The Space of Density Continuous Functions

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Abstract

A function $f : \mathbf{R} \to \mathbf{R}$ is *density continuous* if it is continuous when using the density topology on both its domain and range. The set of density continuous functions contains all convex functions and all real analytic functions, but does not contain all C^∞ functions. It is not closed under addition.

We denote by $\mathbf{R}_d$ the set of real numbers, $\mathbf{R}$, endowed with the density topology. A function $f : \mathbf{R}_d \to \mathbf{R}_d$ is said to be *density continuous*, if it is continuous with respect to the topology on $\mathbf{R}_d$ in both the domain and range. The set of density continuous functions has been studied in several limited ways. Bruckner [1] and Niewiarowski [3] have studied density continuous functions which are homeomorphisms under the standard topology on $\mathbf{R}$. Ostaszewski has investigated the local behavior of density continuous functions [4] and has investigated their behavior as a semigroup [5].

In this paper, we consider the composition of the set of density continuous functions. The structure of this set seems to be quite complicated. Ostaszewski [5] has noted that it is not closed under uniform convergence. In Example 2 we show that it is not a vector space. Corollary 3 shows that each real-analytic function is density continuous, but Example 1 is a C^∞ function which is not density continuous. It is not difficult to construct a density continuous function which is not continuous. On the other hand, every density continuous function must be approximately continuous.

In what follows, the right (left) unilateral derivatives of a function f are represented as f^+ (f^-). The Lebesgue measure of a set A is denoted by $|A|$ and the Lebesgue density (right, left Lebesgue density) of A at a point

x is written as $d(A,x)$ ($d^+(A,x)$, $d^-(A,x)$). The set of functions which are infinitely differentiable on $\mathbf{R}$ is written as C^∞. Finally, if A and B are two sets such that $\sup A \le \inf B$, then we write $A \ll B$.

Before stating the main result, we first present the following lemma.

Lemma 1 *Suppose I is a compact interval and $f : I \to \mathbf{R}$. If there exist numbers α and β such that*

$$0 < \alpha < \frac{f(x) - f(y)}{x - y} < \beta < \infty, \ \forall x, y \in I, \ x \neq y, \tag{1}$$

then f is density continuous on I.

Proof. From (1) it is easy to see that f is strictly increasing and continuous on I. If $g = f^{-1}$, then it follows from (1) that

$$0 < \frac{1}{\beta} < \frac{g(u) - g(v)}{u - v} < \frac{1}{\alpha}, \ \forall u, v \in f(I), \ u \neq v. \tag{2}$$

The right-hand inequality in (2) implies that g is a Lipschitz function on $f(I)$ and hence g is absolutely continuous and g' is bounded above a.e. The left-hand inequality in (2) shows that g' is bounded away from 0 on $f(I)$ a.e. Now a result of Bruckner [1, Corollary 1] shows that g preserves density points. This implies the density continuity of f .

Theorem 1 *If I is an open interval and $f : I \to \mathbf{R}$ is convex, then f is density continuous.*

Proof. Fix a point $a \in I$. It will be shown that f is right density continuous at a. To do this, we lose no generality in supposing that $f(a) = a = 0$, because the translation of a density continuous function is obviously density continuous.

According to [6, Theorem 10.11], there exists a nondecreasing function $h : I \to \mathbf{R}$ such that

$$f(x) = \int_0^x h(t)\,dt, \ \forall x \in I. \tag{3}$$

Because of this, it is easy to see that there must exist a real number $b > 0$ such that f is monotone on $[0, b]$. We may assume that f is strictly monotone on $[0, b]$ because if it is not, f must be constant on some right neighborhood of 0, and right density continuity at 0 follows at once. With this assumption, f is a homeomorphism from $[0, b]$ onto $f([0, b])$. Denote $g = (f|_{[0,b]})^{-1}$.

There are now two cases to consider, depending upon whether f is strictly increasing or strictly decreasing on $[0, b]$.

