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Abstract

We study differentiable functions f from compact perfect subsets $\mathfrak{C}$ of $\mathbb{R}$ *onto* $\mathfrak{C}$ with vanishing derivative, that is, with $f'(x) = 0$ for every $x \in \mathfrak{C}$. We show that the domain of such a function can have Hausdorff dimension d for any $d \in [0, 1)$ and that it can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ such that F is α-Hölder for every $\alpha \in (0, 1)$. This last part is deduced from a novel generalization of Jarník's differentiable extension theorem stating that every differentiable map $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is compact, admits a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ which preserves Hölder continuity of f.

1 Introduction and preliminaries

A *dynamical system* is any continuous function f from a metric (or, more generally, topological) space $\langle X, d\rangle$ into itself. It is a *minimal system* when the orbit $O(x) := \{x, f(x), f^2(x), \ldots\}^1$ of every $x \in X$ is dense in X and it is a *Cantor system* when X is homeomorphic to the Cantor ternary set.

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[1] The symbol f^n stands for the n times composition of the function f.

We say that a function $f\colon \langle X,d\rangle \to \langle X,d\rangle$ is *extremely slow*[2] provided, for every $\lambda \in (0,1)$, f is pointwise contractive with constant λ, that is, such that for every $x \in X$ there is an open $U \ni x$ (depending on λ) for which $d(f(x),f(y)) \leq \lambda d(x,y)$ for every $y \in U$. Notice that if $X \subset \mathbb{R}$ is considered with the standard distance, then f is extremely slow if, and only if, $f'(x) = 0$ for every non-isolated $x \in X$.

The first extremely slow minimal dynamical system f from a compact perfect $\mathfrak{C} \subset \mathbb{R}$ onto $\mathfrak{C}$ was described in 2016 paper by the first author and Jakub Jasinski [6]. Consecutively, such systems were studied in [1, 4, 2, 14]. Specifically, in [1] the authors proved that every odometer may be embedded into the real line as an extremely slow dynamical system, that is, as a differentiable function with zero derivative at every point.

Paper [2], written by the same authors, is devoted to generalizing the result from [1] by proving that in fact every minimal dynamical system on the Cantor set may be embedded into the real line with vanishing derivative everywhere. It also contains solid description of the consequences of their result, which are paradoxical from the dynamical point of view. It is also worth to mention [2, Question 1.4], where the authors ask whether every such embedding can be extended to a differentiable Hölder continuous function on $\mathbb{R}$. In the present note we provide a partial answer to this question. Finally, the main result of [14] is that a Cantor dynamical system can be embedded into $\mathbb{R}$ as an extremely slow dynamical system if, and only if, all its finite orbits are attracting.

The interest in the slow minimal dynamical systems comes from their paradoxical properties: being pointwise contractive they "shrink" their compact domains around every point; nevertheless, they map their domains onto themselves, that is, there is no global shrinking, see [7, Figure 25]. The restrictions on the format of slow minimal dynamical systems f on compact $X \subset \mathbb{R}$ are emphasized by the following two propositions.

Proposition 1.1. *[6, Fact 2] Assume that $X \subset \mathbb{R}$ and $f\colon X \to \mathbb{R}$. Then $X \not\subset f[X]$, provided that at least one of the following conditions holds:*

- *X is a bounded closed interval and $|f'| \leq \lambda$ for some $\lambda < 1$;*
- *X has positive finite Lebesgue measure and $|f'| \leq \lambda$ for some $\lambda < 1$;*
- *X is perfect compact, $|f'| < 1$, and f can be extended to continuously differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.*

In particular, if f is a slow minimal dynamical system on a perfect compact $X \subset \mathbb{R}$, then X must be of Lebesgue measure 0 and f has no continuously differentiable extension to $F\colon \mathbb{R} \to \mathbb{R}$. Also, f cannot be *locally contractive* in the sense that "there exists a $\lambda \in [0,1)$ such that for every $x \in X$ there is an open $U \ni x$ for which $d(f(y),f(z)) \leq \lambda d(x,y)$ for every $y,z \in U$." This follows from the following result, which is an easy corollary from Edelstein's generalizations of the Banach Fixed-Point Theorem, [10, Remark 5.1] and [9, Theorem 5.2], and the fact that all orbits in an infinite minimal dynamical system are infinite.

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[2] In papers [2, 14] the authors refer to extremely slow dynamical systems simply as "slow dynamical systems." However, neither of these papers formally contains a definition of this notion and the commonly used notion of "slow-fast dynamics" (see e.g. [18]) treats slowness as "the magnitude of the derivative being less than 1," rather than being equal to 0. So, adding adjective "extremely" to the term we describe seems appropriate.

Proposition 1.2. *[6, Theorem 6] Assume that X is compact and $f\colon X \to X$ is locally contractive. Then:*

- *f has a periodic point;*
- *f has a fixed point provided that X is connected.*

By Proposition 1.1 any compact $\mathfrak{C} \subset \mathbb{R}$ that admits an extremely slow dynamical system has Lebesgue measure 0, so it is nowhere dense in $\mathbb{R}$. The first goal of this paper is to show, in Theorem 2.1, that such sets $\mathfrak{C}$ can still be relatively large: they can have Hausdorff dimension arbitrarily close to 1.

All extremely slow dynamical systems described so far in the literature are of the form

$$f = h \circ \sigma \circ h^{-1}\colon \mathfrak{C} \to \mathfrak{C}, \tag{1.1}$$

where $\mathfrak{C} \subset \mathbb{R}$ is compact, $\sigma\colon X \to X$ is a dynamical system, and $h\colon X \to \mathbb{R}$ is an embedding (i.e., h is a homeomorphism between X and $h[X]$) with $\mathfrak{C} = h[X]$. Following [2] we say that a function f given by (1.1) constitutes an *embedding of σ into $\mathbb{R}$.*

The originally constructed extremely slow f from [6], its considerably simplified form described in [4], as well as all examples constructed in this manuscript are of the form (1.1), with σ being a minimal dynamical system on 2^ω (where 2^ω denotes the set of all sequences from $\omega := \{0,1,2,\ldots\}$ into $2 := \{0,1\}$) known as a *binary odometer* or *add-one-and-carry adding machine* and defined as

$$\sigma(\langle 1,1,\ldots,1,0,s_{k+1},s_{k+2},\ldots\rangle) := \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle \tag{1.2}$$

and $\sigma(\langle 1,1,1,\ldots\rangle) := \langle 0,0,0,\ldots\rangle$. However, this choice of σ is dictated only by its simplicity and other dynamical systems can also be embedded into $\mathbb{R}$ as extremely slow systems. In fact, as we mentioned before, Boroński, Kupka, and Oprocha [2] proved that every minimal Cantor dynamical system admits an extremely slow embedding into $\mathbb{R}$. In addition, in [14] Gangloff and Oprocha characterized all Cantor dynamical systems that admit extremely slow embeddings into $\mathbb{R}$.

