{M}_0=\{P^I\subset I: I\in\mathcal{B}\}$ *of pairwise disjoint perfect sets such that* $\bigcup\mathcal{M}_0$ *is algebraically independent and the transcendence degree of* $\mathbb{R}$ *over* $\bar{\mathbb{Q}}(\bigcup\mathcal{M}_0)$ *is* $\mathfrak{c}$.

Proof. Let $K\subset\mathbb{R}$ be an algebraically independent compact perfect set, see [31] or [25]. Choose a family $\{K^I\subset K: I\in\mathcal{B}\}$ of pairwise disjoint perfect sets such that the set $K\setminus\bigcup_{I\in\mathcal{B}}K^I$ has cardinality $\mathfrak{c}$. For every $I\in\mathcal{B}$ choose non-zero $p_I,q_I\in\mathbb{Q}$ such that the set $P^I:=p_I+q_IK^I$ is contained in I. Then $\mathcal{M}_0:=\{P^I\subset I: I\in\mathcal{B}\}$ is as needed. $\square$

Next, let $\mathcal{M}_0=\{P^I\subset I: I\in\mathcal{B}\}$ be as in Lemma 3.2. For every $I\in\mathcal{B}$ let $\{P^{I,r}_\xi\subset P^I: \xi<\mathfrak{c}\ \&\ r\in\mathbb{R}\}$ be a family of pairwise disjoint perfect sets and put $\mathcal{S}:=\bigcup_{I\in\mathcal{B}}\{P^{I,r}_\xi\subset P^I: \xi<\mathfrak{c}\ \&\ r\in\mathbb{R}\}$. Applying Lemma 2.8 to $\mathcal{S}$ and numbers $r_S=\mathrm{id}_\star(r)$ for $S=P^{I,r}_\xi$ we can decrease each $P^{I,r}_\xi$, if necessary, to a perfect set such that the sets in the family $\{\mathrm{id}_\star(r)\cdot P^{I,r}_\xi: r\in\mathbb{R}\ \&\ \xi<\mathfrak{c}\ \&\ I\in\mathcal{B}\}$ are pairwise disjoint. For every $\xi<\mathfrak{c}$ and $r\in\mathbb{R}$ put $S^r_\xi:=\bigcup_{I\in\mathcal{B}}P^{I,r}_\xi$ and $D^r_\xi:=\mathrm{id}_\star(r)\cdot S^r_\xi$.

Let $\mathcal{F}(\varphi):=\{\varphi_\xi: \xi<\mathfrak{c}\}$, where the functions φ_ξ are defined as in (φ), that is, $\varphi_\xi:=\sum_{r\in\mathbb{R}}r\cdot\chi_{D^r_\xi}$ for the sets D^r_ξ as above.

Theorem 3.3. *If* $\mathcal{F}(\varphi)$ *is defined as above, then* $W_{\mathcal{F}(\varphi)}$ *justifies* $2^{\mathfrak{c}}$*-lineability of* $\mathrm{ES}\cap\neg\,\mathrm{Conn}\cap\mathrm{SCIVP}$.

Proof. $W_{\mathcal{F}(\varphi)}$ has dimension $2^{\mathfrak{c}}$ by Remark 2.1. Also, each $\varphi_\xi\in\mathcal{F}(\varphi)$ is in SCIVP. Indeed, every set S^r_ξ contains a perfect subset of every $I\in\mathcal{B}$ (namely $P^{I,r}_\xi$), so the same is true for each set $D^r_\xi=\mathrm{id}_\star(r)\cdot S^r_\xi$. In particular, for every $I\in\mathcal{B}$ and perfect $K\subset\mathbb{R}$ there exists a perfect $P\subset I$ such that $\varphi_\xi\restriction P$ is constant (so continuous) with value $r\in K$, proving that indeed $\varphi_\xi\in\mathrm{SCIVP}$. Therefore, $\mathcal{F}(\varphi)\subset\mathrm{SCIVP}$, so by Proposition 2.4(iii) used with $B:=\mathbb{R}$ and $\mathcal{H}=\mathcal{F}(\varphi)$, $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$-lineability of SCIVP.

