Sets on which measurable functions are determined by their range

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Abstract

We study sets on which measurable real-valued functions on a measurable space with negligibles are determined by their range.

1 Introduction

In [BD, theorem 8.5], it is shown that, under the Continuum Hypothesis (CH), in any separable Baire topological space X there is a set M such that for any two continuous real-valued functions f and g on X, if f and g are not constant on any nonvoid open set then $f[M] \subseteq g[M]$ implies $f = g$. In particular, if $f[M] = g[M]$ then $f = g$. Sets having this last property with respect to entire (analytic) functions in the complex plane were studied in [DPR] where they were called *sets of range uniqueness (SRU's)*. We study the properties of such sets in measurable spaces with negligibles. (See below for the definition.) We prove a generalization of the aforementioned result [BD, theorem 8.5] to such spaces (Theorem 4.3) and answer Question 1 from [BD] by showing that CH cannot be omitted from the hypothesis of their theorem (Example 5.17). We also study the descriptive nature of SRU's for the nowhere constant continuous functions on Baire Tychonoff topological spaces.

When $X = \mathbf{R}$, the result of [BD, theorem 8.5] states that, under CH, there is a set $M \subseteq \mathbf{R}$ such that for any two nowhere constant continuous functions $f, g\colon \mathbf{R} \to \mathbf{R}$, if $f[M] \subseteq g[M]$ then $f = g$. It is shown in [BD, theorem 8.1] that there is (in ZFC) a set $M \subseteq \mathbf{R}$ such that for any continuous functions $f, g\colon \mathbf{R} \to \mathbf{R}$, if f has countable level sets and $g[M] \subseteq f[M]$ then g is constant on the connected components of $\{x \in \mathbf{R}\colon f(x) \neq g(x)\}$. In the case where g is the identity function, these properties of M are similar to various properties that have been considered in the literature. Dushnik and Miller [DM] showed that, under CH, there is an uncountable set $M \subseteq \mathbf{R}$ such that for any monotone (nonincreasing or nondecreasing) function $f\colon \mathbf{R} \to \mathbf{R}$, if $\{x \in \mathbf{R}\colon f(x) = x\}$ is nowhere dense, then $f[M] \cap M$ is countable. Building on this result from [DM], Büchi [Bü] showed that, under CH, there is a set $M \subseteq \mathbf{R}$ of cardinality $\mathfrak{c}$ such that for any $X \subseteq M$ and for any Borel function $f\colon X \to M$, $f[\{x \in X\colon f(x) \neq x\}]$ has cardinality less than $\mathfrak{c}$. He calls such sets *totally heterogeneous*. He also observes that if instead of saying "for any Borel function $f\colon X \to M$"

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we say "for any Borel function belonging to collection A" for for special classes A of Borel functions, the use of CH can be avoided. This is true, e.g., for the class of all Borel functions $f\colon \mathbf{R} \to \mathbf{R}$ such that $f[X] \subseteq M$ and for each $y \in \mathbf{R}$, $f^{-1}(\{y\})$ is either countable or of positive Lebesgue measure (a class which contains monotone functions, for example). In the last section of the paper, we shall discuss the relationships between some of these properties and the SRU property for various classes of functions.

2 Preliminaries

Our set-theoretic terminology is standard: see [Ku1] or [Je]. In particular, $\mathbf{R}$, $\mathbf{Q}$ and $\mathbf{Z}$ will stand for the sets of real numbers, rational numbers and integers, respectively. We will write $\mathcal{B}or$ for the Borel σ-algebra of $\mathbf{R}$.

A triple $\langle X, \Sigma, \mathcal{N}\rangle$ is a *measurable space with negligibles* if Σ is a σ-algebra of subsets of X and $\mathcal{N}$ is a proper σ-ideal of subsets of X generated by $\Sigma \cap \mathcal{N}$. (See [F] for the basic properties of such spaces.) By analogy with the case where Σ and $\mathcal{N}$ are respectively the σ-algebra of measurable sets and the σ-ideal of null sets for a measure on X, we will call the elements of Σ *measurable sets* and refer to the elements of $\mathcal{N}$ as *negligible sets*. We will also call the members of $\Sigma \setminus \mathcal{N}$ *positive sets*. (So, in particular, positive sets are measurable.) If Σ and $\mathcal{N}$ are clear from the context, we will write X in places where it would be more appropriate to write $\langle X, \Sigma, \mathcal{N}\rangle$. In particular, when $M \subseteq X$ is not negligible, we shall identify M with the space $\langle M, \Sigma_M, \mathcal{N}_M\rangle$ where $\Sigma_M = \{E \cap M\colon E \in \Sigma\}$ and $\mathcal{N}_M = \{N \cap M\colon N \in \mathcal{N}\}$. We say that $\langle X, \Sigma, \mathcal{N}\rangle$ is $\aleph_1$-*saturated* if every pairwise disjoint family of positive sets is countable. An *atom* of $\langle X, \Sigma, \mathcal{N}\rangle$ is a positive set which does not have two disjoint positive subsets. $\langle X, \Sigma, \mathcal{N}\rangle$ is *nonatomic* if it has no atoms. The *completion* of $\langle X, \Sigma, \mathcal{N}\rangle$ is the space $\langle X, \hat{\Sigma}, \mathcal{N}\rangle$ where $\hat{\Sigma} = \{E \mathbin{\triangle} N\colon E \in \Sigma,\ N \in \mathcal{N}\}$.

Now, let $\langle X, \Sigma, \mathcal{N}\rangle$ be a measurable space with negligibles. A function $f\colon X \to \mathbf{R}$ is Σ-*measurable* (or simply *measurable* if Σ is clear from the context) if $f^{-1}(U) \in \Sigma$ for each open set $U \subseteq \mathbf{R}$. The family of all measurable functions from X into $\mathbf{R}$ will be denoted by $\mathcal{M}_\Sigma(X)$. We will often write $\mathcal{M}(X)$ in place of $\mathcal{M}_\Sigma(X)$ when Σ is clear from the context. If $E \subseteq X$, then $f\restriction E$ denotes the restriction of f to E. For $f, g \in \mathcal{M}(X)$ we write $f \equiv g$ to mean that $\{x \in X\colon f(x) \neq g(x)\}$ is negligible. The *level sets* of $f \in \mathcal{M}(X)$ are the sets $f^{-1}(y)$ for $y \in \mathrm{range}(f)$. We say that f is *nowhere constant* if it is not constant on any positive set, or equivalently, if its level sets are negligible.

Definition 2.1 Let $\langle X, \Sigma, \mathcal{N}\rangle$ be a measurable space with negligibles and let $\mathcal{F} \subseteq \mathcal{M}(X)$ be a family of measurable functions. A set $M \subseteq X$ is *an SRU (a set of range uniqueness) for* $\mathcal{F}$ if whenever $f, g \in \mathcal{F}$ are nowhere constant functions, $f[M] = g[M]$ implies $f \equiv g$. A set M is a *strong SRU for* $\mathcal{F}$ if for any positive set E and any nowhere constant $f, g \in \mathcal{F}$, if $f[M \cap E] \subseteq g[M]$ then $f\restriction E \equiv g\restriction E$. We will frequently have $\mathcal{F} = \mathcal{M}(X)$. It will be convenient, when M is an SRU (resp. a strong SRU) for $\mathcal{M}(X)$, to call M an SRU (resp. a strong SRU) for $\langle X, \Sigma, \mathcal{N}\rangle$ or, when the structure is clear from the context, simply an SRU (resp. a strong SRU).

Note that if $\langle X, \Sigma, \mathcal{N}\rangle$ is a measurable space with negligibles and $\mathcal{F} \subseteq \mathcal{M}(X)$ then every strong SRU for $\mathcal{F}$ is an SRU for $\mathcal{F}$, and if $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{M}(X)$ then any SRU (resp. strong SRU) for $\mathcal{G}$ is also an SRU (resp. a strong SRU) for $\mathcal{F}$.

All topological spaces considered in this paper are assumed to be Tychonoff. A topological space is *Baire* if it satisfies the Baire Category Theorem, i.e., if every meager subset of X has empty interior. A *Cantor set* is a nonvoid zero-dimensional compact metrizable space with no isolated points, i.e., a homeomorphic copy of the Cantor middle third set. If X is a topological space, we write $\mathrm{C}(X)$ for the family of all continuous real-valued functions on X. An s_0-*set* (in X) is a set $S \subseteq X$ with the property that for every Cantor set $P \subseteq X$, there is a Cantor set $Q \subseteq P$ such that $Q \cap S = \emptyset$. (See [Mi2] for the basic facts about s_0-sets.) We will say that $S \subseteq X$ is a *strong* s_0-*set* if $f[S]$ is an s_0-set in $\mathbf{R}$ for every $f \in \mathrm{C}(X)$.

To any Baire topological space X we can associate a natural measurable space with negligibles $\langle X, \Sigma, \mathcal{N}\rangle$, where $\mathcal{N}$ is the σ-ideal of meager subsets of X and Σ is the σ-algebra of Baire subsets of X, i.e., the σ-

algebra generated by the family $\{f^{-1}((a,\infty)): a \in \mathbf{R} \;\&\; f \in \mathrm{C}(X)\}$. When we consider a topological space as a measurable space with negligibles, it is this structure we have in mind, unless otherwise stated. Note that $\mathrm{C}(X) \subseteq \mathcal{M}(X)$. When X is Baire, the term "nowhere constant" applied to a continuous function $f \in \mathrm{C}(X)$ considered as a member of $\mathcal{M}(X)$ coincides with the usual meaning of "not constant on any nonvoid open set." On the few occasions where we consider spaces which are not Baire, we will clarify the meaning of "nowhere constant." (See also Remark 5.16.)

We record the following simple observation for future reference.

Proposition 2.2 *Let X be a Baire topological space. If $M \subseteq X$ is an SRU (strong SRU) for $\mathcal{M}(X)$ then M is an SRU (strong SRU) for $\mathrm{C}(X)$. Moreover, for any $f, g \in \mathrm{C}(X)$ and any positive $E \in \Sigma$,*

$$f \restriction E \equiv g \restriction E \quad \textit{if and only if} \quad f \restriction E = g \restriction E. \qquad \blacksquare$$

Proposition 2.2 will be our main tool for producing SRU's for Baire spaces. However, we shall also see (Example 5.20) that the converse to the first part of Proposition 2.2 can fail.

Finally, note that if in some measurable space with negligibles $\langle X, \Sigma, \mathcal{N}\rangle$ there are no nowhere constant measurable functions $f: X \to \mathbf{R}$ then every subset of X is (vacuously) a strong SRU. This happens, for example, if X has an atom. (See also Corollary 3.8.) Of course, this situation is of no interest and we shall be interested in spaces in which it does not happen.

Definition 2.3 Let $\langle X, \Sigma, \mathcal{N}\rangle$ be a measurable space with negligibles. We shall say that X is *flexible* if there is a nowhere constant measurable function $f: X \to \mathbf{R}$. If X is a Baire topological space then we say that X is *locally flexible* if for every open set $U \subseteq X$ there is a continuous function $f: X \to \mathbf{R}$ which is identically equal to zero outside U and nowhere constant in U.

Note that if $\langle X, \Sigma, \mathcal{N}\rangle$ is the natural measurable space with negligibles associated with some atomless countably additive nontrivial σ-finite measure on X then X is flexible.

The notion of local flexibility is better suited than the notion of flexibility to the study of SRU's the class $\mathrm{C}(X)$. This will be more apparent in Section 5. Notice, however, that there are compact topological spaces that are flexible but not locally flexible. The space $X = [0,1] \times (\omega_1 + 1)$ with its natural topology is one example. Its flexibility is witnessed by the projection onto the first coordinate. The failure of local flexibility is witnessed by the open set $U = [0,1] \times \omega_1$. (See [BS, examples 1 and 2].)

The next proposition is established as part of the proof of [BS, theorem 1]. It shows, for example, that a space with no isolated points which is either separable or metrizable is locally flexible.

Proposition 2.4 *Let X be a space with no isolated points such that either* (i) *X is separable or* (ii) *X is normal and has a dense set which is a countable union of closed discrete sets. Then X is locally flexible.* $\blacksquare$

3 Properties of SRU's for $\mathcal{M}(X)$.

In this section we establish some results concerning the nature of SRU's.

Proposition 3.1 *Let $\langle X, \Sigma, \mathcal{N}\rangle$ be a measurable space with negligibles, and let $M \subseteq X$.*

(a) *Let X be flexible. If $M \subseteq X$ is an SRU (a strong SRU), then M is not empty, and for every positive E, $M \cap E$ is an SRU (a strong SRU) for $\mathcal{M}(E)$. In particular, an SRU must meet every positive subset of X.*

(b) *If M is an SRU and $f, g: X \to \mathbf{R}$ are nowhere constant measurable functions such that $f \not\equiv g$, then $f[M] \triangle g[M]$ is uncountable.*

If M is a strong SRU, E is a positive set, and $f \restriction E \not\equiv g \restriction E$, then $f[M \cap E] \setminus g[M]$ is uncountable.

(c) *If M is an SRU (for $\mathcal{M}(X)$), $N \in \mathcal{N} \cap \Sigma$, and $C \subseteq X$ is countable then $(M \setminus N) \cup C$ is an SRU (for $\mathcal{M}(X)$).*

If M is a strong SRU (for $\mathcal{M}(X)$), $N \in \mathcal{N}$, and $C \subseteq X$ is countable then $(M \setminus N) \cup C$ is a strong SRU (for $\mathcal{M}(X)$).

Proof. (a) Let $f: X \to \mathbf{R}$ be a nowhere constant function from the definition of flexibility of X and let $g: X \to \mathbf{R}$ be given by $g(x) = f(x) + 1$. Then g is a nowhere constant measurable function which does not agree with f anywhere. So clearly M cannot be empty.

Let E be a positive set. We will show that $M \cap E$ is an SRU for $\mathcal{M}(X)$. The proof for the strong SRU case is similar. Let $h_1, h_2: E \to \mathbf{R}$ be nowhere constant measurable functions such that $h_1[M \cap E] = h_2[M \cap E]$. For $i = 1, 2$, extend h_i to X by letting $h_i(x) = f(x)$ when $x \in X \setminus E$. Then $h_1[M] = h_2[M]$ and hence $h_1 \equiv h_2$. In particular, $h_1 \restriction E \equiv h_2 \restriction E$.

(b) We prove the first statement; the second is proven similarly. By way of contradiction suppose that $C = f[M] \triangle g[M]$ is countable. By (a), M is not negligible, and hence $f[M]$ and $g[M]$ are uncountable. So, we can choose $y \in f[M] \cap g[M]$. Define $\bar{f}, \bar{g}: X \to \mathbf{R}$ as follows. $\bar{f}(x) = f(x)$ if $x \notin f^{-1}(C)$, and $\bar{f}(x) = y$ if $x \in f^{-1}(C)$. Similarly, $\bar{g}(x) = g(x)$ if $x \notin g^{-1}(C)$, and $\bar{g}(x) = y$ if $x \in g^{-1}(C)$. We have $\bar{f} \equiv f \not\equiv g \equiv \bar{g}$. Thus, it is enough to show that $\bar{f}[M] = \bar{g}[M]$, since this contradicts that M is SRU for $\mathcal{M}(X)$.