The definition of h in the first paper [6] was quite intricate. However, in [4] its definition is of the following considerably simpler format, very similar to the one we use in this article:

$$h(s) := \textstyle\sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)} \text{ for every } s \in 2^\omega, \tag{1.3}$$

where $s \restriction n$, identified with the sequence $\langle s_0,\ldots,s_{n-1}\rangle$, is the restriction of s to $n := \{0,1,\ldots,n-1\}$ and, for any binary sequence $u = \langle s_0,\ldots,s_{n-1}\rangle$, $N(u)$ is the natural number for which $\langle t_n,t_{n-1},\ldots,t_0\rangle := \langle 1, 1-s_{n-1}, s_{n-2},\ldots,s_0\rangle$ is its binary representation,[3] that is,

$$N(u) := \sum_{i\le n} t_i 2^i = \sum_{i<n-1} s_i 2^i + (1-s_{n-1})2^{n-1} + 2^n. \tag{1.4}$$

Notice that

$$2^n \le N(s\restriction n) \le \textstyle\sum_{i\le n} 2^i < 2^{n+1} \text{ for every } s \in 2^\omega. \tag{1.5}$$

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[3] Note that ordering s_i's is reversed, but only one of them is changed to the opposite.

1.1 Hausdorff dimension

Recall that the Hausdorff dimension $\dim_H(E)$ of a set $E \subset \mathbb{R}$ is defined as follows, see e.g. [11]. For $\rho \geq 0$ and $\delta > 0$ let

$$\mathcal{H}^{\rho}_{\delta}(E) := \inf \left\{ \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{\rho} \colon \mathcal{U} \text{ is a } \delta\text{-cover of } E \right\},$$

where $\operatorname{diam}(U)$ denotes the diameter of U and by a *δ-cover* of E we mean a cover of E by sets of diameter less than or equal to δ. It is well known and easy to see that the value of $\mathcal{H}^{\rho}_{\delta}(E)$ remains unchanged when in its definition we allow only δ-covers by open intervals. The *Hausdorff ρ-dimensional measure* of E is defined as

$$\mathcal{H}^{\rho}(E) := \lim_{\delta \to 0^+} \mathcal{H}^{\rho}_{\delta}(E). \tag{1.6}$$

It is well defined, as the map $\delta \mapsto \mathcal{H}^{\rho}_{\delta}(E)$ is monotone. The *Hausdorff dimension* $\dim_H(E)$ is defined as the only number $d \geq 0$ such that

$$\mathcal{H}^{\rho}(E) = \infty \text{ for every } \rho \in [0, d) \text{ and } \mathcal{H}^{\rho}(E) = 0 \text{ for every } \rho > d.$$

Remark 1.3. If $\mathfrak{C}_0 := h[2^{\omega}]$ for h given by (1.3), then $\dim_H(\mathfrak{C}_0) = 0$.

Proof. For every $n < \omega$ and $t \in 2^n$ (where 2^n stands here for the family of all functions from $n = \{0, \ldots, n-1\}$ into $2 = \{0, 1\}$) let $[t] := \{s \in 2^{\omega} \colon t \subset s\}$. Notice that if $s \in [t]$, then

$$\sum_{i<n} 2 s_i 3^{-(i+1)N(s \restriction i)} \leq h(s) \leq \sum_{i<n} 2 s_i 3^{-(i+1)N(s \restriction i)} + \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)N(s \restriction i)}.$$

So, as $N(s \restriction i) \geq N(s \restriction n) \geq 2^n$ for any $i \geq n$,

$$\begin{aligned} \operatorname{diam}\left(h\big[[t]\big]\right) &\leq \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)N(s \restriction i)} \leq \sum_{i=n}^{\infty} 2 \cdot 3^{-(i+1)2^n} \\ &\leq \sum_{i=0}^{\infty} 2 \cdot 3^{-(n+1)2^n - i} = 3 \cdot 3^{-(n+1)2^n}. \end{aligned}$$

Therefore, $\mathcal{U} = \{h\big[[t]\big] \colon t \in 2^n\}$ is a 3^{-n}-cover of $\mathfrak{C}_0$ and, for every $d > 0$,

$$\mathcal{H}^{d}_{3^{-n}}(\mathfrak{C}_0) \leq \sum_{t \in 2^n} \operatorname{diam}\left(h\big[[t]\big]\right)^d \leq 2^n \cdot 3^d \cdot 3^{-d(n+1)2^n} \xrightarrow{n \to \infty} 0$$

Hence $\mathfrak{C}_0$ has Hausdorff dimension 0. ∎

The remark shows that $\mathfrak{C}_0$ is, in the sense of Hausdorff dimension, as small as it gets. One of the goals of this paper is to show, see Theorem 2.1, that there are compact perfect sets $\mathfrak{C}$ admitting extremely slow minimal dynamical systems that are as big as possible, that is, of Hausdorff dimension arbitrarily close to 1.

1.2 α-Hölder property

Let $\alpha \in (0,1]$ and let f be a function from $P \subset \mathbb{R}$ into $\mathbb{R}$. Recall that f *is* α*-Hölder* if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x-y|^{\alpha} \text{ for all } x, y \in P.$$

In [2, Question 1.4] the authors ask if every minimal Cantor dynamical system can be embedded into $\mathbb{R}$ as (extremely) slow Cantor dynamical system f such that f can be extended to a differentiable α-Hölder $F\colon \mathbb{R} \to \mathbb{R}$ for some $\alpha \in (0,1)$. The second goal of this paper is to show, see Theorem 3.1, that the answer to this question is positive for the binary odometer σ defined in (1.2).
In the sequel we will use the following two simple and certainly known facts on Hölder continuity. For the reader's convenience we include their proofs.

Fact 1.4. Let $\alpha \in (0,1]$ and $P \subset \mathbb{R}$ closed. Assume that there is a $C > 0$ such that $F\colon \mathbb{R} \to \mathbb{R}$ is α-Hölder with constant C on the closure $\mathrm{cl}(J)$ of every connected component J of $\mathbb{R} \setminus P$, that is,

$$|F(x) - F(y)| \leq C|x-y|^{\alpha}, \tag{1.7}$$

for all $x, y \in \mathrm{cl}(J)$. If the restriction $F \restriction P$ of F to P is α-Hölder, then F is also α-Hölder.