Finally, notice that $\mathcal{S}=\{S^r_\xi\subset\mathbb{R}: r\in\mathbb{R}\ \&\ \xi<\mathfrak{c}\}$ and $\Delta:=\{\mathrm{id}_\star(r)\cdot S^r_\xi: r\in\mathbb{R}\ \&\ \xi<\mathfrak{c}\}$ satisfy the assumptions of Proposition 2.7. Hence $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$-lineability of $\mathrm{ES}\setminus\mathrm{Conn}$. $\square$

The consistency of the $2^{\mathfrak{c}}$-lineability of $\mathrm{ES}\cap\neg\,\mathrm{Conn}\cap\mathrm{SCIVP}$ can be also deduced from the theorem proved in [4] that this class is $\mathfrak{c}^+$-lineabile if $\mathfrak{c}$ is a regular cardinal. Thus, Theorem 3.3 can be viewed as a generalization of the result from [4].

3.3. **Proof of** $2^{\mathfrak{c}}$**-lineability of** $\mathrm{PES}\cap\neg\,\mathrm{Conn}$**.** It is worth to notice that the developed machinery, which we used to prove the previous theorem, gives also a direct proof of Theorem 3.1.

To see this, let T be an algebraically independent set intersecting every perfect set (i.e., T is a Bernstein set).[2] Let $\{S^r_\xi\subset T: \xi<\mathfrak{c}\ \&\ r\in\mathbb{R}\}$ be a family of pairwise disjoint Bernstein sets. Applying Lemma 2.8 we can ensure that the sets in the

[2]If $\{P_\xi: \xi<\mathfrak{c}\}$ is a list of all perfect subsets of $\mathbb{R}$ and $t_\xi\in P_\xi\setminus\bar{\mathbb{Q}}(\{t_\zeta: \zeta<\xi\})$ for every $\xi<\mathfrak{c}$, then $T:=\{t_\xi: \xi<\mathfrak{c}\}$ is as needed: it cannot contain any perfect $P\subset\mathbb{R}$, since then, for any $p\in P$, T would be disjoint from perfect $p+P$.

family $\Delta := \{\mathrm{id}_\star(r)\cdot S^r_\xi : r\in\mathbb{R}\ \&\ \xi<\mathfrak{c}\}$ are pairwise disjoint. Let $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi<\mathfrak{c}\}$, where functions φ_ξ are defined as in (φ), that is, $\varphi_\xi := \sum_{r\in\mathbb{R}} r\cdot\chi_{D^r_\xi}$ for the sets D^r_ξ as above. Then the family $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$-lineability of $\mathrm{PES}\cap\neg\,\mathrm{Conn}$.

Indeed the argument used in the proof of Theorem 3.3 immediately implies that $W_{\mathcal{F}(\varphi)}$ justifies $2^{\mathfrak{c}}$-lineability of $\neg\,\mathrm{Conn}$. Also, the fact that all sets S^r_ξ and $D^r_\xi = \mathrm{id}_\star(r)\cdot S^r_\xi$ are Bernstein, ensures that $\mathcal{F}(\varphi)\subset\mathrm{PES}$. So, by Proposition 2.4(v) used with $B=\mathbb{R}$ and $\mathcal{H}=\mathcal{F}$, $W_{\mathcal{F}(\varphi)}\subset\mathrm{PES}\cup\{0\}$, as needed.

4. Lineability of $\mathrm{ES}\cap\neg\,\mathrm{Conn}\cap\mathrm{PR}\cap\neg\,\mathrm{CIVP}$ and $\mathrm{ES}\cap\neg\,\mathrm{Conn}\cap\mathrm{CIVP}\cap\neg\,\mathrm{SCIVP}$

The following lemma will be used to establish $2^{\mathfrak{c}}$-lineability of both of these classes.