Assume first that f is strictly decreasing on $[0, b]$. Then by (3), $h < 0$ on $[0, b)$. There is no generality lost in assuming $h(b) < 0$. If $0 \leq x < y \leq b$, then considering the average value of h on (x, y) and recalling that h is nondecreasing, it is obvious that

$$0 > h(b) \geq \frac{\int_x^y h}{y - x} = \frac{f(y) - f(x)}{y - x} \geq h(0).$$

This implies

$$0 < -h(b) < \frac{(-f(y)) - (-f(x))}{y - x} < -h(0) < \infty, \ \forall x, y \in [0, b].$$

($h(0)$ is finite because h is monotone on a neighborhood of 0.) Lemma 1 now shows that $-f$ is density continuous on $[0, b]$. Since density continuity is easily shown to be preserved under constant multiplication, it follows that f is density continuous on $[0, b]$ and therefore right density continuous at 0.

Next, assume that f is strictly increasing on $(0, b)$ and that $I_n = [a_n, b_n]$ is a sequence of disjoint intervals from $(0, f(b))$ such that I_n decreases to 0 and

$$\frac{| \cup_{n=1}^{\infty} I_n \cap (0, t)|}{t} > \rho > 0, \ \forall t \in (0, f(b)). \tag{4}$$

Let $S = \bigcup_{n=1}^{\infty} I_n$, $J_n = g(I_n)$ and $G_n = (b_{n+1}, a_n)$. From (4), it follows that

$$\frac{\bigcup_{k=n}^{\infty} I_k}{| \bigcup_{k=n-1}^{\infty} G_k|} > \frac{\rho}{1 - \rho}, \ \forall n > 1. \tag{5}$$

Before proceeding with the proof, we make the following useful observations. From (3) and the assumption that f is increasing we see that $h > 0$ on $(0, b)$. Let A and B be intervals contained in $(0, b)$ such that $A \ll B$. Then, because h is nondecreasing,

$$\frac{|f(A)|}{|A|} = \frac{\int_A h}{|A|} \leq \sup_{t \in A} h(t) \leq \inf_{t \in B} h(t) \leq \frac{\int_B h}{|B|} = \frac{|f(B)|}{|B|}.$$

This implies the statement

$$|g(C)| \geq |g(D)| \frac{|C|}{|D|} \tag{6}$$

for all intervals C and D from $(0, f(b))$ such that $C \ll D$.

We define an infinite partition S_n of S as follows.

Let $\alpha_1 = a_1$.

By (5), there exists an $\alpha_2' < \alpha_1$ such that

$$\frac{|(\alpha_2', \alpha_1) \cap S|}{|G_1|} = \frac{\rho}{1-\rho}.$$

Let $\alpha_2 = \min\{\alpha_2', a_2\}$.

Assume that α_k has been chosen for $n = 1, 2, \ldots, n-1$ so that either $\alpha_k = a_k$ or $\alpha_k < a_k$ and

$$\frac{|(\alpha_k, \alpha_{k-1}) \cap S|}{|G_{k-1}|} = \frac{\rho}{1-\rho}.$$

Choose $\alpha_n' < \alpha_{n-1}$ such that

$$\frac{|(\alpha_n', \alpha_{n-1}) \cap S|}{|G_{n-1}|} = \frac{\rho}{1-\rho}.$$

To see that such a choice is possible, there are two cases to consider, depending on α_{n-1}. If $\alpha_{n-1} = a_{n-1}$, it can be seen immediately from (5). In case $\alpha_{n-1} < a_{n-1}$, let

$$m = \max\{k < n : \alpha_k = a_k\}.$$

Then $|(\alpha_k, \alpha_{k-1}) \cap S| = \rho |G_{k-1}|/(1-\rho)$ for $m+1 \leq k \leq n-1$ so that

$$|(\alpha_{n-1}, \alpha_m) \cap S| = \frac{\rho}{1-\rho} \sum_{k=m}^{n-1} |G_{k-1}| \tag{7}$$

According to (5), there is a $t < \alpha_{n-1}$ such that

$$|(t, \alpha_m) \cap S| = \frac{\rho}{1-\rho} \sum_{k=m}^{n} |G_{k-1}|. \tag{8}$$

Subtracting (7) from (8) gives

$$|(t,\alpha_{n-1})\cap S| = \frac{\rho}{1-\rho}|G_{n-1}|.$$

We set $\alpha'_n = t$ in this case.

Then let $\alpha_n = \min\{\alpha'_n, a_n\}$.