By symmetry, it is enough to prove that $\bar{f}[M] \subseteq \bar{g}[M]$. So, let $x \in M$. If $x \in f^{-1}(C)$ then $\bar{f}(x) = y \in g[M] \setminus C = \bar{g}[M]$. If $x \notin f^{-1}(C)$ then $\bar{f}(x) = f(x) \in f[M] \setminus C = g[M] \setminus C = \bar{g}[M]$. Thus, $\bar{f}[M] \subseteq \bar{g}[M]$.

(c) First we will prove this in the case when $C = \emptyset$.

Let $f, g: X \to \mathbf{R}$ be nowhere constant measurable functions. Suppose M is an SRU, N is a measurable negligible set, and $f[M \setminus N] = g[M \setminus N]$. Let $\bar{f}, \bar{g}: X \to \mathbf{R}$ be the functions which agree on $X \setminus N$ with f, g respectively, and are identically equal to 0 on N. Then $\bar{f}, \bar{g} \in \mathcal{M}(X)$ and $\bar{f}[M] = \bar{g}[M]$. Since M is an SRU, we have $\bar{f} \equiv \bar{g}$ and hence $f \equiv g$. Thus $M \setminus N$ is an SRU for $\mathcal{M}(X)$.

Next suppose M is a strong SRU for $\mathcal{M}(X)$, N is negligible, E is a positive set and $f[(M \setminus N) \cap E] \subseteq g[M \setminus N]$. Choose $N' \in \Sigma \cap \mathcal{N}$ such that $N \subseteq N'$. Then $f[M \cap (E \setminus N')] \subseteq f[(M \setminus N) \cap E] \subseteq g[M \setminus N] \subseteq g[M]$. Since $E \setminus N'$ is positive and M is a strong SRU for $\mathcal{M}(X)$, we have $f \restriction [E \setminus N'] \equiv g \restriction [E \setminus N']$ and hence $f \restriction E \equiv g \restriction E$. Thus $M \setminus N$ is a strong SRU for $\mathcal{M}(X)$.

Now, to prove the general case we can assume by what we proved above that $N = \emptyset$. But then, the desired result follows easily from (b). ■

Proposition 3.1(c) suggests that possibly $M \triangle N$ is an SRU (strong SRU) for X if $N \in \mathcal{N} \cap \Sigma$ and M is an SRU (strong SRU) for X. However, this is false already for $X = \mathbf{R}$, since if $K \subseteq \mathbf{R}$ is any Cantor set, then $M \cup K$ is not an SRU for $\mathrm{C}(\mathbf{R})$ for any strong SRU for $\mathcal{M}(\mathbf{R})$. (See Theorem 5.6(5).) However, we do not know the answer to the following question.

Problem 3.2 If X is a Baire topological space, is Proposition 3.1(c) true with $\mathcal{M}(X)$ replaced by $\mathrm{C}(X)$?

(It is consistent that there is a strong SRU for $\mathrm{C}(\mathbf{R})$ which is not an SRU for $\mathcal{M}(\mathbf{R})$, as we will show in Example 5.20.)

The next proposition gives special circumstances in which no SRU can exist. It will be useful later.

Proposition 3.3 *Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a measurable space with negligibles. In any of the following circumstances, there is no SRU for $\mathcal{M}(X)$.*

(a) *Σ is the collection of all subsets of X, the cardinality of X is at most $\mathfrak{c}$, and X is nonatomic.*

(b) *X is flexible, $\mathfrak{c}$ is regular, $\mathcal{N} \subseteq \Sigma$, $\mathcal{N}$ contains all sets of cardinality less than $\mathfrak{c}$, and X is covered by less than $\mathfrak{c}$ negligible sets.*

(c) *There are more than $2^{\mathfrak{c}}$ pairwise nonequivalent (modulo $\mathcal{N}$) nowhere constant measurable functions $X \to \mathbf{R}$.*

Proof. If (a) holds, we argue as follows. Since X is nonatomic, singletons are negligible. Thus any one-to-one function $f: X \to \mathbf{R}$ is a nowhere constant measurable function. Clearly there are such functions since X has cardinality at most $\mathfrak{c}$. Thus, X is flexible. Suppose $M \subseteq X$ were an SRU. By Proposition 3.1(a), M is not negligible. Fix any one-to-one function $f: X \to \mathbf{R}$. Write M as the union of two disjoint sets A and B of equal cardinality and find a bijection $h: X \to X$ such that $h[A] = B$ and $h[B] = A$. Define $g: X \to \mathbf{R}$ by $g = f \circ h$. Then $f[M] = g[M]$ but $f \not\equiv g$ and hence M is not an SRU.

If (b) holds, suppose $M \subseteq X$ were an SRU. Since sets of cardinality less than $\mathfrak{c}$ are negligible, M has cardinality at least $\mathfrak{c}$. Since X is covered by less than $\mathfrak{c}$ negligible sets and $\mathfrak{c}$ is regular, there is a negligible set $N \subseteq M$ of cardinality $\mathfrak{c}$. Let $h: N \to \mathbf{R}$ be any surjection. Since X is flexible and N is measurable, there are many distinct nowhere constant measurable extensions of h to X. All of these extensions map M onto $\mathbf{R}$, and hence M is not an SRU.

If (c) holds, then for any $M \subseteq X$ there are, by the pigeonhole principle, two nonequivalent nowhere constant measurable functions $f, g: X \to \mathbf{R}$ such that $f[M] = g[M]$. Hence M is not an SRU. ■

To state the next result we need the following definition.

Definition 3.4 Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a measurable space with negligibles. A set $E \in \Sigma$ is a *measurable cover of* $M \subseteq X$ if $M \subseteq E$ and every positive subset of E meets M.

Lemma 3.5 *Let $M \subseteq X$ and let $f: M \to \mathbf{R}$ be a measurable function.*

(a) *The function f extends to a measurable function $\bar{f}: X \to \mathbf{R}$.*

(b) *Suppose M has a measurable cover. If f is nowhere constant in M, and X is flexible, then f extends to a nowhere constant measurable function $\bar{f}: X \to \mathbf{R}$.*

Proof. (a) Of course this is well-known. Here is a sketch of the proof. For each rational number q, let $E_q \in \Sigma$ be such that $\{x \in M: f(x) < q\} = E_q \cap M$. For $x \in E = \bigcup\{E_q: q \in \mathbf{Q}\}$, we let $\bar{f}(x) = \inf\{q \in \mathbf{Q}: x \in E_q\}$ and for $x \in X \setminus E$ we let $\bar{f}(x) = 0$. It is straightforward to check that $\bar{f}$ is as desired.

(b) Fix a nowhere constant measurable function $g: X \to \mathbf{R}$. First notice that every positive subset of E must have a nonnegligible intersection with M. This is the case, since otherwise there exists a positive $F \subseteq E$ and $N \in \Sigma \cap \mathcal{N}$ with $F \cap M \subseteq N$ and then $F \setminus N$ is a positive subset of E which does not meet M.

Extend f to E using (a). By the above remark the extension of f to E is necessarily nowhere constant. Now extend f to X by letting it agree with g outside of E. ■

We leave the straightforward proof of the following consequence of Lemma 3.5(b) to the reader.

Corollary 3.6 *Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a flexible measurable space with negligibles and suppose that every subset of X has a measurable cover. If $M \subseteq X$ is an SRU (a strong SRU) for $\mathcal{M}(X)$, then M is also an SRU (a strong SRU) for $\mathcal{M}(M)$.* ■

The next proposition and its corollary show that in an $\aleph_1$-saturated measurable spaces with negligibles, the assumption of flexibility is not very restrictive when we are considering SRU's for $\mathcal{M}(X)$.

Recall that a *Souslin algebra* is a ccc nonatomic complete Boolean algebra in which the intersection of any countable collection of dense open sets is a dense open set (or equivalently, every countable collection of maximal antichains has a common refinement). See [Je, pp. 220, 274] for the basic properties of these algebras.

Proposition 3.7 *Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a nonatomic, $\aleph_1$-saturated, measurable space with negligibles. Then X is flexible if and only if $\Sigma_E/\mathcal{N}_E$ is not a Souslin algebra for any positive E. In particular, if Souslin's Hypothesis is true then X is flexible.*

Proof. Suppose $\Sigma_E/\mathcal{N}_E$ is a Souslin algebra for some positive $E \in \Sigma$. Let $f: E \to \mathbf{R}$ be measurable. For each $q \in \mathbf{Q}$, let $\mathcal{A}_q$ be the collection of positive subsets A of E such that $A \subseteq f^{-1}(-\infty, q)$ or $A \subseteq f^{-1}(q, \infty)$. If we denote by $\mathcal{A}_q^\bullet$ the image of $\mathcal{A}_q$ in $\Sigma_E/\mathcal{N}_E$, then $\mathcal{A}_q^\bullet$ is dense open in $\Sigma_E/\mathcal{N}_E$. Since $\Sigma_E/\mathcal{N}_E$ is a Souslin algebra, there is an $A \in \bigcap_{q \in \mathbf{Q}} \mathcal{A}_q$. Clearly f is constant on A, and thus E (and hence X as well) is not flexible.

Suppose, conversely, that none of the algebras $\Sigma_E/\mathcal{N}_E$ is a Souslin algebra. Since $\Sigma/\mathcal{N}$ is ccc, to show that X is flexible it suffices to show that for each positive set E there is a positive $E' \subseteq E$ which is flexible. So fix a positive set E, and let $\mathcal{A}_n \subseteq \Sigma_E \setminus \mathcal{N}_E$, $n < \omega$, be partitions of E which have no common refinement. Give each $\mathcal{A}_n$ the discrete topology and let $f: E \to \Pi_n \mathcal{A}_n$ be given by $f(x) =$ the unique sequence in $\Pi_n \mathcal{A}_n$ whose n-th term contains x for each n. Clearly f is measurable and its range is homeomorphic to a subspace of $\mathbf{R}$. Let $\mathcal{A} = \{f^{-1}(y): y \in \Pi_n \mathcal{A}_n,\ f^{-1}(y) \text{ positive}\}$. $\mathcal{A}$ is an antichain which refines each $\mathcal{A}_n$, and hence $\mathcal{A}$ is not maximal. Thus $E' = E \setminus \bigcup \mathcal{A}$ is positive and is flexible since $f \restriction E'$ is nowhere constant. ■

The following structural result follows easily from the previous proposition.

Corollary 3.8 *If $\langle X, \Sigma, \mathcal{N} \rangle$ is an $\aleph_1$-saturated measurable space with negligibles, then X admits an essentially unique decomposition into three pieces $X = A \cup S \cup F$, where A is a countable union of atoms, $\Sigma_S/\mathcal{N}_S$ is a Souslin algebra if S is not empty, and F is either empty or flexible.* ■

4 Existence of SRU's for $\mathcal{M}(X)$

The next lemma states that sufficiently generic generalized Lusin sets are strong SRU's. Since the conditions given here will be used several times in the rest of the paper, we make the following definition.

Definition 4.1 Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a measurable space with negligibles and let κ be an uncountable cardinal. A set $L \subseteq X$ is called a *κ-strong SRU* (for X) if for each pair $f, g \in \mathcal{M}_\Sigma(X)$ of nowhere constant functions there exists a subset $C_{f,g}$ of L satisfying the following properties.

(i) $\mathrm{Card}(L \cap E \setminus f^{-1}(g[C_{f,g}])) \geq \kappa$ for each positive set E, and $\mathrm{Card}(L \cap N) < \kappa$ for each negligible set N.

(ii) $\mathrm{Card}(C_{f,g}) < \kappa$, and $f(x) \neq g(y)$ for every distinct $x, y \in L \setminus C_{f,g}$.

Notice that if κ is regular, then we can drop the "$\setminus f^{-1}(g[C_{f,g}])$" in clause (i) of Definition 4.1, as it is taken care of by the other parts of clauses (i) and (ii). (The other parts of (i) and (ii) imply that $L \cap f^{-1}(g[C_{f,g}]) = \bigcup\{L \cap f^{-1}(y): y \in g[C_{f,g}]\}$ is the union of less than κ many sets of cardinality less than κ.)

Lemma 4.2 *Let $\langle X, \Sigma, \mathcal{N} \rangle$ be a measurable space with negligibles and κ be an uncountable cardinal. If $L \subseteq X$ is a κ-strong SRU, then for any nowhere constant functions $f, g \in \mathcal{M}_{\hat{\Sigma}}(X)$ and any positive set E such that $f \restriction E \not\equiv g \restriction E$, $f[L \cap E] \setminus g[L]$ has cardinality at least κ. In particular L is a strong SRU for $\mathcal{M}_{\hat{\Sigma}}(X)$.*

Proof. For each pair $u, v \in \mathcal{M}_\Sigma(X)$ of nowhere constant functions fix a set $C_{u,v}$ as given by Definition 4.1. Let f, g, E be as in the hypothesis of the lemma. Let N be a negligible set on whose complement f and g agree with Σ-measurable functions $\bar{f}$ and $\bar{g}$ respectively. By shrinking E we can assume that $E \cap N = \emptyset$ and $E \subseteq \{x \in X: f(x) \neq g(x)\}$. But then E is the countable union of the measurable sets $\{x \in E: f(x) < q < g(x)\}$ and $\{x \in E: g(x) < q < f(x)\}$, where $q \in \mathbf{Q}$. So, at least one of these sets is positive, and shrinking E even further we can assume that $f[E] \cap g[E] = \emptyset$.

Now, since L is a κ-strong SRU, the set $K = (L \cap E) \setminus \left(\bar{f}^{-1}(\bar{g}[C_{\bar{f},\bar{g}}]) \cup C_{\bar{f},\bar{g}} \cup C_{\bar{f},\bar{f}}\right)$ has cardinality at least κ. Since $K \subseteq L \setminus C_{\bar{f},\bar{f}}$, $\bar{f}$ is one-to-one on K and hence $\bar{f}[K]$ has cardinality at least κ. We have also $\emptyset = \bar{f}[K] \cap \bar{g}[L] = f[K] \cap \bar{g}[L]$, since $K \subset E$ is disjoint from $N \cup \bar{f}^{-1}(\bar{g}[C_{\bar{f},\bar{g}}]) \cup C_{\bar{f},\bar{g}}$. Hence, $f[K] \cap g[L] \subseteq g[L \cap N]$ has cardinality less than κ. So, $f[L \cap E] \setminus g[L] \supset f[K] \setminus (f[K] \cap g[L])$ has cardinality at least κ. ■

Theorem 4.3 *Let $\langle X, \Sigma, \mathcal{N}\rangle$ be a measurable space with negligibles. Suppose that Σ has cardinality $\mathfrak{c}$ and no positive set can be covered by less than $\mathfrak{c}$ negligible sets. Then there is a $\mathfrak{c}$-strong SRU for X. In particular, there is a strong SRU for $\langle X, \hat{\Sigma}, \mathcal{N}\rangle$.*

Proof. Let $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be a list of all nowhere constant functions from $\mathcal{M}_\Sigma(X)$, let $\langle E_\alpha : \alpha < \mathfrak{c}\rangle$ be a list of all positive sets in which each set is listed $\mathfrak{c}$ times and let $\langle N_\alpha : \alpha < \mathfrak{c}\rangle$ be a list of $\mathcal{N} \cap \Sigma$.