Lemma 4.1. *Let $\mathcal{M}_0=\{P^I\subset I : I\in\mathcal{B}\}$ be a family of pairwise disjoint perfect sets as in Lemma 3.2, that is, such that $\bigcup\mathcal{M}_0$ is algebraically independent and the transcendence degree of $\mathbb{R}$ over $\bar{\mathbb{Q}}(\bigcup\mathcal{M}_0)$ is $\mathfrak{c}$. Then there exist a family $\mathcal{S}_0=\{S^r\subset\mathbb{R}\setminus\bigcup\mathcal{M}_0 : r\in\mathbb{R}\}$ of pairwise disjoint $\mathfrak{c}$-dense sets and a set $Z\subset\mathbb{R}\setminus\mathbb{Q}(\bigcup\mathcal{S}_0)$ such that*

- (i) *$\bigcup(\mathcal{M}_0\cup\mathcal{S}_0)$ is algebraically independent;*
- (ii) *$\bigcup\{\mathrm{id}_\star(r)\cdot S^r : r\in\mathbb{R}\}$ contains no perfect set;*
- (iii) *Z intersects every perfect set and $Z\cup\bigcup\mathcal{S}_0$ is algebraically independent.*

Proof. Let $\langle\langle r_\xi, J_\xi\rangle : \xi<\mathfrak{c}\rangle$ be an enumeration of $\mathbb{R}\times\mathcal{B}$ with $\mathfrak{c}$-many repetitions and let $\{P_\xi : \xi<\mathfrak{c}\}$ be an enumeration of all perfect subsets of $\mathbb{R}$. By induction on $\xi<\mathfrak{c}$ choose a sequence $\langle\langle x_\xi, y_\xi, z_\xi\rangle : \xi<\mathfrak{c}\rangle$ so that

- (A_ξ) $x_\xi\in J_\xi\setminus\bar{\mathbb{Q}}(\bigcup\mathcal{M}_0\cup\{x_\zeta:\zeta<\xi\}\cup\{y_\zeta/\mathrm{id}_\star(r_\xi):\zeta<\xi\}\cup\{z_\zeta:\zeta<\xi\})$;
- (C_ξ) $y_\xi\in P_\xi\setminus\{\mathrm{id}_\star(r_\zeta)\cdot x_\zeta:\zeta\le\xi\}$;
- (Z_ξ) $z_\xi\in P_\xi\setminus\bar{\mathbb{Q}}(\{x_\zeta:\zeta\le\xi\}\cup\{z_\zeta:\zeta<\xi\})$.

Such x_ξ can be chosen, as otherwise the transcendence degree of $\mathbb{R}$ over $\bar{\mathbb{Q}}(\bigcup\mathcal{M}_0)$ would be less than $\mathfrak{c}$.

For each $r\in\mathbb{R}$ define $S^r := \{x_\xi : \xi<\mathfrak{c}\ \&\ r_\xi=r\}$ and let $Z=\{z_\xi:\xi<\mathfrak{c}\}$. We claim that these definitions ensure that $\mathcal{S}_0=\{S^r : r\in\mathbb{R}\}$ and Z are as needed.

Indeed, clearly the sets in $\mathcal{S}_0$ are pairwise disjoint and $\mathfrak{c}$-dense, since the sequence $\langle x_\xi:\xi<\mathfrak{c}\rangle$ is one-to-one and each pair $\langle r,J\rangle\in\mathbb{R}\times\mathcal{B}$ appears in the sequence $\langle\langle r_\xi,J_\xi\rangle:\xi<\mathfrak{c}\rangle$ $\mathfrak{c}$-many times. Also $\bigcup\mathcal{S}_0=\{x_\xi:\xi<\mathfrak{c}\}$ is contained in $\mathbb{R}\setminus\bar{\mathbb{Q}}(\bigcup\mathcal{M}_0)$. So, (A_ξ) ensures (i).