Define $S_n = [\alpha_{n+1},\alpha_n)\cap S$. From the choice of $\alpha_n \le a_n$, and the fact that $a_n \notin S_n$, we see $\sup S_n \le b_{n+1}$. So $S_n \ll G_n = (b_{n+1}, a_n)$ and

$$\frac{|S_n|}{|G_n|} \ge \frac{\rho}{1-\rho}.$$

Finally, we use (6) and the preceding inequality to see

$$\begin{aligned} \frac{|g(\cup_{n=1}^{\infty} S_n)|}{|g(\cup_{n=1}^{\infty} G_n)|} &= \frac{\sum_{n=1}^{\infty}|g(S_n)|}{\sum_{n=1}^{\infty}|g(G_n)|} \\ &\ge \frac{\sum_{n=1}^{\infty}|g(G_n)|\frac{|S_n|}{|G_n|}}{\sum_{n=1}^{\infty}|g(G_n)|} \\ &\ge \frac{\rho}{1-\rho} \end{aligned}$$

Hence,

$$\frac{g(\cup_{n=1}^{\infty} S_n)|}{|g((0,a_1))|} \ge \rho$$

Because ρ can be made as close to 1 as desired, we see that f is right density continuous at 0.

Similar arguments show that f is left density continuous at every point of I. This completes the proof of the theorem.

Corollary 1 *If $g:[a,b]\to\mathbf{R}$ is convex on (a,b) and $\{g^+(a), g^-(b)\}\subset\mathbf{R}$, then g is density continuous.*

Proof. Define

$$f(x) = \begin{cases} g^+(a)(x-a)+g(a) & x<a \\ g(x) & a\le x\le b \\ g^-(b)(x-b)+g(b) & x>b \end{cases}$$

and apply Theorem 1

By using $g=-f$ in Theorem 1 and Corollary 1. we arrive at the following corollary.

Corollary 2 *If g is concave downward on an open interval I, then g is density continuous on I. Further, if g is concave downward on the interval $[a, b]$ with both $g^+(a)$ and $g^-(b)$ finite, then g is density continuous on $[a, b]$.*

Ostaszewski [5, Question 4] asked whether polynomials are density continuous. The following corollary provides an affirmative answer to this question.

Corollary 3 *Real analytic functions are density continuous.*

Proof. If f is real analytic, then f' is finite everywhere and f'' has only a finite number of zeroes in every interval, so applications of Corollaries 1 and 3 suffice to establish this corollary.

Corollary 4 *If $f(x) = x^\alpha$ for $\alpha \in \mathbf{R}$, then f is density continuous on its domain.*

Proof. If $\alpha \leq 0$, then this follows directly from Theorem 1. If $\alpha \geq 1$, then this corollary is a consequence of Corollary 1.

Suppose $0 < \alpha < 1$. It is clear that Theorem 1 implies f is density continuous on $\mathrm{Dom}(f) \setminus \{0\}$. So, it must be shown that f is density continuous at 0.

Let $h > 0$ and suppose $A \subset (0, h)$. Then, we use the fact that $(f^{-1})'$ is an increasing function to see

$$
\begin{aligned}
\frac{|f^{-1}(A)|}{f^{-1}(h)} &= \frac{1}{h^{1/\alpha}} \int_A \frac{x^{(1/\alpha)-1}}{\alpha} \\
&\geq \frac{1}{h^{1/\alpha}} \int_0^{|A|} \frac{x^{(1/\alpha)-1}}{\alpha} \\
&= \frac{|A|^{1/\alpha}}{h^{1/\alpha}} \\
&= (|A|/h)^{1/\alpha}
\end{aligned}
$$

It follows from this inequality that f is right density continuous at 0. A similar argument holds from the left.

Example 1 *There is a function $f \in C^\infty$ which is not density continuous.*

Choose any sequence of disjoint intervals $J_n = [a_n, b_n] \subset [0,1]$ decreasing to 0 such that

$$d^+(\bigcup_{n=1}^{\infty} J_n, 0) = 0 \tag{9}$$

and let h be a C^∞ function satisfying

$$h(0) = 0,\ h(1) = 1,\ \text{and } h^{(n)}(0) = h^{(n)}(1) = 0,\ \forall n \in \mathbf{N}. \tag{10}$$