Inductively choose points $x_\alpha \in X$ as follows. Let $S_\alpha = \{x_\beta : \beta < \alpha\}$. Since the level sets of f_α are negligible, and E_α cannot be covered by less than $\mathfrak{c}$ negligible sets, there is a point $x_\alpha \in E_\alpha$ which does not belong to any sets of the form N_β or $f_\beta^{-1}(f_\gamma(x_\delta))$ with $\beta, \gamma \le \alpha$ and $\delta < \alpha$.

We shall show that $L = \{x_\alpha : \alpha < \mathfrak{c}\}$ is a $\mathfrak{c}$-strong SRU with witnessing sets given by $C_{f,g} = S_\alpha$ for any $\alpha < \mathfrak{c}$ such that $f, g \in \{f_\beta : \beta \le \alpha\}$. First notice that $x_\alpha \ne x_\beta$ for every $\beta < \alpha < \mathfrak{c}$, since $x_\alpha \notin f_\alpha^{-1}(f_\alpha(x_\beta))$.

To see clause (ii) of Definition 4.1, suppose that x and y are distinct elements of $L \setminus S_\alpha$. Thus $x = x_\beta$, $y = x_\gamma$ for some distinct $\beta, \gamma > \alpha$, say $\beta < \gamma$. Then $x_\gamma \notin g^{-1}(f(x_\beta))$ and hence $f(x) = f(x_\beta) \ne g(x_\gamma) = g(y)$.

Clause (i) of Definition 4.1 now follows easily. ■

Corollary 4.4 *Consider the measurable space with negligibles $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$. If either*

(i) *$\mathcal{N}$ is the σ-ideal of null sets in $\mathbf{R}$ and the union of less than $\mathfrak{c}$ many null sets does not cover $\mathbf{R}$; or*

(ii) *$\mathcal{N}$ is the σ-ideal of meager sets in $\mathbf{R}$ and the union of less than $\mathfrak{c}$ many meager sets does not cover $\mathbf{R}$,*

then there exists a strong SRU for the completion of $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$. ■

Note that the conclusion of Corollary 4.4 cannot be proved in ZFC for either of the ideals. This follows from Proposition 3.3(b), since in the Cohen model every subset of $\mathbf{R}$ of cardinality $< \mathfrak{c}$ is null and $\mathbf{R}$ is the union of less then $\mathfrak{c}$ many null sets, while in the random real model every subset of $\mathbf{R}$ of cardinality $< \mathfrak{c}$ is meager and $\mathbf{R}$ is the union of less then $\mathfrak{c}$ many meager sets. (See e.g., [Mi1].) But what about the existence of a strong SRU, or an SRU, just for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$, with $\mathcal{N}$ as in the corollary?

To answer this question we will need the following theorem of P. Corazza [C].

Proposition 4.5 [C] *It is consistent with ZFC that $\mathfrak{c} = \omega_2$, every subset of $\mathbf{R}$ of cardinality less than $\mathfrak{c}$ has strong (so Lebesgue) measure zero, and for every subset M of $\mathbf{R}$ cardinality $\mathfrak{c}$ there is a uniformly continuous function $f : \mathbf{R} \to [0,1]$ such that $f[M] = [0,1]$.* ■

From this we can easily deduce the following.

Corollary 4.6 *It is consistent with ZFC that there is no SRU for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$, where $\mathcal{N}$ is the σ-ideal of null sets in $\mathbf{R}$.*

Proof. This happens in the model from Proposition 4.5. First recall, that in this model the real line is covered by less than $\mathfrak{c}$ sets of measure zero, i.e., $\mathbf{R} = \bigcup_{\xi<\omega_1} N_\xi$ for some null sets N_ξ. (See e.g. [C] or [Mi3].) Now, if M were an SRU then, by Proposition 3.1(a), M cannot be null. So, it has cardinality $\mathfrak{c}$. Hence, there is $\xi < \omega_1$ such that $M' = M \cap N_\xi \in \mathcal{N}$ has cardinality $\mathfrak{c}$. Let $h : \mathbf{R} \to [0,1]$ be a uniformly continuous map such that $h[M'] = [0,1]$. There are many pairwise nonequivalent extensions of $h \restriction M'$ to nowhere constant Borel functions from $\mathbf{R}$ into $[0,1]$. Since these extensions all have the same image of M, M is not an SRU, contradiction. ■

We do not know whether there is a model of ZFC in which there is no SRU for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$, where $\mathcal{N}$ is the σ-ideal of meager sets. (See Problem 5.18.) However, in models where every set of reals of cardinality $\mathfrak{c}$ maps uniformly continuously onto $[0,1]$, there is no SRU for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$ of cardinality $\mathfrak{c}$. (See Theorem 5.6(5) and the comments before Lemma 5.5.) On the other hand, the following observations, that arose in conversations with S. Todorcevic, show that there is an $\aleph_1$-strong SRU for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$ in the models from [C] and [Mi3].

For any set L, let $L^{(2)} = \{\langle a, b\rangle \in L^2 : a \ne b\}$.

Definition 4.7 [To, §6] Let $\mathcal{N}$ be the ideal of subsets of $\mathbf{R}$ consisting either of the meager sets or of the sets of Lebesgue measure zero. A set $L \subseteq \mathbf{R}$ is called 2-*Lusin* for $\mathcal{N}$ if L is uncountable, but for every $N \in \mathcal{N}$, the set $N \cap L^{(2)}$ does not contain an uncountable disjoint set, where two ordered pairs $\langle a, b\rangle$ and $\langle c, d\rangle$ are *disjoint* if $\{a, b\} \cap \{c, d\} = \emptyset$.

Proposition 4.8 *Consider the measurable space with negligibles $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$ where $\mathcal{N}$ is either the ideal of meager sets or the ideal of sets of Lebesgue measure zero. If L is a 2-Lusin set for $\mathcal{N}$ which has uncountable intersection with every positive set, then L is an $\aleph_1$-strong SRU for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$.*

Proof. Let L be a 2-Lusin set and let $f, g\colon \mathbf{R} \to \mathbf{R}$ be nowhere constant Borel functions. We claim there is a countable set $C_{f,g}$ such that $f(x) \neq g(y)$ for any distinct $x, y \in L \setminus C$. (Then the rest of Definition 4.1 is easily checked.) By way of contradiction assume that this is not the case. Inductively choose distinct points $x_\alpha, y_\alpha \in L \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$ such that $f(x_\alpha) = g(y_\alpha)$. The set $F = \{\langle x, y\rangle \in \mathbf{R}^2\colon f(x) = g(y)\}$ is Borel and contains all the points $\langle x_\alpha, y_\alpha\rangle$. Since L is 2-Lusin, it follows that F is not negligible. By Fubini's theorem (by which we mean [Ox, theorem 15.4] if $\mathcal{N}$ is the meager ideal), there is an $x \in \mathbf{R}$ such that $F_x = \{y \in \mathbf{R}\colon \langle x, y\rangle \in F\}$ is not negligible. But $g^{-1}(f(x)) = F_x$, contradicting the fact that g is nowhere constant. ∎

Under CH there is a 2-Lusin set having uncountable intersection with every positive set (by a minor modification of the proof of [To, Proposition 6.0]). It now follows from Proposition 4.8 that there is an $\aleph_1$-strong SRU's for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$ in the iterated perfect set model for $\mathcal{N}$ equal to either the meager or the null ideal, and in the model of [C] for $\mathcal{N}$ equal to the meager ideal. The point is that in all of these situations, (i) the ground model coded negligible Borel subsets of the plane are cofinal in the ideal of negligible subsets of the plane in the extension and (ii) every positive Borel subset in the extension contains a ground model coded positive Borel set, and hence a ground model set which is 2-Lusin and has uncountable intersection with every positive set retains these properties in the extension.

We now give a version of Theorem 4.3 for several measurable spaces with negligibles simultaneously.

Theorem 4.9 *Let $\langle X, \Sigma, \mathcal{N}_i\rangle$, $i \in \mathbf{N}$, be $\aleph_1$-saturated measurable spaces with negligibles. Suppose that Σ has cardinality $\mathfrak{c}$ and, for each $i \in \mathbf{N}$, no set in $\Sigma \setminus \mathcal{N}_i$ can be covered by less than $\mathfrak{c}$ members of $\mathcal{N}_i$. Then there is a set $M \subseteq X$ which is a strong SRU for all the spaces $\langle X, \Sigma, \mathcal{N}_i\rangle$ simultaneously.*

Proof. Let $\langle\langle i_\alpha, f_\alpha, g_\alpha, E_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$ be a list of all quadruples $\langle i, f, g, E\rangle$ where $i \in \mathbf{N}$, $f, g\colon X \to \mathbf{R}$ are nowhere constant measurable functions with respect to $\langle X, \Sigma, \mathcal{N}_i\rangle$, $E \in \Sigma \setminus \mathcal{N}_i$, and $f[E] \cap g[E] = \emptyset$.

Inductively choose points $x_\alpha \in X$ so that $g_\alpha^{-1}(f_\alpha(x_\alpha)) \in \bigcap_{i\in\mathbf{N}} \mathcal{N}_i$, as follows. The set $C_{g_\alpha} = \{y \in \mathbf{R}\colon g_\alpha^{-1}(y) \notin \bigcap_{i\in\mathbf{N}} \mathcal{N}_i\}$ is countable since each $\langle X, \Sigma, \mathcal{N}_i\rangle$ is $\aleph_1$-saturated. Hence $f_\alpha^{-1}(C_{g_\alpha}) \in \mathcal{N}_{i_\alpha}$. The sets $g_\beta^{-1}(f_\beta(x_\beta))$, for $\beta < \alpha$ are in $\bigcap_{i\in\mathbf{N}} \mathcal{N}_i$ by the induction hypothesis. The sets $f_\alpha^{-1}(g_\alpha(x_\beta))$, for $\beta < \alpha$, are in $\mathcal{N}_{i_\alpha}$ since f_α is nowhere constant. Thus we may choose a point $x_\alpha \in E_\alpha$ which avoids all these less that $\mathfrak{c}$ members of $\mathcal{N}_{i_\alpha}$. We have $g_\alpha^{-1}(f_\alpha(x_\alpha)) \in \bigcap_{i\in\mathbf{N}} \mathcal{N}_i$, since $x_\alpha \notin f_\alpha^{-1}(C_{g_\alpha})$.

Now let $i < \mathbf{N}$ and, with respect to $\langle X, \Sigma, \mathcal{N}_i\rangle$, let $f, g\colon X \to \mathbf{R}$ be nowhere constant measurable functions and $E \subseteq X$ a positive set such that $f{\restriction}E \not\equiv g{\restriction}E$. By shrinking E we may assume that $f[E] \cap g[E] = \emptyset$. Now, let $\alpha < \mathfrak{c}$ be such that $i_\alpha = i$, $E_\alpha = E$, $f_\alpha = f$ and $g_\alpha = g$. It is straightforward to verify that $f(x_\alpha) \notin g[M]$. ∎

Remark 4.10 Consider the spaces $\langle X, \Sigma, \mathcal{N}\rangle$ where $X = \mathbf{R}$, $\Sigma = \mathcal{B}or$, and $\mathcal{N}$ is either the ideal of meager sets or the ideal of sets of Lebesgue measure zero.

(a) It follows from Theorem 4.9 that if $\mathbf{R}$ can be covered neither by less than $\mathfrak{c}$ meager sets, nor by less than $\mathfrak{c}$ sets of measure zero then there is a set $M \subseteq X$ which is a strong SRU simultaneously for both spaces under consideration.

(b) Another way to get a set which is a strong SRU for both spaces simultaneously, is to force with an $\aleph_1$-stage finite support iteration $\langle P_\alpha, Q_\alpha\colon \alpha < \omega_1\rangle$ where Q_α is a P_α-name for Cohen forcing if α is even and for random forcing if α is odd. The generic set of reals $M = \{r_\alpha : \alpha < \omega_1\}$ is a strong SRU for both spaces simultaneously. Since there are Lusin and Sierpiński sets in this model, the covering assumption of Theorem 4.9 fails for both ideals if $\mathfrak{c} > \aleph_1$. The proof that M is a strong SRU is left as an exercise for the reader.

(c) A set which is a strong SRU for both spaces simultaneously satisfies the following stronger property. Let $f, g: \mathbf{R} \to \mathbf{R}$ be Borel functions. If $f[M] \triangle g[M]$ is countable, then there is a Borel set E such that $f[\mathbf{R} \setminus E]$ and $g[\mathbf{R} \setminus E]$ are countable and $f(x) = g(x)$ for all $x \in E$ except for x belonging to a meager set of measure zero. (Compare this with Theorem 4.13.)

(d) No set $M \subseteq \mathbf{R}$ can be an SRU for the completion of both spaces simultaneously. The reason is that there is a set $H \subseteq \mathbf{R}$ of measure zero and whose complement is meager. One of $M \cap H$, $M \cap (\mathbf{R} \setminus H)$ would have the same cardinality as M, say the former. Then given any two nonequivalent Lebesgue measurable functions $f, g: \mathbf{R} \to \mathbf{R}$, we can easily modify $f \restriction (M \cap H)$ and $g \restriction (M \cap H)$ to arrange $f[M] = g[M]$.

The next example shows that an SRU need not be a strong SRU. (The assumptions on the ideal $\mathcal{N}$ are satisfied by the ideal of countable sets. They are also consistently satisfied by the meager and null ideals.)

Example 4.11 *Consider a measurable space with negligibles $\langle X, \Sigma, \mathcal{N}\rangle$ in which $X = \mathbf{R}$, Σ is the Borel σ-algebra of $\mathbf{R}$ and $\{-x : x \in N\} \in \mathcal{N}$ for every $N \in \mathcal{N}$. Assume that singletons are negligible and no positive set can be covered by less than $\mathfrak{c}$ negligible sets. Then there is a set $M \subseteq \mathbf{R}$ which is an SRU for $\langle X, \hat{\Sigma}, \mathcal{N}\rangle$ and such that $\{|x| : x \in M\} \subseteq M$. In particular, M is not a strong SRU for the piecewise linear functions.*

Proof. Clearly, such a set M is not a strong SRU, for if we let $f(x) = |x|$ and $g(x) = x$ for every $x \in \mathbf{R}$, we have that f and g are nowhere constant (since $\mathcal{N}$ contains singletons), $f[M] \subseteq g[M]$, and $f \not\equiv g$.