To show (ii), for a perfect set $P\subset\mathbb{R}$ choose a $\xi<\mathfrak{c}$ such that $P_\xi=P$ and notice that $y_\xi\in P_\xi=P$ does not belong to $\bigcup\{\mathrm{id}_\star(r)\cdot S^r : r\in\mathbb{R}\}=\{\mathrm{id}_\star(r_\zeta)\cdot x_\zeta:\zeta<\mathfrak{c}\}$: $y_\xi\notin\{\mathrm{id}_\star(r_\zeta)\cdot x_\zeta:\zeta\le\xi\}$ is ensured by (C_ξ), while $y_\xi\notin\{\mathrm{id}_\star(r_\zeta)\cdot x_\zeta:\xi<\zeta\}$ by the conditions (A_ζ) with $\zeta>\xi$.

Finally, the first part of (iii)—the fact that Z intersects every perfect set—is ensured by (Z_ξ), while its second part—an algebraic independence of the set $Z\cup\bigcup\mathcal{S}_0=\{z_\xi:\xi<\mathfrak{c}\}\cup\{x_\xi:\xi<\mathfrak{c}\}$—by the choice as in (A_ξ) and (Z_ξ). □

In what follows $\mathcal{M}_0=\{P^I\subset I : I\in\mathcal{B}\}$, $\mathcal{S}_0=\{S^r\subset\mathbb{R}\setminus\bigcup\mathcal{M}_0 : r\in\mathbb{R}\}$, and Z are always as in Lemma 4.1. Removing from each set S^r one number, if necessary, we can assume that

- (A) the transcendence degree of $\mathbb{R}$ over both $\bar{\mathbb{Q}}(\bigcup(\mathcal{M}_0\cup\mathcal{S}_0))$ and $\bar{\mathbb{Q}}(Z\cup\bigcup\mathcal{S}_0)$ is $\mathfrak{c}$.

Notice that

(B) $M := \bigcup \mathcal{M}_0$ is a meager F_σ-set and that $\bigcup \mathcal{S}_0$ is contained in $M^c := \mathbb{R} \setminus M$.

For every $r \in \mathbb{R}$ let $\{S^r_\xi \subset S^r : \xi < \mathfrak{c}\}$ be a family of pairwise disjoint $\mathfrak{c}$-dense sets. Since $\bigcup(\mathcal{M}_0 \cup \mathcal{S}_0)$ is algebraically independent, we can apply Lemma 2.8 to the family $\mathcal{S} := \{S^r_\xi : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\}$ and numbers $r_S = \mathrm{id}_\star(r)$ for $S = S^r_\xi$ and $r_M = 1$ to slightly decrease sets S^r_ξ, if necessary, to ensure that the sets in the family $\Delta := \{\mathrm{id}_\star(r) \cdot S^r_\xi : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\}$ are pairwise disjoint. In summary

(C) the sets in $\{S^r_\xi \subset \bigcup \mathcal{S}_0 : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ are $\mathfrak{c}$-dense, pairwise disjoint, and the sets in $\Delta := \{\mathrm{id}_\star(r) \cdot S^r_\xi : r \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\} \cup \{M\}$ are pairwise disjoint.

(D) Let $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$, where functions $\varphi_\xi := \sum_{r \in \mathbb{R}} r \cdot \chi_{D^r_\xi}$ are as in (φ) with $D^r_\xi := \mathrm{id}_\star(r) \cdot S^r_\xi$ for the sets S^r_ξ as in (C).

The following fact will be used in the proofs of our two remaining theorems.