Proof. Increasing C, if necessary, we can assume that (1.7) holds for all $x, y \in P$. Choose any $p < q$. It suffices to show that $|F(p) - F(q)| \leq 3C|p-q|^{\alpha}$.
If $[p,q] \cap P = \emptyset$, then $[p,q]$ is contained in a single connected component of $\mathbb{R} \setminus P$, so the inequality holds. Therefore assume that $[p,q] \cap P \neq \emptyset$, and let $x = \min([p,q] \cap P)$ and $y = \max([p,q] \cap P)$. Then

$$\begin{aligned} |F(p) - F(q)| &\leq |F(p) - F(x)| + |F(x) - F(y)| + |F(y) - F(q)| \\ &\leq C|p-x|^{\alpha} + C|x-y|^{\alpha} + C|y-q|^{\alpha} \leq 3C|p-q|^{\alpha} \end{aligned}$$

as needed. ■

Fact 1.5. Let $a < b$, $P \subset [a,b]$ compact, and $F\colon (-\infty, a] \cup P \cup [b, \infty) \to \mathbb{R}$ such that $F \restriction (-\infty, a] \equiv F(\min P)$ and $F \restriction [b, \infty) \equiv F(\max P)$. Then, for every $\alpha \in (0,1]$, F is α-Hölder if and only if $F \restriction P$ is α-Hölder.

Proof. The forward implication is obvious. Conversely, assume that $F \restriction P$ is α-Hölder with constant C, choose $x_1, x_2 \in (-\infty, a] \cup P \cup [b, \infty)$ such that $x_1 \leq x_2$, and define $r := \max\{x_1, \min P\}$ and $s := \min\{x_2, \max P\}$. Then

$$|F(x_1) - F(x_2)| = |F(r) - F(s)| \leq C|r-s|^{\alpha} \leq C|x_1 - x_2|^{\alpha},$$

that is, F is α-Hölder. ■

It is well known (see e.g. [13]) and easy to see that

$$\text{if } P \text{ is compact, } f \text{ is } \alpha\text{-Hölder, and } \beta \in (0, \alpha), \text{ then } f \text{ is also } \beta\text{-Hölder.} \tag{1.8}$$

2 Slow systems of any Hausdorff dimension

The entire content of this section is devoted to the proof of the following result.

Theorem 2.1. *For every $d \in [0,1)$ there exists an extremely slow minimal dynamical system $f_d \colon \mathfrak{C}_d \to \mathfrak{C}_d$ such that $\mathfrak{C}_d \subset \mathbb{R}$ is compact and has Hausdorff dimension d.*

By Remark 1.3, we can assume that $d > 0$. So, fix $d \in (0,1]$ and define $p := 2^{-1/d}$. Notice that $p \in (0,1/2]$ and

$$d = \log_p(1/2).$$

For $n < \omega$ define $\hat{n} := 0$ when $n < 2$ and $\hat{n} := \lfloor \log_2 \log_2 n \rfloor$ for $n \geq 2$. Thus $\hat{n} < n$ for $n > 0$ and the map $n \mapsto \hat{n}$ is nondecreasing. Also, for any $n \geq 2$ we have $\hat{n} \leq \log_2 \log_2 n < \hat{n} + 1$, so $2^{\hat{n}} \leq \log_2 n < 2 \cdot 2^{\hat{n}}$ and, by (1.5), for every $s \in 2^\omega$,

$$\frac{1}{2} \log_2 n \leq 2^{\hat{n}} \leq N(s \restriction \hat{n}) < 2^{\hat{n}+1} \leq 2 \log_2 n, \tag{2.1}$$

where N is as defined in (1.4). The function f_d is defined similarly as f given by (1.1), (1.3), and (1.4):

$$f_d := h_d \circ \sigma \circ h_d^{-1} \colon \mathfrak{C}_d \to \mathfrak{C}_d, \tag{2.2}$$

where $\sigma \colon 2^\omega \to 2^\omega$ is the add-one-and-carry adding machine defined in (1.2) while $h_d \colon 2^\omega \to \mathbb{R}$ is defined as

$$h_d(s) := \sum_{n=0}^{\infty} s_n p^{n+\psi(s \restriction n)},$$

where $\psi(t) := N(t \restriction \hat{n})^2$ for any $t \in 2^n$. The fact that h_d is indeed an embedding is proved in Section 2.1. Notice that the sequence $\langle \psi(s \restriction n) \rangle_n$ is nondecreasing. Similarly as earlier, we put $\mathfrak{C}_d := h_d[2^\omega]$.

In the rest of this section we prove that f_d, for $d \in (0,1)$, is as claimed in Theorem 2.1. We also indicate where the argument does not work for $d = 1$.

2.1 Geometrical description of $\mathfrak{C}_d$ and continuity of h_d

For $n < \omega$ and $t \in 2^n$ let

$$I_t := [a_t, b_t] \quad \text{where} \quad a_t := \inf h_d\big[[t]\big] \quad \text{and} \quad b_t := \sup h_d\big[[t]\big].$$

Then, by the definition of h_d, for $s \in 2^\omega$ extending t with $s(i) = 1$ for all $i \geq n$,

$$a_t = \sum_{i<n}^{\infty} t_i p^{i+\psi(t \restriction i)} \;\text{ and }\; b_t = a_t + \sum_{i=n}^{\infty} p^{i+\psi(s \restriction i)}.$$

Let $\psi_n := (2 \log_2 n)^2$ and notice that, by (2.1), $\psi_n > \psi(t)$. Next, we will show that

$$p^{n+\psi_n} \leq \operatorname{diam}(I_t) < \frac{1}{1-p}\, p^{n+\psi(t)} \leq p^{n-1+\psi(t)}. \tag{2.3}$$

To see this, let $s \in 2^\omega$ be an extension of t such that $s_i = 1$ for every $i \geq n$ and notice that $\operatorname{diam}(I_t) = b_t - a_t = \sum_{i=n}^\infty p^{i+\psi(s\restriction i)}$. Then, the lower estimate of $\operatorname{diam}(I_t)$ in (2.3) is justified by $\sum_{i=n}^\infty p^{i+\psi(s\restriction i)} > p^{n+\psi(s\restriction n)} \geq p^{n+\psi_n}$, while the upper estimates by

$$\sum_{i=n}^{\infty} p^{i+\psi(s\restriction i)} < \sum_{i=n}^{\infty} p^{i+\psi(t)} = \frac{1}{1-p}\, p^{n+\psi(t)} \leq p^{n-1+\psi(t)},$$

where the last inequality holds since $\frac{1}{1-p} \leq p^{-1}$ for any $p \in [0, 1/2]$.

