Model with no magic set

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Abstract

We will prove that there exists a model of ZFC+"$\mathfrak{c} = \omega_2$" in which every $M \subseteq \mathbb{R}$ of cardinality less than continuum $\mathfrak{c}$ is meager, and such that for every $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f[X] = [0, 1]$.

In particular in this model there is no magic set, i.e., a set $M \subseteq \mathbb{R}$ such that the equation $f[M] = g[M]$ implies $f = g$ for every continuous nowhere constant functions $f, g : \mathbb{R} \to \mathbb{R}$.

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1 Introduction

The main goal of this paper is to prove the following theorem.

Theorem 1.1 *There exists a model of ZFC in which $\mathfrak{c} = \omega_2$,*

($\star$) *for every $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[X] = [0,1]$, and*

($\star\star$) *every $M \subseteq \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is meager.*

Note that ($\star$) of Theorem 1.1 is known to hold in the iterated perfect set model. (See A. W. Miller [Mi].) This result was also generalized by P. Corazza [Co] by finding another model leading to the following theorem.

Theorem 1.2 (Corazza) *It is consistent with ZFC that ($\star$) holds and*

($\star\star'$) *every $M \subseteq \mathbb{R}$ of cardinality less than $\mathfrak{c}$ is of strong (so Lebesgue) measure zero.*

Note that the condition ($\star\star$) is false in the iterated perfect set model and in Corazza model. (See [BuCi].)

Corazza noticed also that Theorem 1.2 implies the following corollary (since there exists a universal measure zero set of cardinality non($\mathcal{L}$), where non($\mathcal{L}$) is the smallest cardinality of a nonmeasurable set).

Corollary 1.3 (Corazza [Co, Thm 0.3]) *It is consistent with ZFC that ($\star$) holds and there is a universal measure zero set of cardinality $\mathfrak{c}$. In particular in this model there are $2^{\mathfrak{c}}$ many universal measure zero sets of cardinality $\mathfrak{c}$.*

He asked also whether the similar statement is true with "always first-category set" replacing "universal measure zero set." The positive answer easily follows from Theorem 1.1, since (in ZFC) there exists an always first-category set of cardinality non($\mathcal{M}$), where non($\mathcal{M}$) is the smallest cardinality of a nonmeager set.

Corollary 1.4 *It is consistent with ZFC that ($\star$) holds and there is an always first-category set of cardinality $\mathfrak{c}$. In particular in this model there are $2^{\mathfrak{c}}$ many always first-category sets of cardinality $\mathfrak{c}$.* ∎

Clearly Theorem 1.1 can be viewed as dual to Theorem 1.2. However, our original motivation for proving Theorem 1.1 comes from another source. In [BeDi] A. Berarducci and D. Dikranjan proved that under the Continuum Hypothesis (abbreviated as CH) there exists a set $M \subseteq \mathbb{R}$, called a *magic set*, such that for any two continuous nowhere constant functions $f, g\colon \mathbb{R} \to \mathbb{R}$ if $f[M] \subseteq g[M]$ then $f = g$. Different generalizations of a magic set were also studied by M. R. Burke and K. Ciesielski in [BuCi]. In particular they examined the *sets of range uniqueness for the class* $C(\mathbb{R})$, i.e., sets which definition is obtained from the definition of a magic set by replacing the implication "if $f[M] \subseteq g[M]$ then $f = g$" with "if $f[M] = g[M]$ then $f = g$." They proved [BuCi, Cor. 5.15 and Thm. 5.6(5)] that if $M \subseteq \mathbb{R}$ is a set of range uniqueness for $C(\mathbb{R})$ then M is not meager and there is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ for which $f[M] = [0, 1]$. This and Theorem 1.1 imply immediately the following corollary, which solves the problems from [BeDi] and [BuCi].

Corollary 1.5 *There exists a model of ZFC in which there is no set of range uniqueness for $C(\mathbb{R})$. In particular there is no magic set in this model.*

Finally, it is worthwhile to mention that for the class of nowhere constant differentiable function the existence of a magic set is provable in ZFC, as noticed by Burke, Ciesielski and Larson [BCL].

2 Preliminaries

Our terminology is standard and follows that from [BaJu], [Ci], or [Ku].