Note that since $\mathcal{N}$ is a proper ideal, $\mathbf{R}$ is not negligible. Thus, from the symmetry assumption on $\mathcal{N}$, neither $(-\infty, 0)$ nor $(0, \infty)$ is negligible. Let $h: \mathbf{R} \to \mathbf{R}$ be given by $h(x) = -x$ for all $x \in \mathbf{R}$. To construct M, let $\mathcal{F}$ be the family of all triples $\langle f, g, E\rangle$ such that f and g are nowhere constant Borel functions, E is a positive Borel set, $f[E] \cap g[E] = \emptyset$ and either

(i) $E \subseteq (0, \infty)$, or

(ii) $E \subseteq (-\infty, 0)$ and $f[E] \cap g[-E] = \emptyset$, where $-E = \{-x : x \in E\}$.

Let $\langle\langle f_\alpha, g_\alpha, E_\alpha\rangle : \alpha < \mathfrak{c}\rangle$ list all elements of $\mathcal{F}$ with each triple appearing $\mathfrak{c}$ many times and let $\langle N_\alpha : \alpha < \mathfrak{c}\rangle$ be a list of all negligible Borel sets.

By induction on $\alpha < \mathfrak{c}$ define a sequence $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ of real numbers such that

$$x_\alpha \in E_\alpha \setminus \bigcup_{\beta<\alpha} \left[N_\beta \cup f_\alpha^{-1}(f_\alpha(x_\beta)) \cup f_\alpha^{-1}(g_\alpha(\pm x_\beta)) \cup \pm[f_\beta^{-1}(g_\beta(x_\beta))]\right].$$

Such a choice can be made, since each set $N_\beta \cup f_\alpha^{-1}(f_\alpha(x_\beta)) \cup f_\alpha^{-1}(g_\alpha(\pm x_\beta)) \cup \pm[f_\beta^{-1}(g_\beta(x_\beta))]$ is negligible and no positive set is the union of less than $\mathfrak{c}$ many negligible sets.

Let $M = \{x_\alpha : \alpha < \mathfrak{c}\} \cup \{|x_\alpha| : \alpha < \mathfrak{c}\}$. Clearly $\{|x| : x \in M\} \subseteq M$.

To see that M is an SRU for $\langle X, \hat{\Sigma}, \mathcal{N}\rangle$ let $f, g \in \mathcal{M}_{\hat{\Sigma}}(\mathbf{R})$ be nonequivalent nowhere constant. Let N be a negligible Borel set on whose complement f and g agree with Σ-measurable functions $\bar{f}$ and $\bar{g}$ respectively. In particular, $\bar{f}$ and $\bar{g}$ are $\mathcal{N}$-nowhere constant and not equivalent. Note that

there is a positive Borel set $E \subseteq X \setminus N$ such that either $\langle \bar{f}, \bar{g}, E\rangle \in \mathcal{F}$ or $\langle \bar{g}, \bar{f}, E\rangle \in \mathcal{F}$.

It is clear if $\bar{f} \restriction (0, \infty) \not\equiv \bar{g} \restriction (0, \infty)$ since then $\langle \bar{f}, \bar{g}, E\rangle \in \mathcal{F}$ for any positive Borel set $E \subseteq (0, \infty) \setminus N$ such that $\bar{f}[E] \cap \bar{g}[E] = \emptyset$. So, assume that $\bar{f} \restriction (0, \infty) \equiv \bar{g} \restriction (0, \infty)$. Then $\bar{f} \restriction (-\infty, 0) \not\equiv \bar{g} \restriction (-\infty, 0)$. Choose a positive Borel set $E \subseteq (-\infty, 0) \setminus N$ such that $\bar{f}[E] \cap \bar{g}[E] = \emptyset$. By shrinking E, if necessary, we can also assume that $(\bar{f} \circ h)\restriction E = (\bar{g} \circ h)\restriction E$, since $\bar{f}\restriction(0,\infty) \equiv \bar{g}\restriction(0,\infty)$. Now either $\bar{f}\restriction E \not\equiv (\bar{f} \circ h)\restriction E$ or $\bar{g}\restriction E \not\equiv (\bar{g} \circ h)\restriction E$, since otherwise we would have $\bar{f}\restriction E \equiv (\bar{f} \circ h)\restriction E = (\bar{g} \circ h)\restriction E \equiv \bar{f}\restriction E$. Assume the former case, the other being similar. Then we can shrink E further to arrange $\bar{f}[E] \cap (\bar{f} \circ h)[E] = \emptyset$. But $(\bar{f} \circ h)[E] = (\bar{g} \circ h)[E] = \bar{g}[-E]$. So, $f[E] \cap g[-E] = \emptyset$ and $\langle \bar{f}, \bar{g}, E\rangle \in \mathcal{F}$.

By symmetry we can assume that $\langle \bar{f}, \bar{g}, E\rangle \in \mathcal{F}$. Now, let $\alpha < \mathfrak{c}$ be such that $\langle \bar{f}, \bar{g}, E\rangle = \langle f_\alpha, g_\alpha, E_\alpha\rangle$. Then, $\bar{f}(x_\alpha) = f_\alpha(x_\alpha) \notin g_\alpha[M] = \bar{g}[M]$, because of the following.

– $f_\alpha(x_\alpha) \neq g_\alpha(\pm x_\beta)$ for $\beta < \alpha$ since $x_\alpha \notin f_\alpha^{-1}(g_\alpha(\pm x_\beta))$.

– $f_\alpha(x_\alpha) \neq g_\alpha(x_\alpha)$ since $x_\alpha \in E_\alpha$ and $f_\alpha[E_\alpha] \cap g_\alpha[E_\alpha] = \emptyset$.

– $f_\alpha(x_\alpha) \neq g_\alpha(|x_\alpha|)$. This follows from the previous line if $x_\alpha > 0$. If $x_\alpha < 0$ then $E_\alpha \subseteq (-\infty, 0)$ and $f_\alpha[E_\alpha] \cap g_\alpha[-E_\alpha] = \emptyset$. So and $f_\alpha(x_\alpha) \neq g_\alpha(-x_\alpha) = g_\alpha(|x_\alpha|)$, since $x_\alpha \in E_\alpha$.

– $f(x_\alpha) \neq g(\pm x_\beta)$ for $\beta > \alpha$ since then $x_\beta \notin \pm[g_\alpha^{-1}(f_\alpha(x_\alpha))]$.

Since each $f_\alpha(x_\alpha) \neq f_\alpha(x_\beta)$ for every $\beta < \alpha < \mathfrak{c}$ and each element of $\mathcal{F}$ is listed in our enumeration $\mathfrak{c}$ many times, we conclude that $\bar{f}[M] \setminus \bar{g}[M]$ has cardinality continuum. But $M \cap N$ has cardinality less than continuum since $N = N_\alpha$ for some $\alpha < \mathfrak{c}$. Hence, $f[M]\triangle\bar{f}[M]$ and $g[M]\triangle\bar{g}[M]$ have cardinality less than $\mathfrak{c}$ and so, $f[M] \setminus g[M]$ is nonempty. ∎

When $\langle X, \Sigma, \mathcal{N}\rangle$ is $\aleph_1$-saturated, we can give a version of Theorem 4.3 which shows that *any* measurable function is essentially determined by its range on a strong SRU. First we prove the following lemma.

Lemma 4.12 *Let $\langle X, \Sigma, \mathcal{N}\rangle$ be a flexible $\aleph_1$-saturated measurable space with negligibles and let κ be a cardinal such that the union of any collection of less than κ many negligible sets is negligible. If $M \subseteq X$ is a strong SRU then M has the following properties.*

(a) *If $f, g \in \mathcal{M}(X)$ are such that $f[M] \setminus g[M]$ has cardinality less than κ then $E = X \setminus (f^{-1}(C_f) \cup \{x \in X\colon f(x) = g(x)\})$ is negligible, where $C_f = \{y \in \mathbf{R}\colon f^{-1}(y) \text{ is positive}\}$.*

(b) *If $f, g \in \mathcal{M}(X)$ are such that $f[M] \triangle g[M]$ has cardinality less than κ, then there is a measurable set U such that $f[X \setminus U]$ and $g[X \setminus U]$ are countable, and $f\restriction U \equiv g\restriction U$.*

Proof. (a) Let $C = f[M] \setminus g[M]$. Note that C_f, C_g are countable and C has cardinality less than κ. By way of contradiction suppose that E is positive. Then $f\restriction E$ is nowhere constant. Let $E_1 \subseteq E$ be a positive set on which the ranges of f and g are separated by some rational number q, say $f(x) < q < g(x)$ for all $x \in E_1$. Since the set $E_1 \cap f^{-1}(y)$ is negligible for every $y \in \mathbf{R}$ and $C \cup C_g$ has cardinality less than κ we can find a positive subset K of $E_1 \setminus f^{-1}(C \cup C_g)$. Let $A = g^{-1}(C_g)$. Define a measurable function $h\colon X \to \mathbf{R}$ so that $h\restriction(K \cup A)$ is any nowhere constant measurable function such that $h[K] \cap f[K] = \emptyset$, and h agrees with g on $X \setminus (K \cup A)$. Since M is a strong SRU, there must be a point $x \in M \cap K$ such that $f(x) \notin h[M]$. From the definition of K, $f(x) \notin C_g = g[A]$. So, $f(x) \notin g[A] \cup g[K] \cup h[M \setminus (K \cup A)] = g[A \cup K] \cup g[M \setminus (K \cup A)] \supseteq g[M]$. Thus $f(x) \in f[M] \setminus g[M] = C$, contradicting $x \in K$.

(b) Take $U = X \setminus (f^{-1}(C_f) \cap g^{-1}(C_g))$. By (a) applied as stated, and also with f and g interchanged, $U \setminus \{x \in X\colon f(x) = g(x)\} = \big(X \setminus (f^{-1}(C_f) \cup \{x \in X\colon f(x) = g(x)\})\big) \cup \big(X \setminus (g^{-1}(C_g) \cup \{x \in X\colon f(x) = g(x)\})\big)$ is negligible. ∎

Theorem 4.13 *Let $\langle X, \Sigma, \mathcal{N}\rangle$ be an $\aleph_1$-saturated measurable space with negligibles. Suppose $\mathfrak{c}$ is regular, Σ has cardinality $\mathfrak{c}$ and the union of less than $\mathfrak{c}$ negligible sets is negligible. Then there is a set $L \subseteq X$ such that for any two $\hat{\Sigma}$-measurable functions $f, g\colon X \to \mathbf{R}$, $f[L]\triangle g[L]$ has cardinality less than $\mathfrak{c}$ if and only if there is a measurable set E such that $f[\mathbf{R} \setminus E]$ and $g[\mathbf{R} \setminus E]$ are both countable and $f\restriction E \equiv g\restriction E$.*

Proof. First note that by Corollary 3.8, we may assume that X is flexible, since f and g have essentially countable range on the atomic and Souslin parts of the space. Let L be a $\mathfrak{c}$-strong SRU (Theorem 4.3). Then L has the desired properties by Lemma 4.12 and the assumption that the union of less than $\mathfrak{c}$ negligible sets is negligible. ■

Remark 4.14 The saturation assumption in Theorem 4.13 cannot be deleted. Consider $X = \mathbf{R}^2$, Σ the Borel σ-algebra of $\mathbf{R}^2$, $\mathcal{N} =$ the ideal of countable subsets of X. By Theorem 4.3, X has a strong SRU M. By Proposition 3.1(a), $M \cap B \neq \emptyset$ for every uncountable Borel set B. Let π_1, π_2 be the projection maps $\mathbf{R}^2 \to \mathbf{R}$ onto the first and second coordinates, respectively. Then $\pi_1[M] = \pi_2[M] = \mathbf{R}$. However, $\pi_1(x) \neq \pi_2(x)$ except when x is on the main diagonal, and π_1 and π_2 are not both constant on any set with more than one point.

The existence theorems we have given for SRU's in measurable spaces with negligibles deal with structures in which there are only $\mathfrak{c}$ equivalence classes of measurable functions. As was pointed out in Proposition 3.3(c), there can be no SRU for a measurable space with negligibles which has more than $2^{\mathfrak{c}}$ pairwise nonequivalent measurable functions. This leaves the question of what happens when there are κ equivalence classes of measurable functions and $\mathfrak{c}^+ \leq \kappa \leq 2^{\mathfrak{c}}$. The next theorem and the remark which follows it partially address this question.

Theorem 4.15 *Assume GCH. Let $X = 2^{\omega_2}$, let Σ be the σ-algebra of Baire sets for the usual topology on X, and let $\mathcal{N}$ be such that $\langle X, \Sigma, \mathcal{N}\rangle$ is a measurable space with negligibles (i.e., $\mathcal{N}$ is a proper σ-ideal and $\Sigma \cap \mathcal{N}$ is cofinal in $\mathcal{N}$). There is a countably closed $\aleph_2$-cc forcing notion $\mathbf{P}$ which preserves GCH and such that in the extension V^P there exists an $\aleph_1$-strong SRU for X.*

In particular, it is consistent that there exists a strong SRU for a flexible measurable space with negligibles in which there are $2^{\mathfrak{c}}$ equivalence classes of measurable functions.

Proof. The last statement follows from the rest of the theorem by taking $\mathcal{N}$ to be the σ-ideal of meager sets in 2^{ω_2}, since any two projections onto subproducts of the form $2^{[\alpha,\alpha+\omega)} \cong 2^\omega$, $\alpha < \omega_1$ a limit ordinal, are nonequivalent nowhere constant members of $\mathcal{M}(X)$. (We identify here 2^ω with the Cantor middle third set $C \subseteq \mathbf{R}$.) The space 2^ω is flexible by Proposition 2.4, since it is separable.

We can assume that X is flexible since otherwise the theorem is trivial.

We start with few remarks on the structure of Baire sets and Baire functions in X. First recall that for each Baire set $B \subseteq X$, there exists a countable set $A \subseteq \omega_2$ on which B "lives" in the sense that

$$p \in B \text{ iff } q \in B \qquad \text{for every } p, q \in X \text{ with } p\restriction A = q\restriction A. \tag{1}$$

(See [Ku2] for example.) Similarly, since every Baire function $f \in \mathcal{M}(X)$ is fully described by the sets $f^{-1}((a,\infty))$ $(a \in \mathbf{Q})$ we can find a countable set $A \subseteq \omega_2$ on which f "lives" in the sense that

$$f(p) = f(q) \qquad \text{for every } p, q \in X \text{ with } p\restriction A = q\restriction A. \tag{2}$$

Note also that if f and A satisfy (2) then the function $f_A\colon 2^A \to \mathbf{R}$, $f_A(p\restriction A) = f(p)$ for $p \in X$, is well defined and it codes f.