Next, notice that if $m = n+1$ and $t\hat{\ }j \in 2^m$ is an extension of t such that $(t\hat{\ }j)_n = j$, then $(t\hat{\ }0) \restriction \hat{m} = (t\hat{\ }1) \restriction \hat{m}$ and $\psi(t\hat{\ }0) = \psi(t\hat{\ }1)$. This equation and (2.3) imply that

$$\begin{aligned} b_{t\hat{\ }0} &= a_{t\hat{\ }0} + \operatorname{diam}(I_{t\hat{\ }0}) = a_t + \operatorname{diam}(I_{t\hat{\ }0}) \\ &< a_t + p^{m-1+\psi(t\hat{\ }0)} = a_t + p^{n+\psi(t\hat{\ }1)} = a_{t\hat{\ }1}. \end{aligned} \tag{2.4}$$

This means that the intervals in the family $\mathcal{C}_n := \{I_t \colon t \in 2^n\}$ are pairwise disjoint, so that if $s, t \in 2^\omega$ are distinct, then the sets $\{h_d(s)\} = \bigcap_{n<\omega} I_{s\restriction n}$ and $\{h_d(t)\} = \bigcap_{n<\omega} I_{t\restriction n}$ are disjoint. This clearly implies that h_d is injective. It is also continuous, since for any distinct $s, t \in 2^\omega$ and $n := \min\{i \colon s(i) \neq t(i)\}$, we have that $|h_d(s) - h_d(t)| \leq \operatorname{diam}(I_{s\restriction n}) \leq p^{n-1+\psi(t)} \leq p^{n-1} \xrightarrow{n\to\infty} 0$. Together with the compactness of 2^ω, this implies that h_d is indeed an embedding, what was implicitly assumed in definition (2.2) for the function f_d. Moreover, $\mathfrak{C}_d = h_d[2^\omega]$ can also be represented in the standard geometric format of the Cantor ternary set:

$$\mathfrak{C}_d = \bigcap_{n<\omega} \bigcup \mathcal{C}_n.$$

We will use this representation when calculating the Hausdorff dimension of $\mathfrak{C}_d$.

2.2 Hausdorff dimension of $\mathfrak{C}_d$

So far, we proved that f_d is a well defined Cantor dynamical system. It is minimal since $D := \{s \in 2^\omega \colon s^{-1}(1) \text{ is finite}\}$ is an orbit of σ dense in 2^ω. We will show that $\dim_H(\mathfrak{C}_d) = d$. This will be deduced from the following lemma.

Lemma 2.2. *Let $p \in (0, 1/2]$ and, for $n < \omega$, let $\mathcal{C}_n$ be a family of 2^n pairwise disjoint closed intervals such that each $I \in \mathcal{C}_n$ contains two intervals from $\mathcal{C}_{n+1}$. Assume that there is a sequence $\langle \psi_n \in [0,\infty) \colon n < \omega \rangle$ such that $\lim_{n\to\infty} \frac{\psi_n}{n+1} = 0$ and*

- *the length of every $I \in \mathcal{C}_n$ is between $p^{n+\psi_n}$ and p^{n-1}.*

Then $C := \bigcap_{n<\omega} \bigcup \mathcal{C}_n$ has Hausdorff dimension $\rho := \log_p(1/2)$.

Notice that our set $\mathfrak{C}_d$ satisfies the assumptions of this lemma: by (2.3) the length of every $I_t \in \mathcal{C}_n$ is between $p^{n+\psi_n}$ and $p^{n-1+\psi(t)} \leq p^{n-1}$. Clearly

$$\lim_{n\to\infty} \frac{\psi_n}{n+1} = \lim_{n\to\infty} \frac{(2\log_2 n)^2}{n+1} = 0,$$

so the lemma implies that $\dim_H(\mathfrak{C}_d) = \log_p(1/2) = d$.

The last paragraph of the following proof comes from [12, Mass distribution principle 4.2].

Proof of Lemma 2.2. To see that $\dim_H(C) \leq \rho$, notice that, by $\bullet$, for every $n < \omega$ all sets in $\mathcal{C}_n$ have diameters less than or equal to p^{n-1}. So $\mathcal{C}_n$ is a p^{n-1}-cover of C. Moreover,

$$\sum_{I \in \mathcal{C}_n} \operatorname{diam}(I)^\rho \leq 2^n (p^{n-1})^\rho = 2^n \cdot (1/2)^{n-1} = 2.$$

Therefore, by (1.6), $\mathcal{H}^\rho(C) = \lim_{n\to\infty} \mathcal{H}^\rho_{p^n}(C) \leq 2$.

To see that $\dim_H(C) \geq \rho$, fix $\eta \in (0, \rho)$. It suffices to show that $\mathcal{H}^\eta(C) > 0$. Let μ_0 be the standard product measure on 2^ω (i.e., such that $\mu_0([s]) = 2^{-n}$ for every $s \in 2^n$) and define a measure μ on $\mathbb{R}$ (referred sometimes as a mass distribution of C) such that $\mu(U) = \mu_0(\bigcup\{[t] \colon I_t \subset U\})$ for every open $U \subset \mathbb{R}$. In particular, $\mu(I) = 2^{-n}$ for every $I \in \mathcal{C}_n$.

Since $\lim_{n\to\infty} \dfrac{n+\psi_n}{n+1} = 1 < \dfrac{\rho}{\eta}$ we can find $m < \omega$ such that

$$\frac{n+\psi_n}{n+1} \leq \frac{\rho}{\eta} \text{ for } n \geq m. \tag{2.5}$$

The number $M := \sup\left\{\frac{\psi_n}{n+1} \colon n < \omega\right\}$ is finite, since $\lim_{n\to\infty} \frac{\psi_n}{n+1} = 0$. Define $\delta := p^{(M+1)(m+1)} > 0$ and notice that

$$\operatorname{diam}(I)^\eta \geq \frac{1}{2}\mu(I) \text{ whenever } \operatorname{diam}(I) \leq \delta \text{ and } I \in \mathcal{C}_n \text{ for some } n < \omega. \tag{2.6}$$

Indeed, our assumption on the lengths of the intervals $I \in \mathcal{C}_n$ implies that $p^{n+\psi_n} \leq \operatorname{diam}(I) \leq \delta = p^{(M+1)(m+1)}$. Since

$$n + \psi_n = \left(\frac{n+\psi_n}{n+1}\right)(n+1) \leq \left(1 + \frac{\psi_n}{n+1}\right)(n+1) \leq (M+1)(n+1),$$

we obtain that $p^{(M+1)(n+1)} \leq p^{n+\psi_n} \leq p^{(M+1)(m+1)}$. Hence $n \geq m$ and, by (2.5), $(n+\psi_n)\eta \leq \rho(n+1)$. So, $\operatorname{diam}(I)^\eta \geq p^{(n+\psi_n)\eta} \geq p^{\rho(n+1)} = 2^{-(n+1)} = \frac{1}{2}\mu(I)$.