A model satisfying Theorem 1.1 will be obtained as a generic extension of a model V satisfying CH. The forcing used to obtain such an extension will be a countable support iteration $\mathbb{P}_{\omega_2}$ of length ω_2 of a forcing notion $\mathbb{P}$ defined below. Note that $\mathbb{P}$, which is a finite level version of Laver forcing,[1] is a version of a tree-forcing $\mathbb{Q}_1^{\text{tree}}(K, \Sigma)$ from [RoSh 470, sec. 2.3] (for a 2-big finitary local tree-creating pair (K, Σ); it is also a relative of the forcing notion defined in [RoSh 470, 2.4.10]) and most of the results presented in this section is a variation of general facts proved in this paper. To define $\mathbb{P}$, we need the following terminology.

[1] Note that Theorem 1.1 is false in Laver model, since in this model there is a $\mathfrak{c}$-Lusin set (there is a scale) and such a set cannot be mapped continuously onto $[0, 1]$.

A subset $T \subseteq \omega^{<\omega}$ is a *tree* if $t \restriction n \in T$ for every $t \in T$ and $n < \omega$. For a tree $T \subseteq \omega^{<\omega}$ and $t \in T$ we will write $\mathrm{succ}_T(t)$ for the set of all immediate successors of t in T, i.e.,

$$\mathrm{succ}_T(t) = \{s \in T \colon t \subseteq s \;\&\; |s| = |t| + 1\}.$$

We will use the symbol $\mathcal{T}$ to denote the set of all nonempty trees $T \subseteq \omega^{<\omega}$ with no finite branches, i.e.,

$$\mathcal{T} = \{T \subseteq \omega^{<\omega} \colon\; T \neq \emptyset \text{ is a tree } \;\&\; \mathrm{succ}_T(t) \neq \emptyset \text{ for every } t \in T\}.$$

For $T \in \mathcal{T}$ we will write $\lim T$ to denote the set of all branches of T, i.e.,

$$\lim T = \{s \in \omega^\omega \colon s \restriction n \in T \text{ for every } n < \omega\}.$$

Also if $t \in T \in \mathcal{T}$ then we define

$$T^t = \{s \in T \colon s \subseteq t \;\text{ or }\; t \subseteq s\}.$$

Now define inductively the following "very fast increasing" sequences $\langle b_i, n_i < \omega \colon i < \omega \rangle$ by putting $n_{-1} = 1$, and for $i < \omega$

$$b_i = (i+2)^{(n_{i-1}!)^i} \qquad \text{and} \qquad n_i = (b_i)^{(b_i)^i}.$$

In particular $b_0 = 2$, $n_0 = 2$, $b_1 = 9$, $n_1 = 9^9$, $b_2 = 4^{[(9^9)!]^2}$, etc. (For the purpose of our forcing any sequences that grows at least "as fast" would suffice.) Also let

$$T^\star = \bigcup_{k<\omega} \prod_{i<k} n_i = \{s \restriction k \colon k < \omega \;\&\; s \in \prod_{i<\omega} n_i\}$$

and

$$\mathcal{T}^\star = \{T \in \mathcal{T} \colon T \subseteq T^\star\}.$$

Forcing $\mathbb{P}$ is defined as a family of all trees $T \in \mathcal{T}^\star$ that have "a lot of branching." To define this last term more precisely we need the following definition for every $i < \omega$, $T \in \mathcal{T}$ and $t \in T \cap \omega^i$:

$$\mathrm{norm}_T(t) = \log_{b_i} \log_{b_i} |\mathrm{succ}_T(t)| \in [-\infty, \infty).$$

Note that $\mathrm{norm}_{T^\star}(t) = i$ for every $t \in T^\star \cap \omega^i$. Now for $T \in \mathcal{T}^\star$ and $k < \omega$ let

$$\underline{\mathrm{norm}}_T(k) = \inf\{\mathrm{norm}_T(t)\colon t \in T \ \&\ |t| \geq k\}$$

and define

$$\mathbb{P} = \left\{T \in \mathcal{T}^\star\colon \lim_{k\to\infty} \underline{\mathrm{norm}}_T(k) = \infty\right\}.$$

The order relation on $\mathbb{P}$ is standard. That is, $T_0 \in \mathbb{P}$ is stronger than $T_1 \in \mathbb{P}$, what we denote by $T_0 \geq T_1$, provided $T_0 \subseteq T_1$. Note also that $\underline{\mathrm{norm}}_T(k) \leq \underline{\mathrm{norm}}_{T^\star}(k) = k$ for every $k < \omega$.