Proposition 4.2. *If $\mathcal{F}(\varphi)$ is as in (D) and functions in $\mathcal{H} \subset \mathbb{R}^\mathbb{R}$ having pairwise disjoint supports are such that $\mathcal{H} \restriction M^c = \mathcal{F}(\varphi) \restriction M^c$, then $W_\mathcal{H} \subset \mathrm{ES} \cup \{0\}$.*

Proof. Our definition of $\mathcal{F}(\varphi)$ ensures that $\mathcal{F}(\varphi) \subset \mathrm{ES}$. Also, $\mathrm{supp}(\mathcal{F}(\varphi)) \subset M^c$ and $M^c \setminus \mathrm{supp}(\mathcal{F}(\varphi))$ is dense in $\mathbb{R}$, since it contains dense sets S^0_ξ. So, Proposition 2.4(iv) implies that $W_\mathcal{H} \subset \mathrm{ES} \cup \{0\}$. □

4.1. **$2^\mathfrak{c}$-lineability of $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \mathrm{PR} \cap \neg\,\mathrm{CIVP}$.** For every $I \in \mathcal{B}$ choose a family $\{P^{I,q}_\xi \subset P^I : \xi < \mathfrak{c} \ \& \ q \in \mathbb{Q}\}$ of pairwise disjoint perfect sets and for every $\xi < \mathfrak{c}$ define

$$\gamma_\xi := \sum_{\langle I,q\rangle \in \mathcal{B} \times \mathbb{Q}} q\, \chi_{P^{I,q}_\xi} \quad \text{and} \quad h_\xi = \varphi_\xi + \gamma_\xi, \tag{3}$$

where maps φ_ξ are from the family $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$ from Proposition 4.2. Clearly the supports of maps in the family $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint as functions in $\mathcal{F}(\varphi)$ have disjoint supports contained in M^c, while the maps in $\{\gamma_\xi : \xi < \mathfrak{c}\}$ have disjoint supports contained in M.

Theorem 4.3. *If $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ for the functions h_ξ from (3), then $W_\mathcal{H}$ justifies $2^\mathfrak{c}$-lineability of $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \mathrm{PR} \cap \neg\,\mathrm{CIVP}$.*

Proof. $W_\mathcal{H}$ has dimension $2^\mathfrak{c}$ by Remark 2.1. The inclusion $W_\mathcal{H} \subset \mathrm{ES} \cup \{0\}$ is ensured by Proposition 4.2, since $\mathcal{H} \restriction M^c = \mathcal{F}(\varphi) \restriction M^c$.

To see that $W_\mathcal{H} \subset \neg\,\mathrm{Conn} \cup \{0\}$ notice that for every $\xi < \mathfrak{c}$

$$(\gamma_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \left\{\frac{q}{r} : q \in \mathbb{Q} \ \& \ r \in \bigcup \mathcal{M}_0\right\} \subset \mathbb{Q}(\bigcup \mathcal{M}_0)$$

and, by (D) and Proposition 2.7, $(\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(\bigcup \mathcal{S}_0)$. Therefore, we have $(h_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset (\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] \cup (\gamma_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(\bigcup(\mathcal{M}_0 \cup \mathcal{S}_0))$. Thus, by (A) and Lemmas 2.6 and 2.5, indeed $W_\mathcal{H} \subset \neg\,\mathrm{Conn} \cup \{0\}$.

Next notice that every non-zero $g \in W_\mathcal{H}$ is in PR. This argument is a variation of one used in [1, fact 2.9]. Indeed, by Remark 2.2, there exists a $\xi < \mathfrak{c}$ and non-zero $c \in \mathbb{R}$ such that $g = c \cdot h_\xi$ on $\mathrm{supp}(h_\xi)$. In particular, for every $x \in \mathbb{R}$ choose a sequence $\langle q_n \in \mathbb{Q} : n \in \mathbb{N}\rangle$ converging to $g(x)/c$ and disjoint intervals $I_n = (a_n, b_n) \in \mathcal{B}$ so that $a_{2n} \nearrow_n x$ and $a_{2n+1} \searrow_n x$. Then $P := \{x\} \cup \bigcup_{n \in \mathbb{N}} P^{I_n, q_n}_\xi$ is perfect having x as a bilateral limit point and $g \restriction P$ is continuous at x, since

$$\lim_{p \to x, p \in P} g(p) = \lim_{p \to x, p \in P} c \cdot h_\xi(p) = \lim_{n \to \infty} c \cdot q_n = g(x).$$

So, indeed $g \in \mathrm{PR}$.