The key fact for the rest of our argument is that

$$\operatorname{diam}(U)^\eta \geq \frac{1}{8}\mu(U) \text{ for every open interval } U \text{ with } \operatorname{diam}(U) \leq \delta. \tag{2.7}$$

To see (2.7), take an open interval U with $\operatorname{diam}(U) \leq \delta$. If $U \cap C = \emptyset$, then $\mu(U) = 0$ and (2.7) holds. So assume that $U \cap C \neq \emptyset$ and take the smalles $n < \omega$ such that U contains some $J \in \mathcal{C}_n$. Then $\operatorname{diam}(J) \leq \operatorname{diam}(U) \leq \delta$. Moreover, by the minimality of n, the family $\mathcal{F}$ of all $I \in \mathcal{C}_n$ intersecting U can have at most 4 elements. In particular, $J \subset U \subset \bigcup \mathcal{F}$ and, by (2.6),

$$\mu(U) \leq \mu(\bigcup \mathcal{F}) = \sum_{I\in\mathcal{F}} \mu(I) \leq 4\mu(J) \leq 8 \operatorname{diam}(J)^\eta \leq 8 \operatorname{diam}(U)^\eta,$$

implying (2.7).

Finally notice that if $\mathcal{U}$ is a δ-cover of C by open intervals then, by (2.7),

$$\sum_{U\in\mathcal{U}} \operatorname{diam}(U)^\eta \geq \sum_{U\in\mathcal{U}} \frac{1}{8}\mu(U) \geq \frac{1}{8}\mu\left(\bigcup \mathcal{U}\right) \geq \frac{1}{8}\mu(C) = \frac{1}{8}.$$

Thus $\mathcal{H}^\eta(C) = \lim_{\delta\to 0^+} \mathcal{H}^\eta_\delta(C) \geq \dfrac{1}{8} > 0$, as needed. ■

2.3 The derivative of f_d for $d < 1$

It remains to show that $f_d'(x) = 0$ for every $x \in \mathfrak{C}_d$. The argument for this is very similar to the one used in [4] to show the same result for f defined by (1.1), (1.3), (1.4).

Notice that $d \in (0,1)$ ensures that $p = 2^{-1/d} < 1/2$ so that $\frac{p}{1-p} < 1$. We start with the following two observations.

(a) For every $s \in 2^\omega$ there is a $k \in \omega$ such that $N(\sigma(s) \restriction \hat{n}) = N(s \restriction \hat{n}) + 1$ for every $n > k$.

(b) If $n = \min\{i \in \omega\colon s_i \neq t_i\}$ for some distinct $s = \langle s_i \rangle$ and $t = \langle t_i \rangle$ from 2^ω, then $\left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)} \leq |h_d(s) - h_d(t)| \leq \left(1 + \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}$.

To see (a) note that $N(\sigma(s) \restriction \ell) = N(s \restriction \ell) + 1$ whenever $s \restriction \ell \neq \langle 1, \ldots, 1, 0 \rangle$. Since $s \restriction \ell = \langle 1, \ldots, 1, 0 \rangle$ for at most one $\ell < \omega$, there exists k_0 such that $N(\sigma(s) \restriction \ell) = N(s \restriction \ell) + 1$ provided $\ell > k_0$. Then $k = 2^{2^{k_0+1}}$ is as needed since then $n > k$ implies $\hat{n} > k_0$.

To see (b), take s and t as in the assumption. Notice that $\widehat{n+1} \leq n$, so

$$\psi(s \restriction n+1) = (N(s \restriction \widehat{n+1}))^2 = (N(t \restriction \widehat{n+1}))^2 = \psi(t \restriction n+1). \tag{2.8}$$

We may assume that $s_n = 1$ and $t_n = 0$. Let $u = t \restriction n = s \restriction n$.

Using the notation from Section 2.1, we have that $h_d(t) \in I_{u\hat{\ }0} = [a_{u\hat{\ }0}, b_{u\hat{\ }0}]$ and $h_d(s) \in I_{u\hat{\ }1} = [a_{u\hat{\ }1}, b_{u\hat{\ }1}]$. So, by (2.4), $a_{u\hat{\ }0} \leq h_d(t) \leq b_{u\hat{\ }0} < a_{u\hat{\ }1} \leq h_d(s) \leq b_{u\hat{\ }1}$. In particular, by (2.3) and (2.8),

$$\begin{aligned}
|h_d(s) - h_d(t)| &\geq a_{u\hat{\ }1} - b_{u\hat{\ }0} = \left(a_u + p^{n+\psi(u\hat{\ }1)}\right) - (a_u + \operatorname{diam}(I_{u\hat{\ }0})) \\
&\geq p^{n+\psi(u\hat{\ }1)} - \frac{1}{1-p}\, p^{n+1+\psi(u\hat{\ }0)} = \left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}; \\
|h_d(s) - h_d(t)| &\leq b_{u\hat{\ }1} - a_{u\hat{\ }0} = (a_u + p^{n+\psi(u\hat{\ }1)} + \operatorname{diam}(I_{u\hat{\ }1})) - a_u \\
&\leq p^{n+\psi(u\hat{\ }1)} + \frac{1}{1-p}\, p^{n+1+\psi(u\hat{\ }0)} = \left(1 + \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)},
\end{aligned}$$

so (b) is proved.