In what follows for $t \in T \in \mathbb{P}$ we will also use the following notation

$$\underline{\mathrm{norm}}_T(t) = \underline{\mathrm{norm}}_{T^t}(|t|) = \inf\{\mathrm{norm}_T(s)\colon s \in T \ \&\ t \subseteq s\}.$$

It is easy to see that

$$\underline{\mathrm{norm}}_T(k) = \min\{\underline{\mathrm{norm}}_T(t)\colon t \in T \cap \omega^k\}.$$

For $n < \omega$ define a partial order $\leq_n$ on $\mathbb{P}$ by putting $T_0 \geq_n T$ if

$$T_0 \geq T \quad \& \quad T_0 \restriction \omega^k = T \restriction \omega^k \quad \& \quad \underline{\mathrm{norm}}_{T_0}(k) \geq n,$$

where $k = \min\{j < \omega\colon \underline{\mathrm{norm}}_T(j) \geq n\}$.

Note that the sequence $\{\leq_n\colon n < \omega\}$ witnesses forcing $\mathbb{P}$ to satisfy the axiom A. (In particular $\mathbb{P}$ is proper.) That is (see [BaJu, 7.1.1] or [RoSh 470, 2.3.7])

(i) $T_0 \geq_{n+1} T_1$ implies $T_0 \geq_n T_1$ for every $n < \omega$ and $T_0, T_1 \in \mathbb{P}$;

(ii) if $\{T_n\colon n < \omega\} \subseteq \mathbb{P}$ is such that $T_{n+1} \geq_n T_n$ for every $n < \omega$ then there exists $T \in \mathbb{P}$ extending each T_n, namely $T = \bigcap_{n<\omega} T_n \in \mathbb{P}$; (such T is often called a fusion of a sequence $\langle T_n\colon n < \omega\rangle$;) and,

(iii) if $\mathcal{A} \subseteq \mathbb{P}$ is an antichain, then for every $T \in \mathbb{P}$ and $n < \omega$ there exists $T_0 \in \mathbb{P}$ such that $T_0 \geq_n T$ and the set $\{S \in \mathcal{A}\colon S \text{ is compatible with } T\}$ is at most countable.

In fact, in case of the forcing $\mathbb{P}$ the set $\{S \in \mathcal{A}\colon S \text{ is compatible with } T\}$ from (iii) is finite. Since this fact will be heavily used in Section 5 we will include here its proof. (See Corollary 2.3.) However, this fact will not be used in the next three sections so it can be skipped in the first reading.

The following definition is a modification of the similar one for the Laver forcing. (See [BaJu, p. 353].)

Let $D \subseteq \mathbb{P}$ be dense below $p \in \mathbb{P}$ and $n < \omega$. For $t \in p$ with $\underline{\mathrm{norm}}_p(t) \geq n$ we define the ordinal number $r^n_D(t) < \omega$ as follows:

(1) $r^n_D(t) = 0$ if there exists $p' \in D$ extending p^t such that $\underline{\mathrm{norm}}_{p'}(t) \geq n-1$;

(2) if $r^n_D(t) \neq 0$ and $t \in \omega^i$ then

$$r^n_D(t) = \min\left\{\alpha\colon \left(\exists U \in [\mathrm{succ}_p(t)]^{\geq (b_i)^{(b_i)^{n-1}}}\right) \left(\forall s \in U\right) \left(r^n_D(s) < \alpha\right)\right\}.$$

Lemma 2.1 *Let $D \subseteq \mathbb{P}$ be dense below $p \in \mathbb{P}$ and $n < \omega$. Then $r^n_D(t)$ is well defined for every $t \in p$ with $\underline{\mathrm{norm}}_p(t) \geq n$.*

Proof. By way of contradiction assume that there exists $t \in p$ with $\underline{\mathrm{norm}}_p(t) \geq n$ for which $r^n_D(t)$ is undefined. Then $n > 1$ (since otherwise we would have $r^n_D(t) = 0$) and for any such t belonging to ω^i the set

$$U = \{s \in \mathrm{succ}_p(t)\colon r^n_D(s) \text{ is defined}\}$$

has cardinality less than $(b_i)^{(b_i)^{n-1}}$. So

$$|\{s \in \mathrm{succ}_p(t)\colon r^n_D(s) \text{ is undefined}\}| = |\mathrm{succ}_p(t) \setminus U| \geq |\mathrm{succ}_p(t)|/2 \qquad (1)$$