To finish the proof, it is enough to show that $W_{\mathcal{H}} \subset \neg\,\mathrm{CIVP} \cup \{0\}$. So, choose a non-zero $g \in W_{\mathcal{H}}$. By Remark 2.2, there exists a finite set $A \subset \mathbb{R}$ such that

$$g \subset \bigcup_{a \in \mathbb{Q}(A)} \bigcup_{\xi < \mathfrak{c}} a \cdot h_\xi. \tag{4}$$

Choose a perfect $K \subset \mathbb{R} \setminus \mathbb{Q}(A)$. Since the graph of g is dense it is enough to show that $g[P] \not\subset K$ for every perfect $P \subset \mathbb{R}$. By way of contradiction, assume that $g[P] \subset K$ for some perfect $P \subset \mathbb{R}$. Since M is Borel, we can assume that either $P \subset M$ or $P \subset M^c$.

However, if $P \subset M$, then $h_\xi[P] = \gamma_\xi[P] \subset \mathbb{Q}$. So, by (4) and the fact that our definition ensures $\gamma_\xi[\mathbb{R}] \subset \mathbb{Q}$, we have $g[P] \subset \bigcup_{a \in \mathbb{Q}(A)} \bigcup_{\xi<\mathfrak{c}} a \cdot \gamma_\xi[P] \subset \mathbb{Q}(A) \subset K^c$, contradicting our assumption that $g[P] \subset K$.

This would mean that $P \subset M^c$. But this is also impossible, since the fact that $\bigcup\{\mathrm{id}_\star(r) \cdot S^r : r \in \mathbb{R}\}$ contains no perfect set (ensured by part (ii) of Lemma 4.1) implies that there exists an $x \in P \setminus \mathrm{supp}(\mathcal{H})$ so that $0 = g(x) \in g[P]$, while $0 \notin K$, a contradiction. □

4.2. **$2^{\mathfrak{c}}$-lineability of** $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \mathrm{CIVP} \cap \neg\,\mathrm{SCIVP}$**.** Although there is a considerable similarity of the argument in this case to the previous one, we need to choose the family $\{P^{I,q}_\xi \subset P^I : \xi < \mathfrak{c}\ \&\ q \in \mathbb{Q}\}$ with considerable more care. For this we will use the following lemma, a slight modification of [1, lemma 2.12]. Let $\mathcal{P}$ be the family of all perfect subsets of $\mathbb{R}$.

Lemma 4.4. *For every $P^I \in \mathcal{P}$ there is a family $\mathcal{P}^I$ of continuum many pairwise disjoint perfect subsets of P^I such that if $P \in \mathcal{P}$ is contained in $\bigcup \mathcal{P}^I$, then there is a $\hat{P} \in \mathcal{P}^I$ such that $P \cap \hat{P}$ is uncountable.*

Proof. Choosing a subset, if necessary, we can assume that P^I is homeomorphic to 2^ω. Let B be a Bernstein subset of 2^ω and $h_I : 2^\omega \times 2^\omega \to P^I$ be an embedding. Then the family $\mathcal{P}^I := \{h_I[\{b\} \times 2^\omega] : b \in B\}$ is as needed. Indeed, assume that $P \subset \bigcup \mathcal{P}^I$ is perfect and let $Q_0 := \{x \in 2^\omega : h_I[\{x\} \times 2^\omega] \cap P \neq \emptyset\}$, that is, Q_0 is the projection of $h_I^{-1}(P \cap P^I)$ onto the first coordinate. The compact set Q_0 must be countable, since otherwise $Q_0 \setminus B \neq \emptyset$, that is, $P \not\subset \bigcup \mathcal{P}^I$, a contradiction. Therefore there is a $b \in B$ so that the intersection of P and $\hat{P} = h_I[\{b\} \times 2^\omega] \in \mathcal{P}^I$ is uncountable. □