Now, for $A \subseteq \omega_2$ and $f\colon 2^A \to \mathbf{R}$ let $\hat{f}\colon X \to \mathbf{R}$ be defined by $\hat{f}(p) = f(p\restriction A)$. Moreover, for $D \subseteq \omega_2$ let $\mathcal{F}(D)$ be the family of all $f\colon 2^A \to \mathbf{R}$ such that $A \in [D]^\omega$ and $\hat{f}$ is a Σ-measurable $\mathcal{N}$-nowhere constant. When $s\colon C \times D \to \{0,1\}$ and $\gamma \in C$ we will write s_γ for the function from D into $\{0,1\}$ given by $s_\gamma(\delta) = s(\gamma,\delta)$ for every $\delta \in D$. Define

$$\mathbf{P} = \{\langle C, D, s, F\rangle\colon C \in [\omega_1]^{\leq\omega} \;\&\; D \in [\omega_2]^{\leq\omega} \;\&\; s\colon C \times D \to \{0,1\} \;\&\; F \in [\mathcal{F}(D)]^{\leq\omega}\}$$

and define a partial order on $\mathbf{P}$ by $\langle C, D, s, F\rangle \leq \langle C', D', s', F'\rangle$ provided $C' \subseteq C$, $D' \subseteq D$, $s' \subseteq s$, $F' \subseteq F$, and for every $\gamma \in C \setminus C'$ and $\alpha \in C \setminus \{\gamma\}$

$$(\forall A \in [D']^\omega)(\forall f, g \in F')[\mathrm{dom}(f) = \mathrm{dom}(g) = 2^A \;\longrightarrow\; f(s_\gamma\restriction A) \neq g(s_\alpha\restriction A)]. \tag{3}$$

It is easy to see that $\mathbf{P}$ is countably closed. $\mathbf{P}$ is $\aleph_2$-cc since, by standard $\triangle$-system arguments, for any sequence of conditions $\langle\langle C^\xi, D^\xi, s^\xi, F^\xi\rangle : \xi < \omega_2\rangle$ we can find $\xi < \zeta < \omega_2$ such that $C^\xi = C^\zeta$ and $s^\xi \cup s^\zeta$ is a function. (See [Ku1, Ch. VII, sec. 6].) Then $\langle C^\xi, D^\xi \cup D^\zeta, s^\xi \cup s^\zeta, F^\xi \cup F^\zeta\rangle \in \mathbf{P}$ extends both $\langle C^\xi, D^\xi, s^\xi, F^\xi\rangle$ and $\langle C^\zeta, D^\zeta, s^\zeta, F^\zeta\rangle$ since condition (3) is viciously satisfied. Thus, $\mathbf{P}$ preserves cardinals and does not add any new countable sequences of ground model elements. In particular, $V^{\mathbf{P}}$ contains neither any new real numbers nor any new code f_A of any Baire functions f.

Let $G \subseteq \mathbf{P}$ be V-generic and let

$$x = \bigcup\{s : \langle C, D, s, F\rangle \in G\}.$$

Clearly x is a function from a subset of $\omega_1 \times \omega_2$ into 2. To see that $\mathrm{dom}(x) = \omega_1 \times \omega_2$ it is enough to notice that for every $\gamma < \omega_1$ and $\delta < \omega_2$ the sets $\{\langle C, D, s, F\rangle \in \mathbf{P} : \gamma \in C\}$ and $\{\langle C, D, s, F\rangle \in \mathbf{P} : \delta \in D\}$ are dense in $\mathbf{P}$. For the latter this is trivial. To see it for the former case, let $\langle C, D, s, F\rangle \in \mathbf{P}$ and $\gamma \in \omega_1$. Pick

$$p \in X \setminus \bigcup\{\hat{f}^{-1}(g(s_\alpha \restriction A)) : \alpha \in C \ \&\ f, g \in F \ \&\ \mathrm{dom}(g) = 2^A\}.$$

Then $\langle C \cup \{\gamma\}, D, t, F\rangle \in \mathbf{P}$ with $t \restriction C \times D = s$ and $t_\gamma = p \restriction D$ extends $\langle C, D, s, F\rangle \in \mathbf{P}$.

Let $L = \{x_\gamma : \gamma < \omega_1\}$ and notice that genericity easily imply that all x_γ's are different. The proof is completed by verifying that for each pair $f, g \in \mathcal{M}_\Sigma(X)$ of nowhere constant functions, there is a countable set $C_{f,g}$ of L such that assumptions (i) and (ii) of Definition 4.1 are satisfied with $\kappa = \aleph_1$. Let $f, g \in \mathcal{M}_\Sigma(X)$ be nowhere constant functions, let E be a positive Baire set, and let N be a negligible Baire set. Let $A \subseteq \omega_2$ be countable and such that $\langle f, A\rangle$ and $\langle g, A\rangle$ satisfy (2) and choose a condition $\langle C, D, s, F\rangle \in G$ such that $f_A, g_A \in F$. (Such conditions are trivially dense.) Define $C_{f,g} = \{x_\gamma : \gamma \in C\}$. It is clear from the definition of the order on $\mathbf{P}$ that Definition 4.1(ii) is satisfied. Since κ is regular, Definition 4.1(i) is equivalent to: $L \cap E$ is uncountable for each positive Baire set E and that $L \cap N$ is countable for each negligible Baire set N. To verify this, let E be a positive Baire set, let N be a negligible Baire set and let and let $\alpha < \omega_1$. Choose a countable set $A \subseteq \omega_2$ such that $\langle B, A\rangle$ and $\langle N, A\rangle$ satisfy (1). The desired properties follow easily from the density of the conditions $\langle C, D, s, F\rangle$ such that $A \subseteq D$ and $s_\gamma \in \{p \restriction D : p \in E\}$ for some $\gamma \in C \setminus \alpha$, and the density of the conditions $\langle C, D, s, F\rangle$ such that $0 \in C$ and that there are $f, g \in F$ with $\mathrm{dom}(f) = \mathrm{dom}(g) = 2^A$ and $N = \hat{f}^{-1}(g(s_0 \restriction A))$. (Such an $\hat{f}$ exists since X is flexible.) ■

Remark 4.16 The measurable space with negligibles $\langle \mathfrak{c}, P(\mathfrak{c}), \mathcal{N}\rangle$, where $\mathcal{N}$ is the ideal of countable subsets of $\mathfrak{c}$, has no SRU by Proposition 3.3(a). Thus, in Theorem 4.3, the cardinality restriction on Σ cannot be relaxed to $2^{\mathfrak{c}}$. If κ is an atomlessly measurable cardinal and $\mathcal{N}$ is the null ideal of a witnessing measure, then $\langle \kappa, P(\kappa), \mathcal{N}\rangle$ is an $\aleph_1$-saturated measurable space with negligibles which illustrates the same point.

5 SRU's for continuous functions

In this section we will examine properties of SRU's for $\mathrm{C}(X)$, for X a Baire topological space considered with its natural structure as a measurable space with negligibles. (See Section 2). Many of the results do not rely on X being Baire, however. (See Remark 5.16.) Recall, that by Proposition 2.2 we can replace the relation $\equiv$ in the definition of (strong) SRU with equality. Proposition 2.2 and Theorem 4.3 ensure that, under suitable hypotheses, strong SRU's exist in many Baire spaces. We will need to assume, for most of our results, that our spaces are locally flexible. Proposition 2.4 provides us with a healthy supply of such spaces.

We begin with several easy lemmas.

Lemma 5.1 *Let X be a Baire topological space considered with its natural measurable structure and let $M \subseteq X$ be an SRU for $\mathrm{C}(X)$. If $f \in \mathrm{C}(X)$ is nowhere constant, $K = f[M]$ and $h : \mathbf{R} \to \mathbf{R}$ is a homeomorphism such that $h[K] = K$ then $h(y) = y$ for every $y \in f[X]$.*

PROOF. Let f, h and K be as above and by way of contradiction assume that $h(y) \neq y$ for some $y \in f[X]$. Then $g = h \circ f \neq f$ while $g \in \mathrm{C}(X)$ is nowhere constant and $g[M] = h[f[M]] = h[K] = K = f[M]$, contradicting the definition of an SRU for $\mathrm{C}(X)$. ■

Lemma 5.2 *Let $a < b$ and $K \subseteq (a,b)$ be a Cantor set. Then, every continuous function $g: K \to (a,b)$ has an extension $G: \mathbf{R} \to \mathbf{R}$ such that $G[(a,b)] = (a,b)$, $G(x) = x$ for all $x \in \mathbf{R} \setminus (a,b)$ and G is countable-to-one on $\mathbf{R} \setminus K$.*

PROOF. Let c and d be the maximum and the minimum of K, respectively. Extend g to a continuous function $g_1: [c,d] \to (a,b)$ by defining it linearly on any component of $[c,d] \setminus K$. Let $U = \bigcup\{\mathrm{int}(g_1^{-1}(y)): y \in [c,d]\}$. If $f: [c,d] \to \mathbf{R}$ is the distance function from $[c,d] \setminus U$ then it is easy to find a constant $k > 0$ such that $G = (g_1 + kf): [c,d] \to (a,b)$. It is also not difficult to check that G is continuous and countable-to-one on $[c,d] \setminus K$. By extending it to the identity function on $\mathbf{R} \setminus (a,b)$ and linearly on each of the intervals (a,c) and (d,b) we obtain the desired function. ■

Lemma 5.3 *Let X be a Baire topological space. Let $f \in \mathrm{C}(X)$ and $g \in \mathrm{C}(\mathbf{R})$ be nowhere constant. If either X is locally connected or g is countable-to-one, then $g \circ f \in \mathrm{C}(X)$ is also nowhere constant.*

PROOF. If g is countable-to-one, then $(g \circ f)^{-1}(y) = \bigcup\{f^{-1}(z): z \in g^{-1}(y)\}$ is meager, as it is a countable union of nowhere dense sets.

If X locally connected and $(g \circ f)^{-1}(y)$ has nonempty interior, then there exists a nonvoid open connected $U \subseteq (g \circ f)^{-1}(y) = f^{-1}(g^{-1}(y))$. So, $f[U] \subseteq g^{-1}(y)$. But $f[U] \subseteq \mathbf{R}$ is connected, as an image of a connected set. So, either $f[U]$ is a singleton, contradicting that f is nowhere constant, or $f[U]$ contains a nonvoid open interval, contradicting the fact that $g^{-1}(y)$ is nowhere dense. ■

Remark 5.4 Notice also that Lemma 5.3 may fail if we require only that g is nowhere constant. To see this, let $K \subseteq \mathbf{R}$ be a Cantor set and let $X = K \times K$. Moreover, let $c: K \to \{0\}$ be the constant map and let G be the extension of c given by Lemma 5.2. Then the conclusion of Lemma 5.3 fails for $g = G$ and $f =$ the projection of X onto the first coordinate.

Recall from Section 2 that $S \subseteq X$ is an *s_0-set* if for every Cantor set $P \subseteq X$, there is a Cantor set $Q \subseteq P$ such that $Q \cap S = \emptyset$. Also, S is a *strong s_0-set* if $f[S]$ is an s_0-set in $\mathbf{R}$ for every $f \in \mathrm{C}(\mathbf{R})$. Note that every set of cardinality less than $\mathfrak{c}$ is a strong s_0-set. There are s_0-sets of cardinality $\mathfrak{c}$ in $\mathbf{R}$ [Mi2], but it is consistent that there are no strong s_0-sets of cardinality $\mathfrak{c}$ [Mi3].

Lemma 5.5 *If $S \subseteq X$ is a strong s_0-set, then S is an s_0-set and S is zero-dimensional.*

PROOF. The zero-dimensionality of S follows easily from the fact that X is completely regular and the image of S by a member of $\mathrm{C}(X)$ cannot contain an interval. To see that S is an s_0-set, let $K \subseteq X$ be a Cantor set, and let $g: K \to \mathbf{R}$ be a homeomorphism onto its range. Since K is compact, g extends to a function $G \in \mathrm{C}(\mathbf{R})$ [En, Exercise 3.2.J]. Since $G[S]$ is an s_0-set, there is a Cantor set $L \subseteq G[K]$ such that $L \cap G[S] = \emptyset$. Then $S \cap (G^{-1}(L) \cap K) = \emptyset$. ■

Theorem 5.6 *Let X be a locally flexible Baire topological space considered with its natural measurable structure. If $M \subseteq X$ is an SRU for $\mathrm{C}(X)$ then M has the following properties.*

(1) *M is dense in X.*

(2) *Let $U \subseteq X$ be nonvoid open and let $f \in \mathrm{C}(X)$ be nowhere constant. Then $f[M \cap U]$ is uncountable provided at least one of the following conditions hold.*

(a) *$U = f^{-1}(W)$ for some open $W \subseteq \mathbf{R}$;*

(b) *M is a strong SRU.*

(3) *$M \cap U$ is uncountable for every nonempty open set $U \subseteq X$.*

(4) *For every $f \in \mathrm{C}(X)$ there is a nowhere constant $\bar{f} \in \mathrm{C}(X)$ such that $f[M] \subseteq \bar{f}[M]$.*

(5) *M is a strong s_0-set. In particular, M is a zero-dimensional s_0-set.*

(6) *If X is a nonvoid analytic metric space then M is not analytic.*

Proof. (1) By way of contradiction, assume that there is a nonvoid open set $U \subseteq X$ disjoint from M. Let $V = X \setminus \mathrm{cl}(U)$ and let h_U and h_V be functions from the definition of local flexibility for U and V respectively. Define $f = h_V + h_U$ and $g = h_V - h_U$. Then f and g are continuous, nowhere constant. Moreover, $f[M] = h_V[M] = g[M]$, while $f \neq g$, since h_U is nowhere constant on U.

(2) By way of contradiction, assume that $f[M \cap U]$ is countable for some nonvoid open U and nowhere constant $f \in \mathrm{C}(X)$. Let $Y = f[M \cap U]$.

First notice that Y has no isolated points. To see this, assume by way of contradiction that Y has an isolated point y and let $I \subseteq \mathbf{R}$ be open such that $I \cap Y = \{y\}$. Then $V = U \cap f^{-1}(I)$ is nonempty and, by (1), $f[V] \subseteq f[\mathrm{cl}(M \cap V)] \subseteq \mathrm{cl}(f[M \cap V]) = \{y\}$ contradicting the fact that f is nowhere constant.

Next notice that

($\star$) there exists $a < b$ and a countable-to-one continuous function $h\colon \mathbf{R} \to \mathbf{R}$ such that $h[Y] = Y$, $h(y_0) \neq y_0$ for some $y_0 \in Y$, and $h(x) = x$ for $x \in \mathbf{R} \setminus (a, b)$; moreover, $(a, b) \subseteq W$ if we are in the case (a).