since $|\mathrm{succ}_p(t)|/2 = (b_i)^{(b_i)^{\mathrm{norm}_p(t)}}/2 \geq (b_i)^{(b_i)^n}/2 \geq (b_i)^{(b_i)^{n-1}} > |U|$. Construct a tree $p_0 \in \mathcal{T}^\star$ such that $p_0 \subseteq p^t$,

$$|\mathrm{succ}_{p_0}(s)| \geq |\mathrm{succ}_p(s)|/2 \qquad (2)$$

and $r^n_D(s)$ is undefined for every $s \in p_0$ with $t \subseteq s$. The construction can be easily done by induction on the levels of a tree, using (1) to make an inductive step. But (2) implies that for every $i < \omega$ and $s \in p_0 \cap \omega^i$ with $t \subseteq s$

$$\mathrm{norm}_{p_0}(s) = \log_{b_i} \log_{b_i} |\mathrm{succ}_{p_0}(s)| \geq \log_{b_i} \log_{b_i} |\mathrm{succ}_p(s)|/2 \geq \mathrm{norm}_p(s) - 1.$$

So $p_0 \in \mathbb{P}$. Take $p' \in D$ with $p' \geq p_0$. We can find $t_1 \in p'$ such that $\underline{\mathrm{norm}}_p(t_1) \geq \underline{\mathrm{norm}}_{p'}(t_1) \geq n-1$. Then $r^n_D(t_1) = 0$, contradicting the fact that $r^n_D(s)$ is undefined for every $s \in p_0 \supseteq p'$. ∎

Lemma 2.2 *Let $D \subseteq \mathbb{P}$ be dense below $p \in \mathbb{P}$ and $n < \omega$. Then for every $t \in p$ with $\underline{\mathrm{norm}}_p(t) \geq n$ there exist $p_t \geq_{n-1} p^t$ and a finite set $A_t \subseteq p_t$ such that $p_t = \bigcup_{s \in A_t} (p_t)^s$ and $(p_t)^s \in D$ for every $s \in A_t$.*

Proof. The proof is by induction on $r^n_D(t)$.

If $r^n_D(t)=0$ then $p_t=p'\in D$ will satisfy the lemma with $A_t=\{t\}$.

If $r^n_D(t)=\alpha>0$ choose $U\in[\mathrm{succ}_p(t)]^{\geq (b_i)^{(b_i)^{n-1}}}$ from the definition of $r^n_D(t)$. By the inductive assumption for every $u\in U$ there exist $q_u\geq_{n-1}p^u$ and a finite set $A_u\subseteq q_u$ such that $q_u=\bigcup_{s\in A_u}(q_u)^s$ and $(q_u)^s\in D$ for every $s\in A_u$. Then $p_t=\bigcup_{u\in U}q_u$ and $A_t=\bigcup_{u\in U}A_u$ satisfy the lemma. ∎

The next corollary can be also found, in general form, in [RoSh 470, 2.3.7, 3.1.1].

Corollary 2.3 *Let $\mathcal{A}\subseteq\mathbb{P}$ be an antichain. Then for every $p\in\mathbb{P}$ and $n<\omega$ there exists $q\in\mathbb{P}$ such that $q\geq_n p$ and the set*

$$\mathcal{A}_0=\{r\in\mathcal{A}\colon r \text{ is compatible with } q\}$$

is finite.

Proof. Extending $\mathcal{A}$, if necessary, we can assume that $\mathcal{A}$ is a maximal antichain. Thus $D=\{q\in\mathbb{P}\colon(\exists p\in\mathcal{A})(q\geq p)\}$ is dense in $\mathbb{P}$.

Let $i<\omega$ be such that $\underline{\mathrm{norm}}_p(i)\geq n+1$. By Lemma 2.2 for every $t\in p\cap\omega^i$ there exists $p_t\geq_n p^t$ and a finite set $A_t\subseteq p_t$ such that $p_t=\bigcup_{s\in A_t}(p_t)^s$ and $(p_t)^s\in D$ for every $s\in A_t$. Put $q=\bigcup_{t\in p\cap\omega^i}p_t$. Then it satisfies the corollary. ∎

3 Proof of the theorem

For $\alpha\leq\omega_2$ let $\mathbb{P}_\alpha$ be a countable support iteration of forcing $\mathbb{P}$ defined in the previous section. Thus $\mathbb{P}_\alpha$ is obtained from a sequence $\langle\langle\mathbb{P}_\beta,\dot{\mathbb{Q}}_\beta\rangle\colon\beta<\alpha\rangle$, where each $\mathbb{P}_\beta$ forces that $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for forcing $\mathbb{P}$. Also we will consider elements of $\mathbb{P}_\alpha$ as functions p which domains are countable subset of α. In particular if $p\in\mathbb{P}_\alpha$ and $0\in\mathrm{dom}(p)$ then $p(0)$ is an element of $\mathbb{P}$ as defined in V.