For every $I \in \mathcal{B}$ let $\mathcal{P}^I$ be as in Lemma 4.4 and let $\{P^{I,K}_\xi \subset P^I : \xi < \mathfrak{c}\ \&\ K \in \mathcal{P}\}$ be its enumeration. Let Z be as in (A). Recall that Z intersects every $P \in \mathcal{P}$. For every $\xi < \mathfrak{c}$ let

(κ) $\kappa_\xi := \sum_{\langle I,K\rangle \in \mathcal{B}\times\mathcal{P}} \kappa^{I,K}_\xi$, where $\kappa^{I,K}_\xi : \mathbb{R} \to Z$ has support contained in $P^{I,K}_\xi$, $\kappa^{I,K}_\xi[P^{I,K}_\xi] \subset K \cap Z$, and $\kappa^{I,K}_\xi$ is discontinuous on any perfect $Q \subset P^{I,K}_\xi$[3]

and let

$$h_\xi = \varphi_\xi + \kappa_\xi, \tag{5}$$

where maps φ_ξ are from the family $\mathcal{F}(\varphi) := \{\varphi_\xi : \xi < \mathfrak{c}\}$ from Proposition 4.2. Clearly the supports of maps in the family $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ are pairwise disjoint as functions in $\mathcal{F}(\varphi)$ have disjoint supports contained in M^c, while the maps in $\{\kappa_\xi : \xi < \mathfrak{c}\}$ have disjoint supports contained in M.

[3] $\kappa^{I,K}_\xi \restriction P^{I,K}_\xi$ can be chosen as a Sierpiński-Zygmund function from $P^{I,K}_\xi$ into $K \cap Z$, that is, a map whose restriction to any set of cardinality $\mathfrak{c}$ is discontinuous, see e.g. [18].

Theorem 4.5. *If $\mathcal{H} := \{h_\xi : \xi < \mathfrak{c}\}$ for the functions h_ξ from (5), then $W_\mathcal{H}$ justifies $2^\mathfrak{c}$-lineability of* $\mathrm{ES} \cap \neg\,\mathrm{Conn} \cap \mathrm{CIVP} \cap \neg\,\mathrm{SCIVP}$.

Proof. Similarly as in the proof of Theorem 4.3 the space $W_\mathcal{H}$ has dimension $2^\mathfrak{c}$ by Remark 2.1 and $W_\mathcal{H} \subset \mathrm{ES} \cup \{0\}$ is ensured by Proposition 4.2, since we have $\mathcal{H} \restriction M^c = \mathcal{F}(\varphi) \restriction M^c$.

To see that $W_\mathcal{H} \subset \neg\,\mathrm{Conn} \cup \{0\}$ notice that for every $\xi < \mathfrak{c}$,

$$(\kappa_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \left\{ \frac{z}{r} : z \in Z \ \&\ r \in \bigcup \mathcal{M}_0 \right\} \subset \mathbb{Q}(Z \cup \bigcup \mathcal{S}_0)$$

and, by (D) and Proposition 2.7, we have $(\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(\bigcup \mathcal{S}_0)$. Therefore, $(h_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset (\varphi_\xi / \mathrm{id}_\star)[\mathbb{R}] \cup (\kappa_\xi / \mathrm{id}_\star)[\mathbb{R}] \subset \mathbb{Q}(Z \cup \bigcup \mathcal{S}_0)$. Thus, by (A) and Lemmas 2.6 and 2.5, indeed $W_\mathcal{H} \subset \neg\,\mathrm{Conn} \cup \{0\}$.

Next notice that every h_ξ is in CIVP. Indeed, for every $I \in \mathcal{B}$ and $K \in \mathcal{P}$ the perfect set $P_\xi^{I,K}$ is contained in I and $h_\xi[P_\xi^{I,K}] = \kappa_\xi[P_\xi^{I,K}] \subset K$. Thus, $\mathcal{H} \subset \mathrm{CIVP}$ so, by Proposition 2.4(ii) used with $B = M$ and $\mathcal{F} = \{\kappa_\xi : \xi < \mathfrak{c}\}$, $W_\mathcal{H} \subset \mathrm{CIVP}$.