Note that this will finish the proof. Indeed, put $g = h \circ f \in \mathrm{C}(X)$ and notice that $f \neq g$ on U, since $U \cap f^{-1}(y_0) \neq \emptyset$. Moreover, by Lemma 5.3, g is nowhere constant and $g[M \cap U] = h[f[M \cap U]] = h[Y] = Y = f[M \cap U] \subseteq f[M]$. This gives a contradiction with M being strong SRU, taking care of (b).

If $U = f^{-1}(W)$ then $g[M \setminus U] = h[f[M \setminus U]] = f[M \setminus U] \subseteq f[M]$, since $h(x) = x$ for all $x \in f[M \setminus U] \subseteq \mathbf{R} \setminus (a, b)$. So, $g[M] = f[M]$ giving us a contradiction with (a) as well.

To prove ($\star$) consider two cases.

Case 1. $\mathrm{cl}(Y)$ is somewhere dense. Let $(a, b) \subseteq \mathrm{cl}(Y)$ be a nonvoid open interval and assume that $(a, b) \subseteq W$ if $U = f^{-1}(W)$. Then, there is a nontrivial homeomorphism $h\colon \mathbf{R} \to \mathbf{R}$ such that $h[Y] = Y$ and $h(x) = x$ for every $x \in \mathbf{R} \setminus (a, b)$. Clearly $h(y) \neq y$ for some $y \in Y$.

Case 2. $\mathrm{cl}(Y)$ is nowhere dense. Let $a < b$ be such that $K = \mathrm{cl}(Y) \cap (a, b)$ is a Cantor set and that $(a, b) \subseteq W$ if $U = f^{-1}(W)$. Let $g\colon K \to K \subseteq (a, b)$ be a homeomorphism such that $g[K \cap Y] = K \cap Y$ and $g(y) \neq y$ for some $y \in K \cap Y$. (See [En, Exercise 4.3.H(3)] for such a homeomorphism.) Let G be an extension of g as in Lemma 5.2. Then $h = G$ satisfies ($\star$).

(3) By way of contradiction, assume that $M \cap U$ is countable for some open nonempty $U \subseteq X$. For an open set $W \subseteq X$ let f_W stand for the absolute value of the function from the definition of local flexibility for W. Put $f = f_U - f_{\mathrm{int}(X \setminus U)}$. Then f is continuous and nowhere constant, and $U \supset f^{-1}((0, \infty)) \neq \emptyset$. An application of (2) case (a) gives a contradiction.

(4) Let $f \in \mathrm{C}(X)$ and let $C_f = \{y \in \mathbf{R}\colon \mathrm{int}(f^{-1}(y)) \neq \emptyset\}$. By (1) for each $y \in C_f$ we can choose $x_y \in \mathrm{int}(f^{-1}(y)) \cap M$. Note that the set $H = [X \setminus \bigcup\{\mathrm{int}(f^{-1}(y))\colon y \in C_f\}] \cup \{x_y\colon y \in C_f\}$ is closed in X. Let $g \in \mathrm{C}(X)$ be as in the definition of local flexibility of X for the set $X \setminus H$ and let $\bar{f} = f + g \in \mathrm{C}(X)$. Then $\bar{f}$ is nowhere constant and $f[M] = f[M \cap H] = \bar{f}[M \cap H] \subseteq \bar{f}[M]$.

(5) The claims made in the second statement follow from Lemma 5.5. For the proof of the first statement, let us first verify that if $f \in \mathrm{C}(\mathbf{R})$, we cannot have a Cantor set contained in $f[M]$. By way of contradiction, assume that there is a Cantor set $K \subseteq f[M]$. By (4) we can assume that f is nowhere constant. Let g be a continuous two-to-one function from K onto $[0, 1]$ and let G be an extension of g as in Lemma 5.2. Then, $G \in \mathrm{C}(\mathbf{R})$ is countable-to-one and, by Lemma 5.3, $F = G \circ f$ is nowhere constant. But then, $F[M] =$

$G[f[M]] \supset G[K] \supset [0,1]$, which contradicts Lemma 5.1, since there are many nontrivial homeomorphisms h of $\mathbf{R}$ with $h(x) = x$ for all $x \in \mathbf{R} \setminus (0,1)$.

Now suppose that $f[M] \cap L \neq \emptyset$ for every Cantor set $L \subseteq K$. Since K is homeomorphic to its square, and the level sets of the projection of the square onto one of the coordinates are all Cantor sets, there is a continuous map $g\colon K \to K$ all of whose level sets are Cantor sets. Extend this function g to a member G of $\mathrm{C}(\mathbf{R})$ and note that $(G \circ f)[M] \supseteq K$, contradicting the result established in the previous paragraph.

(6) This follows immediately from (3) and (5). ■

Problem 5.7 Are the assumptions (a) and (b) essential in (2) of Theorem 5.6?

We will now consider some specific properties of SRU's for $\mathrm{C}(X)$ where X is a separable metric space. We begin with following lemma.

Lemma 5.8 *Let $\langle X, d\rangle$ be a separable metric space, and let $M \subseteq X$ a strong s_0-set. If F is a closed subset of X and $\varepsilon > 0$ then there exists a continuous function $g\colon X \to [0,\varepsilon]$ such that $g^{-1}(\{0\}) = F$ and $g[M]$ is countable.*

Proof. Let $h\colon X \to [0,\varepsilon]$ be given by the formula $h(x) = \min\{\varepsilon, d(x,F)\}$ for $x \in X$ and for $n < \omega$ let $I_n = [\varepsilon/2^{n+1}, \varepsilon/2^n]$. Since M is a strong s_0-set, for every $n < \omega$ there exists a Cantor set $C_n \subseteq I_n$ such that $C_n \cap h[M] = \emptyset$. Let $f_n\colon I_n \to I_n$ be a non decreasing Cantor function such that $f_n[C_n] = I_n$ and let $f\colon [0,\varepsilon] \to [0,\varepsilon]$ be an extension of all these functions. It is easy to see that $h = f \circ g\colon X \to [0,\varepsilon]$ has the desired properties. ■

Lemma 5.9 *Let $\langle X, d\rangle$ be a separable Baire metric space without isolated points. Let $U \subseteq X$ be open and let $M \subseteq U$ be a meager strong s_0-set. Then there is a function $g \in \mathrm{C}(X)$ such that $g(x) = 0$ for all $x \in X \setminus U$, $g(x) \geq 0$ for all $x \in X$, $g\restriction U$ is nowhere constant, and $g[M]$ is countable.*

Proof. Let K_n be closed nowhere dense sets in X such that $M \subseteq \bigcup_{n<\omega} K_n$. Since X is Baire, there is a set $\{d_n\colon n < \omega\} \subseteq U \setminus \bigcup_{n<\omega} K_n$ which is dense in U. We will construct, by induction on $n < \omega$, a sequence $g_n\colon X \to [0,\varepsilon_n)$ of continuous functions such that for every $n < \omega$

(i) $g_n[M]$ is countable

(ii) $g_n^{-1}(\{0\}) = (X \setminus U) \cup \bigcup_{i<n}(K_i \cup \{d_i\})$

(iii) $\varepsilon_n \in (0, 2^{-n})$

(iv) $\left(\sum_{i<n} g_i\right)(d_j) \notin \left(\left(\sum_{i<n} g_i\right)(d_n), \left(\sum_{i<n} g_i\right)(d_n) + \varepsilon_n\right]$ for $n > 0$ and $j < n$.

The construction is easily carried out. It is simply a matter of choosing ε_n satisfying (iii) and (iv), and then defining g_n to be the function g from Lemma 5.8 applied with $F = (X \setminus U) \cup \bigcup_{i<n}(K_i \cup \{d_i\})$ and $\varepsilon = \varepsilon_n$.

Let $g = \sum_{i<\omega} g_i$. Then, by condition (iii), g is continuous. Also, condition (ii) implies that for every $n < \omega$

$$g[M \cap K_n] = \left(\sum_{i\leq n} g_i\right)[M \cap K_n]$$

which is countable by (i). So, $g[M]$ is countable. Moreover, by (ii) and (iv),

$$g(d_j) = \left(\sum_{i<n} g_i\right)(d_j) \notin \left(\left(\sum_{i<n} g_i\right)(d_n), \left(\sum_{i<n} g_i\right)(d_n) + \varepsilon_n\right] \ni \left(\sum_{i\leq n} g_i\right)(d_n) = g(d_n)$$

for every $j < n < \omega$. So g is nowhere constant in U. ■

Theorem 5.10 *Let $\langle X, d\rangle$ be a separable Baire metric space without isolated points. If $M \subseteq X$ is an SRU for $\mathrm{C}(X)$ and $U \subseteq X$ is a nonvoid open set, then $M \cap U$ is not meager.*

Proof. Let M be an SRU for $\mathrm{C}(X)$. By Theorem 5.6(5), M is a strong s_0-set. If $M \cap U$ is meager, then Lemma 5.9 gives a nonnegative function $g_1 \in \mathrm{C}(X)$ which is identically equal to zero outside U, nowhere constant on U, and has a countable image of $M \cap U$. Let $g_2 \in \mathrm{C}(X)$ be a nonpositive function which is identically equal to zero on U and nowhere constant in the exterior of U. (For example, apply Lemma 5.9 with U replaced by the exterior of U and M replaced by the empty set.) Let $g = g_1 + g_2$. Then $g \in \mathrm{C}(X)$ is nowhere constant. Let $W = g^{-1}((0, \infty))$. We have $g \restriction W = g_1 \restriction W$ and hence $W \subseteq U$. Also, $g[M \cap W] \subseteq g_1[M \cap U]$ is countable, contradicting Theorem 5.6(2)(a). ■

Corollary 5.11 *A Sierpiński subset of $\mathbf{R}^n$ is not an SRU for $\mathrm{C}(\mathbf{R}^n)$.*

Proof. Sierpiński sets are meager. ■

Remark 5.12 Theorems 5.10 and 5.6(5) show that an SRU for $\mathrm{C}(\mathbf{R})$ cannot be meager, but also cannot be too big. However, little can be said about the measure of an SRU for $\mathrm{C}(\mathbf{R})$. Under CH, there is a strong SRU for $\mathrm{C}(\mathbf{R})$ which is a Lusin set and hence has strong measure zero and there is another one of full outer measure. (For the full outer measure example, apply Theorem 4.9 to $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}_i\rangle$, $i = 1, 2$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the meager and null ideals.)

Remark 5.13 It is not difficult to see that one can prove Theorems 5.10 for any Baire space without isolated points in which any dense G_δ set contains a countable dense subset. The proof requires only minor changes. This could be used for example to show that any SRU for $[0,1]^{\mathfrak{c}}$ must be nowhere meager, a fact which also follows from the next proposition.

Proposition 5.14 *Let X be a ccc topological space such that X^ω is Baire. If every SRU for $\mathrm{C}(X^\omega)$ is nowhere meager then so is every SRU for $\mathrm{C}(X^\kappa)$ for every infinite cardinal κ.*

Proof. By way of contradiction assume that for some infinite cardinal number κ there is $M \subseteq X^\kappa$ which is an SRU for $\mathrm{C}(X^\kappa)$ and such that $U \cap M$ is meager for some nonvoid open set $U \subseteq X^\kappa$. Decreasing U, if necessary, we can assume that U is a basic open set supported by a finite set T of coordinates. Since X^κ is ccc we can find an F_σ meager set F supported by a countable infinite set S of coordinates and such that $U \cap M \subseteq F$. We can also assume that $T \subseteq S$. Let π be the projection map of X^κ onto X^S. Thus, $\pi[M \cap U] \subseteq \pi[F]$ is meager in $\pi[U]$. So, by our assumption, $\pi[M]$ is not an SRU for X^S, i.e., there are two different nowhere constant functions $f, g \in \mathrm{C}(X^S)$ with $f[\pi[M]] = g[\pi[M]]$. But $f \circ \pi, g \circ \pi \in \mathrm{C}(X^\kappa)$ are different and nowhere constant too, contradicting the assumption that M was an SRU for X^κ. ■

Corollary 5.15 *For any cardinal κ every SRU for $[0,1]^\kappa$ is nowhere meager in $[0,1]^\kappa$.* ■

Remark 5.16 As we pointed out in Section 2, if X is a Baire topological space, the usual notion of "nowhere constant" for a function $f \in \mathrm{C}(X)$ (i.e., "not constant on any nonvoid open set") coincides with the meaning "not constant on any nonmeager Baire set" we gave in Section 2. Most of the results in Section 5 go through even for spaces that are not Baire, if the usual notion of "nowhere constant" is used in the definitions of SRU and of locally flexible. Lemma 5.3 for the case where g is countable-to-one holds by a different argument: the level sets of g are scattered and it is easy to verify by induction on the Cantor-Bendixson height of a scattered set that its preimage under a nowhere constant function is nowhere dense. The proof of Lemma 5.1 goes through with no changes, as does the proof of Theorem 5.6. As for Theorem 5.10, note that if X is any nonvoid separable metric space with no isolated points and for some open set $U \subseteq X$, $M \cap U$ is meager, then by Theorem 5.10 applied to the completion Y of X, there are two distinct nowhere constant functions $f, g \in \mathrm{C}(Y)$ such that $f[M] = g[M]$. But then the restrictions of f and g to X witness that M is not an SRU. In particular, if X itself has a nonvoid meager open set (i.e., is not Baire), then there is no SRU for $\mathrm{C}(X)$.

The next example shows that the assumption of CH cannot be removed from [BD, theorem 8.5].

Example 5.17 *Suppose that*

(i) *there is a nonmeager subset of* $\mathbf{R}$ *of cardinality* $\aleph_1$*; and,*

(ii) *any two nowhere meager subsets of* $\mathbf{R}$ *of cardinality* $\aleph_1$ *are order isomorphic.*

(See [Sh] for a proof that this is consistent.) If $X \subseteq \mathbf{R}$ *is a nowhere meager subset of* $\mathbf{R}$ *of cardinality* $\aleph_1$ *then there is no SRU for* $\mathrm{C}(X)$.

Proof. Suppose $M \subseteq X$ were an SRU for $\mathrm{C}(X)$. By Theorem 5.10 M is nowhere meager. But then, using (ii), it is easy to construct many distinct order isomorphisms of M with itself. Each of these will extend to a homeomorphism of $\mathbf{R}$, and hence M is not an SRU. ∎

The models we are aware of for assumptions (i) and (ii) of Example 5.17 are obtained by κ-stage finite support iterations of ccc forcing notions, where κ is regular. As a result, Cohen reals are added cofinally in the construction and the covering assumption of Theorem 4.3 is satisfied for $X = \mathbf{R}$, $\Sigma =$ the Borel σ-algebra, and $\mathcal{N} =$ the meager ideal. Thus, there is a strong SRU for $\mathrm{C}(\mathbf{R})$ in these models.