In order to prove that there exists $s \in 2^\omega$ such that $f_d'(h_d(s)) = 0$, choose $k \in \omega$ satisfying (a) and let $\delta > 0$ be such that $0 < |h_d(s) - h_d(t)| < \delta$ implies that $n = \min\{i \in \omega\colon s_i \neq t_i\}$ is greater than k. Fix $t \in 2^\omega$ for which $0 < |h_d(s) - h_d(t)| < \delta$. Then we have $n = \min\{i \in \omega\colon s_i \neq t_i\} = \min\{i \in \omega\colon \sigma(s)_i \neq \sigma(t)_i\}$ and, using (b) for the pairs $\langle s, t \rangle$ and $\langle \sigma(s), \sigma(t) \rangle$, we obtain that

$$\frac{|f_d(h_d(s)) - f_d(h_d(t))|}{|h_d(s) - h_d(t)|} = \frac{|h_d(\sigma(s)) - h_d(\sigma(t))|}{|h_d(s) - h_d(t)|} \leq \frac{\left(1 + \frac{p}{1-p}\right) p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1 - \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}}. \tag{2.9}$$

Also, using (a), we obtain that

$$\begin{aligned}\psi(\sigma(s)\restriction n+1)-\psi(s\restriction n+1) &= (N(\sigma(s)\restriction \widehat{n+1}))^2-(N(s\restriction \widehat{n+1}))^2\\ &= (N(s\restriction \widehat{n+1})+1)^2-(N(s\restriction \widehat{n+1}))^2 \geq N(s\restriction \widehat{n+1}).\end{aligned}$$

From this, (2.9), and letting $c := \frac{\left(1+\frac{p}{1-p}\right)}{\left(1-\frac{p}{1-p}\right)}$, we obtain that

$$\frac{|f_d(h_d(s))-f_d(h_d(t))|}{|h_d(s)-h_d(t)|} \leq \frac{\left(1+\frac{p}{1-p}\right)p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1-\frac{p}{1-p}\right)p^{n+\psi(s\restriction n+1)}} \leq c\cdot p^{N(s\restriction \widehat{n+1})}. \tag{2.10}$$

Hence $f_d'(h_d(s))=0$, as $p^{N(s\restriction \widehat{n+1})}$ is arbitrarily small for δ small enough.

3 Hölder property of maps f_d

The aim of this Section is to prove the following result.

Theorem 3.1. *For every $d \in (0,1)$ the extremely slow minimal dynamical system $f_d\colon \mathfrak{C}_d \to \mathfrak{C}_d$ defined in Section 2 is α-Hölder for any $\alpha \in (0,1)$. Moreover, there exists a differentiable extension $F_d\colon \mathbb{R}\to\mathbb{R}$ of f_d such that F_d is α-Hölder for every $\alpha\in(0,1)$.*

The first step in the proof is Lemma 3.2.

Lemma 3.2. *$f_d\colon \mathfrak{C}_d \to \mathfrak{C}_d$ is α-Hölder, for all $d,\alpha \in (0,1]$.*

Proof. Fix $\alpha\in(0,1)$. Since $\mathfrak{C}_d$ is compact, it suffices to prove that

$$\exists\, k<\omega \text{ such that } f_d \text{ is } \alpha\text{-Hölder on any set } h[[u]] \text{ with } u\in 2^k. \tag{3.1}$$

To see that (3.1) implies the lemma, first notice that, by the assumptions of (3.1), $\exists\, C_1>0$ such that f_d is α-Hölder with constant C_1 on any set $h[[u]]$ with $u\in 2^k$. Moreover, if $E=\bigcup_{u\in 2^k}\{\min h[[u]],\max h[[u]]\}$, then there exists $C_2>0$ such that $|f_d(r)-f_d(s)|\leq C_2|r-s|^\alpha$ for all $r,s\in E$. We claim that f_d is α-Hölder with constant $C:=3\max\{C_1,C_2\}$. To see this, choose $x_1,x_2\in\mathfrak{C}_d$ with $x_1\leq x_2$ and let $u_1,u_2\in 2^k$ be such that $x_1\in h[[u_1]]$ and $x_2\in h[[u_2]]$. If $u_1\neq u_2$, let $r:=\max h[[u_1]]$ and $s:=\min h[[u_2]]$; otherwise put $r=s=x_1$. Then

$$\begin{aligned}|f_d(x_1)-f_d(x_2)| &= |f_d(x_1)-f_d(r)|+|f_d(r)-f_d(s)|+|f_d(s)-f_d(x_2)|\\ &\leq C_1|x_1-r|^\alpha+C_2|r-s|^\alpha+C_1|s-x_2|^\alpha\\ &\leq C_1|x_1-x_2|^\alpha+C_2|x_1-x_2|^\alpha+C_1|x_1-x_2|^\alpha\\ &\leq C|x_1-x_2|^\alpha\end{aligned}$$

as needed. To see that (3.1) is satisfied, fix $k<\omega$ and $u\in 2^k$. Take distinct $s=\langle s_i\rangle$ and $t=\langle t_i\rangle$ from $[u]$ with $n=\min\{i\in\omega\colon s_i\neq t_i\}$. Then $n>k$, and, for $c_\alpha := \frac{\left(1+\frac{p}{1-p}\right)}{\left(1-\frac{p}{1-p}\right)^\alpha}$, we obtain the following simple variation of (2.10):

$$\frac{|f_d(h(s))-f_d(h(t))|}{|h(s)-h(t)|^\alpha} \leq \frac{\left(1+\frac{p}{1-p}\right)p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1-\frac{p}{1-p}\right)^\alpha p^{\alpha(n+\psi(s\restriction n+1))}} \leq \frac{c_\alpha p^n}{p^{\alpha(n+\psi(s\restriction n+1))}}. \tag{3.2}$$

Also, by (2.1), we have that $\psi(s \restriction n+1) = (N(s \restriction \widehat{n+1}))^2 \le (2\log_2(n+1))^2$, so

$$\frac{c_\alpha p^n}{p^{\alpha(n+\psi(s\restriction n+1))}} \le \frac{c_\alpha p^n}{p^{\alpha(n+(2\log_2(n+1))^2)}}. \tag{3.3}$$

Since $\frac{\alpha(n+(2\log_2(n+1))^2)}{n} \xrightarrow{n\to\infty} \alpha < 1$, $\exists k < \omega$ such that $\frac{\alpha(n+(2\log_2(n+1))^2)}{n} < 1$ for every $n > k$, that is, $\alpha(n + (2\log_2(n+1))^2) < n$. Therefore, by (3.2) and (3.3), for every $n > k$,

$$\frac{|f_d(h(s)) - f_d(h(t))|}{|h(s) - h(t)|^\alpha} \le \frac{c_\alpha p^n}{p^{\alpha(n+(2\log_2(n+1))^2)}} \le \frac{c_\alpha p^n}{p^n} = c_\alpha,$$

that is, f_d is indeed α-Hölder on any set $h([u])$ with $u \in 2^k$. ■

Theorem 3.1 follows immediately from Lemma 3.2 and Theorem 4.1.

4 Differential extensions preserving Hölder continuity

Jarník's differentiable extension theorem states that *every real valued differentiable function from a closed subset of* $\mathbb{R}$ *into* $\mathbb{R}$ *has a differentiable extension.* For the fascinating history of this theorem and its proof, see [3]. Compare also to [8]. For its generalizations, see [15] and [5]. To prove Theorem 3.1, we will need the following generalization of Jarník's differentiable extension theorem, which is of interest by its own right.