(An example of such a function is

$$h(x) = \rho \int_0^x exp(-1/t^2 - 1/(t-1)^2)\, dt,$$

for suitable ρ.) Let

$$\alpha_n = \max\{|h^{(k)}(x)| :\ 0 \le k \le n \text{ and } 0 \le x \le 1\} \ge 1, \tag{11}$$

$$h_n(x) = \begin{cases} 0 & \text{if } x < a_n, \\ \frac{\alpha_{n+1}}{2^n \alpha_n} h(\frac{x-a_n}{b_n-a_n}) & \text{if } x \in J_n, \text{ and} \\ \frac{\alpha_{n+1}}{2^n \alpha_n} & \text{if } x > b_n \end{cases} \tag{12}$$

and

$$f(x) = \sum_{n=1}^{\infty} h_n(x).$$

From the choice of h, we see that $h_n \in C^\infty$ for each n. Using (9) and (11), it follows that

$$\sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{2^n \alpha_n} < \infty, \tag{13}$$

so that f exists everywhere. Moreover, because the J_n are pairwise disjoint, it follows that f is infinitely differentiable on $\mathbf{R} \setminus 0$ and continuous on $\mathbf{R}$.

To prove that $f^{\ (k+1)}(0)$ exists and equals 0, is differentiable at 0, let us assume that $f^{(k)}(0) = 0$ and choose $a_{n+1} \le s < a_n$ for some $n > k$. Then it follows from (11) and (12) that

$$\frac{f^{(k)}(s) - f^{(k)}(0)}{s - 0} = \frac{1}{s}\sum_{i=n}^{\infty} h_i^{(k)}(s) \le \sum_{i=n}^{\infty} \frac{1}{s}\frac{a_{i+1}}{2^i \alpha_i} \le \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}.$$

Since $a_n \to 0$, this shows $f^{(k+1)}(0) = 0$. Therefore, f is a C^∞ function.

But, f cannot be density continuous because of (9) and the fact that

$$f(\mathbf{R} \setminus \bigcup_{n=1}^{\infty} J_n)$$

is countable.

Example 2 *There is a continuous, density continuous function $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) + x$ is not density continuous.*

To construct such a function, we first choose two differentiable functions h_1 and h_2 satisfying:

(i) $0 < h_1 < h_2$ on $(0, \infty)$;

(ii) $h_1(x) = h_2(x) = x$ for $x \leq 0$; and,

(iii) $1/2 < h_1'(x) < 1 < h_2'(x) < 2$ when $x > 0$.

Let a_n and b_n be any two sequences converging to 0 such that $1 = b_1 > a_1 > b_2 > a_2 > \cdots$, and both

$$\frac{h_2(b_n) - h_1(a_n)}{b_n - a_n} = 2 \quad \text{and} \quad \frac{h_1(a_n) - h_2(b_{n+1})}{a_n - b_{n+1}} = 1/2. \tag{14}$$

Define a piecewise linear function f_0 by letting $f_0(a_n) = h_1(a_n)$, $f_0(b_n) = h_2(b_n)$ and $f_0(x) = x + f_0(b_1) - b_1$ when $x > 1$ and $f_0(x) = x$ when $x \leq 0$. The function f_0 is easily seen to be continuous because h_1 and h_2 are continuous and have value 0 at 0. Equation (14) implies

$$1/2 \leq \frac{f_0(b) - f_0(a)}{b - a} \leq 2, \ \forall a, b \in (0, \infty).$$

It follows from Lemma 1 that f must be density continuous.

Denote by $A(1/2) = \cup_{n=1}^{\infty}[b_{n+1}, a_n]$ and $A(2) = \cup_{n=1}^{\infty}[a_n, b_n]$. Either

$$(-\infty, 0] \cup A(1/2) \text{ or } (-\infty, 0) \cup A(2)$$

has positive upper density at 0. Without loss of generality we assume that it is the former. Then $f_1(x) = f_0(x) - x/2$ is constant on each component of $A(1/2)$. But, this implies that $|f_1(A(1/2))| = 0$ and $A(1/2) =$

$f_1^{-1}(f_1(A(1/2))$ has positive density at 0. Therefore, f_1 is not density continuous at 0. So, it is enough to define $f(x) = -2f_0(x)$ to obtain the desired function.

We note that the f in Example 2 can actually be constructed as a C^∞ function by a method analogous to the construction in Example 1.

This example answers questions posed by Ostaszewski [5, Questions 5 and 6].

We wish to thank Krzysztof Ostaszewski for bringing our attention to several of the questions we have considered here.

References

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