To finish the proof, it is enough to show that $W_\mathcal{H} \subset \neg\,\mathrm{SCIVP} \cup \{0\}$. So, choose a non-zero $g \in W_\mathcal{H}$ and a perfect $K \subset \mathbb{R} \setminus \mathbb{Q}$. Since the graph of g is dense it is enough to show that for every perfect $P \subset \mathbb{R}$, if $g[P] \subset K$, then $g \restriction P$ is discontinuous. By way of contradiction, assume that there exists a perfect $P \subset \mathbb{R}$ such that $g[P] \subset K$ and $g \restriction P$ is continuous. Since M is Borel, we can assume that either $P \subset M$ or $P \subset M^c$.

But $P \subset M^c$ is impossible, since this and the fact that $\bigcup\{\mathrm{id}_\star(r) \cdot S^r : r \in \mathbb{R}\}$ contains no perfect set would imply that there exists an $x \in P \setminus \mathrm{supp}(\mathcal{H})$ with $0 = g(x) \in g[P]$, while $0 \notin K$, a contradiction.

However, the inclusion $P \subset M$ together with $g[P] \subset K \subset \mathbb{R} \setminus \{0\}$ imply that $P \subset M \cap \mathrm{supp}(\{\kappa_\xi : \xi < \mathfrak{c}\}) \subset \bigcup_{I \in \mathcal{B}} P^I$. So, choosing perfect subset of P, if necessary, we can assume that $P \subset P^I$ for some $I \in \mathcal{B}$. But this means that $P \subset \bigcup \mathcal{P}^I$. Since $\mathcal{P}^I$ is as in Lemma 4.4, there exists a $P_\xi^{I,K'} \in \mathcal{P}^I$ such that $P \cap P_\xi^{I,K'}$ is uncountable. In particular, there exists a perfect set $Q \subset P \cap P_\xi^{I,K'}$. Then, by Remark 2.2, there exists an $a \in \mathbb{R}$ such that $g \restriction Q = a\ h_\xi \restriction Q = a\ \kappa_\xi \restriction Q$. But this is impossible, since $a = 0$ implies $g[Q] = \{0\} \subset K^c$, while $a \neq 0$ implies that $g \restriction Q = a\ \kappa_\xi \restriction Q$ is discontinuous by the choice of functions κ_ξ, contradicting our assumption that $g \restriction P$ is continuous. □

5. Final remarks

This work completes the proof that for any class $\mathcal{G}$ in the algebra $\mathcal{A}(\mathbb{D})$ of Darboux-like maps, if $\mathcal{G} \notin \mathrm{Conn} \setminus \mathrm{AC}$, then $\mathcal{G}$ is $2^\mathfrak{c}$-lineable, see [1] and above work. On the other hand for the non-empty classes $\mathcal{G} \in \mathcal{A}(\mathbb{D})$ with $\mathcal{G} \subset \mathrm{Conn} \setminus \mathrm{AC}$, it is only known that $\mathcal{G}$ is $\mathfrak{c}$-lineable, see [2], while potentially each of these classes can be $2^\mathfrak{c}$-lineable. Thus, we have the following open problem.

Problem 5.1. Is the class $\mathrm{Conn} \setminus \mathrm{AC}$ $2^\mathfrak{c}$-lineable? Is the same true for the non-empty classes $\mathcal{G} \in \mathcal{A}(\mathbb{D})$ contained in $\mathrm{Conn} \setminus \mathrm{AC}$? What about $\mathfrak{c}^+$-lineablity of these classes?

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(Gbrel M. Albkwre)
Department of Mathematics,
West Virginia University, Morgantown,
WV 26506-6310, USA.
Email address: `gmalbkwre@mix.wvu.edu`

(Krzysztof C. Ciesielski)
Department of Mathematics,
West Virginia University, Morgantown,
WV 26506-6310, USA.
Email address: `KCies@math.wvu.edu`