Theorem 4.1. *Every differentiable map $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed, admits a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ such that if P is compact, then F preserves Hölder continuity of f, that is, if f is α-Hölder for some $\alpha \in (0,1]$, then so is F.*

Proof. We can assume that P is compact and that the set

$$H := \{\alpha \in (0,1]\colon\ f \text{ is } \alpha\text{-Hölder}\}$$

is not empty, since otherwise the result follows immediately from Jarník's differentiable extension theorem. By (1.8), if $\alpha \in H$ and $\beta \in (0,\alpha)$, then $\beta \in H$.

Let $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ be the linear interpolation[4] of f which is constant on each unbounded connected component of $\mathbb{R} \setminus P$, choose $a < b$ such that $P \subset (a,b)$, and define $\tilde{P} := (-\infty,a] \cup P \cup [b,\infty)$ together with $\tilde{f} := \bar{f} \restriction \tilde{P}$. Then $\tilde{f}$ is still differentiable and, by Fact 1.5 used with $F = \tilde{f}$,[5] $\tilde{f}$ is α–Hölder for every $\alpha \in H$. In addition, $\bar{f}$ is also the linear interpolation of $\tilde{f}$.

Let $\mathcal{J}$ be the family of all connected components of $\mathbb{R} \setminus \tilde{P}$ and Π be the set of all endpoints of the intervals in $\mathcal{J}$. Notice that all $J \in \mathcal{J}$ are bounded.

It is easy to see (compare e.g. [3]) that $\bar{f}$ is differentiable at all points $x \in \mathbb{R} \setminus \Pi$. Also, $\bar{f}$ is differentiable at least from one side at every $x \in \Pi$. Moreover,

$$\bar{f} \text{ is } \alpha\text{- Hölder for every } \alpha \in H. \tag{4.1}$$

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[4] This means that $\bar{f}$ is linear on the closure of every connected component of $\mathbb{R} \setminus P$.

[5] $F \restriction (-\infty,a] \equiv F(\min P)$ since $\bar{f}$ is constant on $(-\infty,\min P]$ with the value $f(\min P) = F(\min P)$. Similarly, $F \restriction [b,\infty) \equiv F(\max P)$.

Indeed, this follows from Fact 1.4 used with $F = \bar{f}$ as long as there is $C > 0$ such that

$$|\bar{f}(x) - \bar{f}(y)| \leq C|x-y|^\alpha \text{ for every } J \in \mathcal{J} \text{ and all } x, y \in \mathrm{cl}(J). \tag{4.2}$$

To see (4.2), fix $\alpha \in H$ and let C be such that $|\tilde{f}(x) - \tilde{f}(y)| \leq C|x-y|^\alpha$ for all $x, y \in \tilde{P}$. Now, if $J = (a,b)$ and $p, q \in \mathrm{cl}(J)$, then

$$\begin{aligned} \frac{|\tilde{f}(p) - \tilde{f}(q)|}{|p-q|} &= \frac{|\tilde{f}(a) - \tilde{f}(b)|}{|a-b|} \quad \text{and} \\ \frac{|\tilde{f}(p) - \tilde{f}(q)|}{|p-q|^\alpha} &= |p-q|^{1-\alpha} \frac{|\tilde{f}(p) - \tilde{f}(q)|}{|p-q|} \\ &\leq |a-b|^{1-\alpha} \frac{|\tilde{f}(a) - \tilde{f}(b)|}{|a-b|} \\ &= \frac{|\tilde{f}(a) - \tilde{f}(b)|}{|a-b|^\alpha} \leq C, \end{aligned}$$

justifying (4.2) and (4.1).

The proof of Jarník's differentiable extension theorem presented in [3] obtains F by modifying $\bar{f}$ on the family $\mathcal{K} = \{K_n \colon n < \omega\}$ of small pairwise disjoint closed intervals, each contained in the closure of an $J \in \mathcal{J}$ and sharing with J one endpoint. More specifically, for each $n < \omega$ one finds a continuous function $f_n \colon \mathbb{R} \to \mathbb{R}$ with support contained in K_n and defines

$$F := \bar{f} + \sum_{n<\omega} f_n. \tag{4.3}$$

This modification ensures differentiability at points $x \in \Pi$ from appropriate sides that needed adjustment, while the small size of each K_n ensures preservation of other (unilateral, pointwise) differentiability of $\bar{f}$. In general, it is not clear that such defined F must preserve Hölder continuity. But we will show that some small modification of the definitions of functions f_n indeed ensures such preservation. Note that without loss of generality we may assume that each $x \in \Pi$ belongs to exactly one K_n, as f_n may be the zero function.

To see this, note that the functions f_n are defined in [3] as $f_n(x) := \int_{-\infty}^{x} h_n(t)\, dt$, where h_n is continuous on K_n and zero on its complement, see [3, Figure 3].

(i) If f is Lipschitz with constant L, then $h_n[\mathbb{R}] \subset [-2L, 2L]$.

In addition, if the lengths of the intervals K_n are further shrinking and new functions h_n are the horizontal proportional shrinking versions of their original selves, then F defined by (4.3) remains differentiable everywhere. Hence, to finish the proof we just need to show that if the numbers $\mathrm{diam}(K_n)$ are small enough, then F is α-Hölder for every $\alpha \in H$.

Towards this goal, let $A = \sup H$ and choose a non-decreasing sequence $\langle \alpha_n \rangle_n$ in H converging to A such that if $A \in H$, then $\alpha_n = A$ for all $n < \omega$. If $1 \in H$, that is, f is Lipschitz, then no change is necessary. Indeed, by (i), F defined by (4.3) is already Lipschitz[6] hence the property (1.8) implies that $F \restriction [a,b]$ is α-Hölder for every $\alpha \in (0,1)$ and, by Fact 1.5 used with $P = [a,b]$, so is F.

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[6] A Lipschitz differentiable extension version of Jarník's theorem can also be found in [16].