The following remains an intriguing problem.

Problem 5.18 Is the existence of an SRU for $\mathrm{C}(\mathbf{R})$ provable in ZFC?

We now give the example promised in Section 2 to show that the converse to the first statement of Proposition 2.2 is false. First we prove the following lemma.

Lemma 5.19 *Let* X *be the space of irrational numbers (with the usual topology). There is a countable dense set* $A \subseteq X$ *and a homeomorphism* $h: X \setminus A \to X \setminus A$ *such that for any* $m \in \mathbf{Z} \setminus \{0\}$ *and any nowhere constant function* $g \in \mathrm{C}(X)$, $g \circ h^m: X \setminus A \to X \setminus A$ *does not extend to a continuous function on any nonempty open subset of* X.

Proof. Fix any countable dense set $A \subseteq \mathbf{R}$. We define open subintervals $I_s = (a_s, b_s)$ of $\mathbf{R}$ for $s \in \mathbf{Z}^{<\omega}$ as follows. Let $I_\emptyset = \mathbf{R}$ and given I_s, define open subintervals $I_{s^\frown n}$, $n \in \mathbf{Z}$, of I_s so that $I_s \setminus \bigcup_{n \in \mathbf{Z}} I_{s^\frown n} \subseteq A$, and $b_{s^\frown n} = a_{s^\frown (n+1)}$. (For $s \in \mathbf{Z}^{<\omega}$ and $n \in \mathbf{Z}$ the symbol $s^\frown n$ denotes the extension of s by n, i.e., $s^\frown n = s \cup \{\langle k, n\rangle\}$, where $k = \mathrm{dom}(s)$.) Also ensure that if we let $U_n = \bigcup \{I_s : s \in \mathbf{Z}^n\}$, then $\bigcap_{n\in\omega} U_n = \mathbf{R} \setminus A$. Identify $\mathbf{Z}^\omega$ with $\mathbf{R} \setminus A$ via the map which sends $u \in \mathbf{Z}^\omega$ to the unique member of $\bigcap\{I_{u\restriction n} : n \in \omega\}$. Let $h: \mathbf{R} \setminus A \to \mathbf{R} \setminus A$ be the homeomorphism which corresponds via this identification to the homeomorphism of $\mathbf{Z}^\omega$ such that for $u \in \mathbf{Z}^\omega$ and $k \in \mathbf{Z}$ we have $h(u)(k) = u(k)$ if $u(k)$ is odd, and $h(u)(k) = u(k) - 2$ if $u(k)$ is even. Let $B \subseteq \mathbf{R} \setminus A$ be a countable dense set such that $h[B] = B$. We may assume $X = \mathbf{R} \setminus B$ since this set is homeomorphic to the set of irrational numbers. Note that $A \subseteq X$ and that h induces a homeomorphism of $X \setminus A$. We will show that X, h and A satisfy the conclusion of the lemma.

Let $g \in \mathrm{C}(X)$ and $m \in \mathbf{Z} \setminus \{0\}$ be such that $g \circ h^m$ extends to a continuous function on a nonvoid open subset $I \subseteq X$. By shrinking I, we may assume that $I = I_{s_0} \cap X$ for some $s_0 \in \mathbf{Z}^{<\omega}$. Let $t_0 \in \mathbf{Z}^{<\omega}$ be such that $h^m[I_{s_0}] = I_{t_0}$. We will show that g is constant on I_{t_0}. First, we claim that for every $t \in \mathbf{Z}^{<\omega}$ such that $t_0 \subseteq t$,

$$g \text{ is constant on the set } S_t = \{a_{t^\frown(2m\ell)} : \ell \in \mathbf{Z}\}.$$

So, let $\ell \in \mathbf{Z}$, $x = a_{t^\frown(2m\ell)}$ and $y = a_{t^\frown(2m(\ell+1))}$. We will show that $g(x) = g(y)$. Let $s \in \mathbf{Z}^{<\omega}$ be such that $h^m[I_s] = I_t$. Note that $s_0 \subseteq s$, so $g \circ h^m$ extends to a continuous function on $I_s \subseteq I_{s_0}$. For $z \in I_{s^\frown(2m(\ell+1))} \cap (X \setminus A)$, as $z \to a_{s^\frown(2m(\ell+1))}$, $h^m(z) \to a_{t^\frown(2m\ell)}$ and hence $g(h^m(z)) \to g(a_{t^\frown(2m\ell)}) = g(x)$. For $z \in I_{s^\frown(2m(\ell+1)-1)} \cap (X \setminus A)$, as $z \to a_{s^\frown(2m(\ell+1))} = b_{s^\frown(2m(\ell+1)-1)}$, $h^m(z) \to b_{t^\frown(2m(\ell+1)-1)} = a_{t^\frown(2m(\ell+1))}$ and hence $g(h^m(z)) \to g(a_{t^\frown(2m(\ell+1))}) = g(y)$. Thus, we must have $g(x) = g(y)$, as desired.

Note that for each $t \supseteq t_0$ and each $n \in \mathbf{Z}$, the constant values of g on $S_{t^\frown n}$ and $S_{t^\frown(n+1)}$ must be the same since these two sets share a cluster point. Let c be the constant value taken by g on T_1 where for each

$m \in (\mathrm{dom}(t), \omega)$ we let $T_m = \bigcup\{S_t : t \supseteq t_0,\ t \in \mathbf{Z}^m\}$. By similar considerations to the case $m = 1$, it follows by induction on m, using the fact that T_{m+1} has cluster points in T_m, that g has the constant value c on each T_m. Since $\bigcup\{T_m : m \in (\mathrm{dom}(t), \omega)\}$ is dense in I_{t_0}, g is constant on I_{t_0}. ∎

Example 5.20 *If* $\mathbf{R}$ *cannot be covered by less than* $\mathfrak{c}$ *meager sets, then in any nonvoid perfect Polish space* X*, there is a set* $M \subseteq X$ *and a Borel isomorphism* $h: X \to X$ *such that* $h[M] = M$*,* $\{x \in \mathbf{R}: h(x) = x\}$ *is meager, and if* $f, g \in \mathrm{C}(X)$ *are nowhere constant and* $E \subseteq X$ *is a nonvoid open set such that* $f \restriction E \neq g \restriction E$*, then* $f[M \cap E] \setminus g[M]$ *has cardinality* $\mathfrak{c}$*. In particular, there is a strong SRU for* $\mathrm{C}(X)$ *which is not an SRU for the Borel isomorphisms.*

Proof. Since every nonvoid perfect Polish space contains a residual copy of the irrationals, we may assume that X is the space of irrational numbers. Let $\langle \langle f_\alpha, g_\alpha, E_\alpha \rangle : \alpha < \mathfrak{c} \rangle$ be a list of all triples $\langle f, g, E \rangle$ where $f, g \in \mathrm{C}(X)$ are nowhere constant maps, E is a nonempty open set, and $f[E] \cap g[E] = \emptyset$; let each such triple appear $\mathfrak{c}$ times in the list. Let A and h be given by Lemma 5.19. Inductively choose points $x_\alpha \in E_\alpha \setminus A$ such that x_α does not belong to any set of the form $f_\alpha^{-1}(g_\alpha(h^m(x_\beta)))$ ($m \in \mathbf{Z}$, $\beta < \alpha$) or $f_\alpha^{-1}(f_\beta(x_\beta))$ ($\beta < \alpha$) or $h^{-m}([g_\beta^{-1}(f_\beta(x_\beta)] \setminus A)$ ($m \in \mathbf{Z}$, $\beta < \mathfrak{c}$) or $\{x \in X \setminus A: f_\alpha(x) = g_\alpha(h^m(x))\}$ ($m \in \mathbf{Z} \setminus \{0\}$). Note that sets of the latter form are nowhere dense by Lemma 5.19, and the remaining sets are nowhere dense because f_α and g_β ($\beta < \alpha$) are nowhere constant. Let $M = \{h^m(x_\alpha): m \in \mathbf{Z},\ \alpha < \mathfrak{c}\}$.

To see that this works, let $f, g \in \mathrm{C}(X)$ be nowhere constant and let $E \subseteq X$ be a nonvoid open set such that $f \restriction E \neq g \restriction E$. By shrinking E we may assume that $f[E] \cap g[E] = \emptyset$. For each α such that $\langle f_\alpha, g_\alpha, E_\alpha \rangle = \langle f, g, E \rangle$, we chose $x_\alpha \in E$. We have that $f_\alpha(x_\alpha) \notin g_\alpha[M]$, i.e., for each $\beta < \mathfrak{c}$ and each $m \in \mathbf{Z}$, $f_\alpha(x_\alpha) \neq g_\alpha(h^m(x_\beta))$. (This is clear from the choice of x_α. Consider separately the cases $\beta < \alpha$, $\beta = \alpha$, $\beta > \alpha$.) The rest of the properties of the example now follow easily. ∎

6 Variations on the theme

As mentioned in the introduction, properties similar to the SRU property have been considered by various authors. We examine some of them in this section.

Let us begin with the results from [DM] and [Bü] mentioned in the introduction. Dushnik and Miller [DM] showed that, under CH, there is an uncountable set $M \subseteq \mathbf{R}$ such that for any monotone (nonincreasing or nondecreasing) function $f: \mathbf{R} \to \mathbf{R}$, if $\{x \in \mathbf{R}: f(x) = x\}$ is nowhere dense, then $f[M] \cap M$ is countable. In a model of set theory where $\mathfrak{c} = \aleph_2$, $\mathbf{R}$ is covered by $\aleph_1$ meager sets and sets of cardinality $\aleph_1$ are meager (e.g., in the random real model), there is no such set. To see this, note that in any such model, for every uncountable set $M \subseteq \mathbf{R}$, there is a set $M' \subseteq M$ of the same cardinality as M such that M' is nowhere dense in $\mathbf{R}$. Then there is a monotone function $f: \mathbf{R} \to \mathbf{R}$ such that $\{x \in \mathbf{R}: f(x) = x\}$ is equal to the closure of M', and hence $f[M] \cap M$ has the same cardinality as M since it contains M'.

Consider now the result from [Bü] that, under CH, there is a totally heterogeneous set. (See the introduction for the definition.) A classical diagonalization argument shows that there is (in ZFC) a set $M \subseteq \mathbf{R}$ of cardinality $\mathfrak{c}$ such that for any Borel function $f: M \to M$, $f[\{x \in M: f(x) \neq x\}]$ is countable. It is not possible, however, to produce a totally heterogeneous set in ZFC. To see this, consider a model in which every set $M \subseteq \mathbf{R}$ of cardinality $\mathfrak{c}$ can be mapped continuously onto $[0,1]$ (e.g., the iterated perfect set model [Mi2]). It is easily seen that a totally heterogeneous set of reals cannot contain a Cantor set, and hence its complement must have cardinality $\mathfrak{c}$. In the model under consideration, if $M \subseteq [0,1]$ is any set such that both M and $[0,1] \setminus M$ have cardinality $\mathfrak{c}$, then let $f: M \to [0,1]$ be a continuous surjection. Choose $M' \subseteq M$ such that $f[M'] = M$. Then $f[\{x \in M': f(x) \neq x\}]$ contains $M \setminus M'$ and hence has cardinality $\mathfrak{c}$. In particular, M is not totally heterogeneous.

For the remainder of this section we will examine two generalizations of the notion of an SRU for $\mathrm{C}(\mathbf{R})$ for pairs of families $\mathcal{F}, \mathcal{G} \subseteq \mathrm{C}(\mathbf{R})$. We are interested particularly in the following subfamilies of $\mathrm{C}(\mathbf{R})$ which we will abbreviate as follows:

- $\mathrm{C} = \mathrm{C}(\mathbf{R})$;
- Const, the class of all constant functions $f \in \mathrm{C}(\mathbf{R})$;
- C_n, the class of all nowhere constant functions $f \in \mathrm{C}(\mathbf{R})$;
- C_c, the class of all countable-to-one functions $f \in \mathrm{C}(\mathbf{R})$.

Definition 6.1 A function $g \in \mathrm{C}$ is said to be a *truncation* of $f \in \mathrm{C}$ if g is constant on every connected component of $\{x \in \mathbf{R}\colon f(x) \neq g(x)\}$.

The following proposition is [BD, theorem 8.1].

Proposition 6.2 *There exists a set $M \subseteq \mathbf{R}$ such that for every $f \in \mathrm{C}_c$ and $g \in \mathrm{C}$ if $g[M] \subseteq f[M]$ then g is a truncation of f.* ■

Notice that a strong SRU M for $\langle \mathbf{R}, \mathcal{B}or, \mathcal{N}\rangle$, $\mathcal{N}$ = the meager ideal, has a similar property. If $f \in \mathrm{C}$ and $g \in \mathrm{C}$ and $g[M] \subseteq f[M]$, then by Lemma 4.12 (with $\kappa = \aleph_1$), the open intervals in $\{x \in \mathbf{R}\colon f(x) \neq g(x)\}$ on which g is constant are dense in $\{x \in \mathbf{R}\colon f(x) \neq g(x)\}$. We do not need to assume that $f \in \mathrm{C}_c$, however we cannot conclude that g is a truncation of f, even if $f \in \mathrm{C}_c$. (For example, we could have $f(x) = x$ for all x, and $g(x) = x$ for $x \notin [0,1]$, $g\restriction[0,1]$ = the Cantor ternary function.) Proposition 6.2 also has the advantage of being a ZFC theorem.

Definition 6.3 For the families $\mathcal{F}, \mathcal{G} \subseteq \mathrm{C}$ we say that $M \subseteq \mathbf{R}$ is an $\langle \mathcal{F}, \mathcal{G}\rangle$*-truncation SRU* if for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, if $g[M] \subseteq f[M]$ then g is a truncation of f. A set $M \subseteq \mathbf{R}$ is an $\langle \mathcal{F}, \mathcal{G}\rangle$*-SRU* if for every $f \in \mathcal{F}$ and $g \in \mathcal{G}$, if $g[M] \subseteq f[M]$ then $g = f$.

We have the following general fact.

Proposition 6.4 *For every $\mathcal{F}, \mathcal{G} \subseteq \mathrm{C}$*

(1) *If $M \subseteq \mathbf{R}$ is an $\langle \mathcal{F}, \mathcal{G}\rangle$-SRU then it is $\langle \mathcal{F}, \mathcal{G}\rangle$-truncation SRU;*

(2) *If $\mathcal{G} \subseteq \mathrm{C}_n$ then every $\langle \mathcal{F}, \mathcal{G}\rangle$-truncation SRU is an $\langle \mathcal{F}, \mathcal{G}\rangle$-SRU.*

Proof. This is obvious since every function is its own truncation and this is the only truncation that could be nowhere constant. ■

Proposition 6.4 shows in particular that for $\mathcal{G} \subseteq \mathrm{C}_n$, $M \subseteq \mathbf{R}$ is an $\langle \mathcal{F}, \mathcal{G}\rangle$-SRU if and only if it is an $\langle \mathcal{F}, \mathcal{G}\rangle$-truncation SRU. Thus, for $\mathcal{G} \subseteq \mathrm{C}_n$ we will examine only $\langle \mathcal{F}, \mathcal{G}\rangle$-SRU's.