Thus, we may assume that $1 \notin H$ and so, $\alpha_n < 1$ for all $n < \omega$. For every $n < \omega$ decrease the length of K_n so that the resulting F satisfies

$$x|F(x) - F(y)| \leq |x - y|^{\alpha_i} \text{ for every } x, y \in K_n \text{ and } i \leq n. \tag{4.4}$$

To see that this can be done, notice that shrinking of K_n does not change the Lipschitz constant L_n of F on K_n, which is bounded by the sum of the Lipschitz constant of $\bar{f}$ on K_n and the supremum of $|h_n|[K_n]$. Since, for any $x, y \in K_n$, we have that

$$|F(x) - F(y)| \leq L_n|x - y| = L_n|x - y|^{1-\alpha_n}|x - y|^{\alpha_n} \leq L_n \mathrm{diam}(K_n)^{1-\alpha_n}|x - y|^{\alpha_i},$$

it is enough to shrink K_n so that its new diameter D_n satisfies $L_n D_n^{1-\alpha_n} \leq 1$.
To finish the proof, it suffices to show that F defined by (4.3) and satisfying (4.4) is as needed. Indeed, clearly F is everywhere differentiable. Next, fix $\alpha \in H$. It suffices to show that F is α-Hölder. For this, choose $i < \omega$ such that $\alpha \leq \alpha_i$. We claim that

$$F \text{ is } \alpha_i\text{-Hölder.} \tag{4.5}$$

This will be proved by applying Fact 1.4 to F and the set $\tilde{P}$. Let C be such that $\bar{f}$ is α_i-Hölder with a constant C and let $\mathcal{J}_0$ be the set of all $J \in \mathcal{J}$ such that $K_n \subset \mathrm{cl}(J)$ for some $n < i$. Notice that

$$F \restriction \mathrm{cl}(J) \text{ is } \alpha_i\text{-Hölder with a constant } C + 2 \text{ for every } J \in \mathcal{J} \setminus \mathcal{J}_0. \tag{4.6}$$

Indeed, let $J = (p, q)$. Find a $K \in \mathcal{K}$ containing p and put $K^p := K$. Analogously let K^q be the unique $K \in \mathcal{K}$ satisfying $q \in K$. Notice that, by (4.4) and the definition of $\mathcal{J}_0$, F on K^p, as well as on K^q, is α_i-Hölder with a constant 1. Finally, to see (4.6), choose $x_1, x_2 \in [p, q]$ with $x_1 \leq x_2$. We need to show that $|F(x_1) - F(x_2)| \leq (C + 2)|x_1 - x_2|^{\alpha_i}$. If both points x_1 and x_2 are in either K^p or K^q, then this inequality holds. So, assume that this is not the case and let $r := \max\{x_1, \max K^p\}$ and $s := \min\{x_2, \min K^q\}$. Then

$$\begin{aligned}
|F(x_1) - F(x_2)| &\leq |F(x_1) - F(r)| + |F(r) - F(s)| + |F(s) - F(x_2)| \\
&\leq |x_1 - r|^{\alpha_i} + C|r - s|^{\alpha_i} + |s - x_2|^{\alpha_i} \\
&\leq (C + 2)|x_1 - x_2|^{\alpha_i}
\end{aligned}$$

justifying (4.6). Next notice that, for every $J \in \mathcal{J}_0$, $F \restriction \mathrm{cl}(J)$ is Lipschitz, so, by (1.8), it is also α_i-Hölder some constant C_J. Combining this with (4.6), we conclude that the assumption (1.7) of Fact 1.4 is satisfied with a constant $\max\{C + 2, \max_{J \in \mathcal{J}_0} C_J\}$. This completes the proof of (4.5).
Finally, to see that F is α-Hölder, notice that, by (4.5), $F \restriction [a, b]$ is α_i-Hölder. So, by (1.8) and the inequality $\alpha \leq \alpha_i$, $F \restriction [a, b]$ is also α-Hölder. So, by Fact 1.5, F is indeed α-Hölder. ■

5 Final remarks and open problems

Although for $d = 1$ our minimal dynamical system $f_d \colon \mathfrak{C}_d \to \mathfrak{C}_d$ is well defined and, by Lemma 2.2, $\mathfrak{C}_d$ has Hausdorff dimension $d = 1$, it is not clear if this f_d is extremely slow. Specifically, the number p associated with $d = 1$ equals to $p = 2^{-1/d} = \frac{1}{2}$, so that the number $1 - \frac{p}{1-p}$ in the estimation (b) from Section 2.3 becomes 0. This renders the estimate useless. Thus, the following problems remain open.

Problem 5.1. Does there exist an extremely slow minimal dynamical system $f \colon C \to C$ such that $C \subset \mathbb{R}$ is compact and of Hausdorff dimension 1?

Notice that for any $n \geq 2$, if a function $F \colon \mathfrak{C}_d^n \to \mathfrak{C}_d^n$ is defined by a formula $F(\langle x_i \rangle_{i=1}^n) = \langle f_d(x_i) \rangle_{i=1}^n$, then clearly F is an extremely slow dynamical system with $\dim_H(\mathfrak{C}_d^n) \geq n \dim_H(\mathfrak{C}_d) = nd$, see e.g. [17]. Thus, an extremely slow dynamical system on a compact subset of $\mathbb{R}^n$ can have Hausdorff dimension arbitrarily close to n, so greater than 1. However, such defined F is not a minimal dynamical system.
Another interesting question, natural in the context of this work, is Problem 5.2.

Problem 5.2. Does there exist an extremely slow dynamical system f on a compact $C \subset \mathbb{R}$ such that f is Lipschitz?

One might wonder if our function $f_d \colon \mathfrak{C}_d \to \mathfrak{C}_d$ with $d = 1$ can give a positive answer for this question, as f_1 is a minimal dynamical system and, by Lemma 2.2, C_1 has Hausdorff dimension 1. However, f_1 is not 1-Hölder, that is, Lipschitz. Indeed, we can deduce from (b) the following variation of (2.10):

$$\begin{aligned} \frac{|f_1(h_1(s)) - f_1(h_1(t))|}{|h_1(s) - h_1(t)|} &\geq \frac{\left(1 - \frac{p}{1-p}\right) p^{n+\psi(\sigma(s)\restriction n+1)}}{\left(1 + \frac{p}{1-p}\right) p^{n+\psi(s\restriction n+1)}} \\ &= c \cdot p^{\psi(\sigma(s)\restriction n+1)-\psi(s\restriction n+1)}, \end{aligned} \tag{5.1}$$

where $c = \frac{1-\frac{p}{1-p}}{1+\frac{p}{1-p}}$. Now, let $m := \widehat{n+1}$ and assume that $s \restriction m = t \restriction m$ is the sequence of 1's followed by a single 0. Then

$$\psi(\sigma(s) \restriction n+1) - \psi(s \restriction n+1) = (2^m)^2 - (2^{m+1} - 1)^2 \xrightarrow{m\to\infty} -\infty\,,$$

so the lower bound in (5.1) can be arbitrarily large, that is, f_1 is not Lipschitz.
Finally note that, by Theorem 4.1, [2, Question 1.4] may be reduced to the following simpler problem.

Problem 5.3. Can every minimal Cantor dynamical system be embedded to the real line in such a way that it is both extremely slow and α-Hölder for some $0 < \alpha < 1$?

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