In the next theorem we seek, for various choices of $\mathcal{G}$, the largest family $\mathcal{F}$ for which there exists an $\langle \mathcal{F}, \mathcal{G}\rangle$-SRU ($\langle \mathcal{F}, \mathcal{G}\rangle$-truncation SRU). The examples in (A′)-(C′) indicate that the families found in (A)-(C) are to some extent the best possible.

Theorem 6.5

(A) *$M = \mathbf{R}$ is a $\langle \mathrm{Const}, \mathrm{C}\rangle$-SRU.*

(A′) *There is no $\langle \{f\}, \mathcal{G}\rangle$-SRU if $f \in \mathrm{C} \setminus \mathrm{Const}$ and $\mathrm{Const} \subseteq \mathcal{G}$.*

(B) *There exists a $\langle \mathrm{C}_c, \mathrm{C}\rangle$-truncation SRU.*

(B′) *There is no $\langle \mathrm{C}_n, \mathrm{C}\rangle$-truncation SRU. More precisely, there is no $\langle \{f\}, \mathrm{C}\rangle$-truncation SRU for every $f \in \mathrm{C}_n$ with the property that $f^{-1}(y)$ is perfect for every $y \in \mathbf{R}$.*

(C) *If* $\mathbf{R}$ *is not the union less than* $\mathfrak{c}$ *many meager sets then there exists a* $\langle \mathrm{C}, \mathcal{G}\rangle$*-truncation SRU for any* $\mathcal{G} \subseteq \mathrm{C}$ *of cardinality less than* $\mathfrak{c}$*.*

(C′) *There exist* $f, g, h \in \mathrm{C}$ *such that there is no* $\langle \{h\}, \{f, g\}\rangle$*-SRU.*

(D) *Any strong SRU for* $\mathrm{C}(\mathbf{R})$ *is a* $\langle \mathrm{C}, \mathrm{C}_n\rangle$*-SRU.*

(E) *There exists an* $\langle \{f\}, \mathrm{C}_n\rangle$*-SRU for every* $f \in \mathrm{C}$*.*

Proof of (A). Obvious.

Proof of (A′). Let $f \in \mathrm{C} \setminus \mathrm{Const}$ and $M \subseteq \mathbf{R}$. If $M = \emptyset$ take an arbitrary $g \in \mathrm{Const} \subseteq \mathcal{G}$. If $M \neq \emptyset$ and $x_0 \in M$ let $g \in \mathcal{G}$ be a constant function equal to $f(x_0)$. Then $g \neq f$ but $g[M] \subseteq f[M]$.

Proof of (B). This is Proposition 6.2.

Proof of (B′). Let $f \in \mathrm{C}_n$ be such that $f^{-1}(y)$ is perfect for every $y \in \mathbf{R}$. For example, if $F = (f_0, f_1)\colon [0,1] \to [0,1]^2$ is a classical Peano curve (see e.g., [CLO, Example 4.3.8]) then we can define f by $f(n + r) = n + f_0(r)$ for every integer n and $r \in [0, 1)$.

Let $M \subseteq \mathbf{R}$. To see that it is not an $\langle \{f\}, \mathrm{C}\rangle$-truncation SRU we will find $g \in \mathrm{C}$ with $g[M] \subseteq f[M]$ which is not a truncation of f. We have two cases to consider.

Case 1. $f[M]$ is not dense in $\mathbf{R}$.

Take $c < d$ such that $(c, d) \cap f[M] = \emptyset$. Since $f[\mathbf{R}] = \mathbf{R}$ there exist $a < b$ such that $(a, b) \subseteq f^{-1}((c, d))$. So, $(a, b) \cap M = \emptyset$. Choose $a_0 < b_0$ such that $a < a_0 < b_0 < b$ and define g on $[a_0, b_0]$ to be nonconstant and such that $g(x) \neq f(x)$ for every $x \in [a_0, b_0]$. Put $g(x) = f(x)$ for every $x \in \mathbf{R} \setminus (a, b)$ and extend it to a continuous function on $\mathbf{R}$. Then g is not a truncation of f, while $g[M] = f[M]$.

Case 2. $f[M]$ is dense in $\mathbf{R}$.

If $f[M] = \mathbf{R}$ it is enough to take as g an arbitrary continuous function which is not a truncation of f. So, without loss of generality we can assume that $f[M] \neq \mathbf{R}$.

Choose $y_0 \in \mathbf{R} \setminus f[M]$ and let $P = f^{-1}(y_0)$. So, P is perfect, nowhere dense and $P \cap M = \emptyset$. Choose a countable dense subset D of $f[M]$ and notice the following fact.

($\star$) For every $d_0, d_1 \in D$, $d_0 < d_1$, there exists a continuous function g from $\mathbf{R}$ onto $[d_0, d_1]$ such that $g[\mathbf{R} \setminus P] \subseteq D \subseteq f[M]$ and g is not constant at on any open interval intersecting P.

To see it, let $h\colon [0,1] \to [0,1]$ be a classical Cantor function, i.e., h is nondecreasing, constant on any component of $[0,1] \setminus C$ and such that $h[C] = [0,1]$, where C is a classical Cantor ternary set. (See [Ro, p. 50].) Extend h to $\mathbf{R}$ by putting $h(x) = 0$ for $x < 0$ and $h(x) = 1$ for $x > 1$ and notice that $h[\mathbf{R} \setminus C] \subseteq \mathbf{Q}$. Let $h_0\colon \mathbf{R} \to \mathbf{R}$ be a homeomorphism such that $h_0[P] = C$ and $h_1\colon \mathbf{R} \to \mathbf{R}$ be an order isomorphism such that $h_1[\mathbf{Q}] = D$, $h_1(0) = d_0$ and $h_1(1) = d_1$. Then $g = h_1 \circ h \circ h_0$ satisfies ($\star$).

To finish the proof notice that for any function g satisfying ($\star$) we have $g[M] \subseteq g[\mathbf{R} \setminus P] \subseteq D \subseteq f[M]$. Now, if $g\colon \mathbf{R} \to [d_0, d_1]$ and $g'\colon \mathbf{R} \to [d_0', d_1']$ are as in ($\star$) and such that $[d_0, d_1] \cap [d_0', d_1'] = \emptyset$ then for every $x \in P$ we have $g(x) \neq g'(x)$. In particular, either g or g' is not a truncation of f.

Proof of (C). Choose $\mathcal{G} \subseteq \mathrm{C}$ of cardinality less than $\mathfrak{c}$. Since any constant function is a truncation of any other function we can assume without loss of generality that $\mathcal{G} \cap \mathrm{Const} = \emptyset$.

For $g \in \mathrm{C}$ let $\mathrm{Const}(g)$ denotes the set of these points at which g is locally constant, i.e.,

$$\mathrm{Const}(g) = \{x \in \mathbf{R}\colon (\exists a, b \in \mathbf{R})[a < x < b \ \&\ g \text{ is constant on } (a, b)]\}.$$

Then for every $g \in \mathcal{G}$ the set $P_g = \mathbf{R} \setminus \mathrm{Const}(g)$ is nonempty and perfect. In particular, it is not a union of less then continuum many its nowhere dense subsets.

Let $\{\langle f_\alpha, g_\alpha\rangle : \alpha < \mathfrak{c}\} = \{\langle f, g\rangle \in \mathrm{C} \times \mathcal{G} : g \text{ is not a truncation of } f\}$. We will construct, by induction on $\alpha < \mathfrak{c}$, a set $M = \{m_\alpha : \alpha < \mathfrak{c}\}$ such that $g_\alpha(m_\alpha) \notin f_\alpha[M]$ for every $\alpha < \mathfrak{c}$. This will finish the proof.

To have $g_\alpha(m_\alpha) \notin f_\alpha[M]$ we will choose m_α such that the following inductive conditions are satisfied.

$g_\alpha(m_\alpha) \notin \{f_\alpha(m_\alpha)\}$, i.e., such that

(I_α) $m_\alpha \in U_\alpha$, where $U_\alpha = \{x \in \mathbf{R} : f_\alpha(x) \neq g_\alpha(x)\}$.

$g_\alpha(m_\alpha) \notin \{f_\alpha(m_\gamma) : \gamma < \alpha\}$, i.e., such that

(II_α) $m_\alpha \notin \bigcup_{\gamma<\alpha} g_\alpha^{-1}(f_\alpha(m_\gamma))$.

$g_\alpha(m_\alpha) \notin \{f_\alpha(m_\gamma) : \gamma > \alpha\}$, i.e., such that $f_\alpha(m_\gamma) \neq g_\alpha(m_\alpha)$ for every $\alpha < \gamma$. By interchanging α and γ in the last condition we obtain $f_\gamma(m_\alpha) \neq g_\gamma(m_\gamma)$ for every $\gamma < \alpha$. So, it is enough to choose

(III_α) $m_\alpha \notin \bigcup_{\gamma<\alpha} f_\gamma^{-1}(g_\gamma(m_\gamma))$.

To make such a choice possible, we will also require that

($\star_\alpha$) $f_\alpha^{-1}(g_\alpha(m_\alpha))$ is nowhere dense in P_g for every $g \in \mathcal{G}$.

We will achieve this by making sure that $g_\alpha(m_\alpha) \notin S_\alpha^g$ for every $g \in \mathcal{G}$, where

$$S_\alpha^g = \{y \in \mathbf{R} : f_\alpha^{-1}(y) \cap P_g \text{ is not nowhere dense in } P_g\}.$$

Notice that each S_α^g is at most countable. So, we will guarantee ($\star_\alpha$) by choosing

(IV_α) $m_\alpha \notin \bigcup_{g\in\mathcal{G}} g_\alpha^{-1}(S_\alpha^g)$.

Clearly it is enough to show that the choice of such m_α is possible. So, assume that for some $\alpha < \mathfrak{c}$ the construction is done till step α. We will choose m_α.

Let

$$V_\alpha = \bigcup_{\gamma<\alpha} [g_\alpha^{-1}(f_\alpha(m_\gamma)) \cup f_\gamma^{-1}(g_\gamma(m_\gamma))] \cup \bigcup_{g\in\mathcal{G}} \bigcup_{y\in S_\alpha^g} g_\alpha^{-1}(y)$$

and put $T_\alpha = U_\alpha \setminus V_\alpha$. It is enough to show that $T_\alpha \cap P_{g_\alpha} \neq \emptyset$. But g_α is not a truncation of f_α. So, $|g_\alpha[U_\alpha]| = \mathfrak{c}$ and the set $U_\alpha \cap P_{g_\alpha}$ is nonempty and open in P_{g_α}. Since V_α is a union of less than continuum many sets, it is enough to argue that each of these sets is nowhere dense in P_{g_α}.

But sets $f_\gamma^{-1}(g_\gamma(m_\gamma))$ are nowhere dense in P_{g_α} by ($\star_\gamma$), i.e., the inductive assumption (IV_γ). To finish the proof it is enough to notice that $g_\alpha^{-1}(y)$ is nowhere dense in P_{g_α} for every $y \in \mathbf{R}$, which follows immediately from the definition of P_{g_α}.

Proof of (C$'$). Let $h(x) = \min\{0, x^2 - 1\}$, $g(x) = 0$, and $f(x) = \max\{h(x), x - 2\}$ for every $x \in \mathbf{R}$. Clearly $f \neq h \neq g$. It is enough to show that for every $M \subseteq \mathbf{R}$ either $f[M] \subseteq h[M]$ or $g[M] \subseteq h[M]$. But if $M \setminus (-1, 1) \neq \emptyset$ then $g[M] = \{0\} \subseteq h[M]$. Otherwise $M \subseteq (-1, 1)$ and $f[M] = h[M]$.

Proof of (D). Apply Lemma 4.12 with $\kappa = \aleph_1$, $g \in \mathrm{C}$ and $f \in \mathrm{C}_n$.

Proof of (E). Let $f \in \mathrm{C}$. If there are $a < b$ such that f is constant on $[a, b]$ it is enough to take $M = [a, b]$.

So, assume that $f \in \mathrm{C}_n$ and let $\{g_\alpha : \alpha < \mathfrak{c}\} = \mathrm{C}_c$. As in the proof of (C) it is enough to find $M = \{m_\alpha : \alpha < \mathfrak{c}\}$ such that

(I_α) $m_\alpha \in U_\alpha = \{x \in \mathbf{R} : f(x) \neq g_\alpha(x)\}$;

(II_α) $m_\alpha \notin \bigcup_{\gamma<\alpha} g_\alpha^{-1}(f(m_\gamma))$;

(III$_\alpha$) $m_\alpha \notin \bigcup_{\gamma<\alpha} f^{-1}(g_\gamma(m_\gamma))$.

But $|f[U_\alpha]| = \mathfrak{c}$, since $U_\alpha \neq \emptyset$ and $f \in \mathrm{C}_n$. Also $\left|\bigcup_{\gamma<\alpha} g_\alpha^{-1}(f(m_\gamma))\right| < \mathfrak{c}$ since $|g_\alpha^{-1}(f(m_\gamma))| \leq \aleph_0$. Thus,

$$\left| f\left[\bigcup_{\gamma<\alpha} g_\alpha^{-1}(f(m_\gamma))\right]\right| < \mathfrak{c}.$$

Moreover,

$$f\left[\bigcup_{\gamma<\alpha} f^{-1}(g_\gamma(m_\gamma))\right] = \bigcup_{\gamma<\alpha} f[f^{-1}(g_\gamma(m_\gamma))] \subseteq \{g_\gamma(m_\gamma)\colon \gamma<\alpha\}$$

has cardinality $< \mathfrak{c}$. So, the set

$$U_\alpha \setminus \left[\bigcup_{\gamma<\alpha} [g_\alpha^{-1}(f(m_\gamma)) \cup f^{-1}(g_\gamma(m_\gamma))]\right]$$

is nonempty. This finishes the proof of Theorem 6.5. ∎

Remark 6.6 There is no set $M \subseteq \mathbf{R}$ such that for every $f, g \in \mathrm{C}(\mathbf{R})$ (not necessarily nowhere constant), if $f[M] = g[M]$ then $f = g$.

Proof. Such a set M would be an SRU for $\mathrm{C}(\mathbf{R})$, so, by theorem 5.6, M is dense and is disjoint from a Cantor set K. But then we can build distinct Cantor-like functions f and g with $f[\mathbf{R}\setminus K] = g[\mathbf{R}\setminus K]$ (a countable set). ∎

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