Now let V be a model of ZFC+CH and let G be a V-generic filter in $\mathbb{P}_{\omega_2}$. We will show that the conclusion of Theorem 1.1 holds in $V[G]$.

In what follows for $\alpha\leq\omega_2$ we will use the symbol G_α to denote $G\cap\mathbb{P}_\alpha$. In particular each G_α is a V-generic filter in $\mathbb{P}_\alpha$ and $V[G_\alpha]\subseteq V[G_{\omega_2}]=V[G]$.

Since CH holds in V, forcing $\mathbb{P}_{\omega_2}$ is ω_2-cc in V. Thus since $\mathbb{P}$ satisfies the axiom A, we conclude that $\mathbb{P}_{\omega_2}$ preserves cardinal numbers and indeed $\mathfrak{c}=\omega_2$ holds in $V[G]$.

To prove that $(\star\star)$ holds in $V[G]$ consider $\prod_{i<\omega} n_i = \lim T^\star$ with the product topology. Since $\prod_{i<\omega} n_i$ is homeomorphic to the Cantor set 2^ω it is enough to show that every subset S of $\prod_{i<\omega} n_i$ of cardinality less than $\mathfrak{c}^{V[G]} = \omega_2$ is meager in $\prod_{i<\omega} n_i$. But every $x \in S$ belongs already to some intermediate model $V[G_\alpha]$ with $\alpha < \omega_2$, since $\mathbb{P}$ satisfies the axiom A (so is proper), and the iteration is with countable support. In particular there exists an $\alpha < \omega_2$ such that $S \subseteq V[G_\alpha]$. So it is enough to prove that $\left(\prod_{i<\omega} n_i\right) \cap V[G_\alpha]$ is meager in $\prod_{i<\omega} n_i$.

Since $V[G_{\alpha+1}]$ is obtained from $V[G_\alpha]$ as a generic extension via forcing $\mathbb{P}$ (in $V[G_\alpha]$) our claim concerning $(\star\star)$ in $V[G]$ follows immediately from the following lemma. (See also [RoSh 470, 3.2.8].)

Lemma 3.1 *Let V be a model of ZFC+CH and H be a V-generic filter in $\mathbb{P}$. Then in $V[H]$ the set $\left(\prod_{i<\omega} n_i\right) \cap V$ is a meager subset of $\prod_{i<\omega} n_i$.*

Proof. Let $r \in \prod_{i<\omega} n_i$ be such that $\{r\} = \bigcap\{\lim T\colon T \in H\}$ and put $M = \bigcup_{j<\omega} M_j$ where

$$M_j = \left\{ s \in \prod_{i<\omega} n_i \colon s(k) \neq r(k) \text{ for every } j \leq k < \omega \right\}.$$

Since clearly every M_j is closed nowhere dense it is enough to show that $\left(\prod_{i<\omega} n_i\right) \cap V \subseteq M$. For this pick $s \in \left(\prod_{i<\omega} n_i\right) \cap V$ and consider a subset $D = \bigcup_{j<\omega} D_j \in V$ of $\mathbb{P}$, where

$$D_j = \{p \in \mathbb{P}\colon (\forall t \in p)(\forall k \in \operatorname{dom}(t) \setminus j)(s(k) \neq t(k))\}.$$

It is enough to prove that D is dense in $\mathbb{P}$, since $H \cap D_j \neq \emptyset$ implies that $s \in M_j$.

So let $p_0 \in \mathbb{P}$ and let $j < \omega$ be such that $p_0 \cap \omega^{j-1} \subseteq p_0(1)$ and define

$$p = \{t \in p_0 \colon (\forall k \in \operatorname{dom}(t) \setminus j)(s(k) \neq t(k))\}.$$

Clearly p is a tree. It is enough to show that $p \in \mathbb{P}$, since then $p \in D_j$ extends p_0. But if $t \in p_0 \cap \omega^k$ for some $k \geq j$ and $t_0 \in p$ is an immediate predecessor of t then

$$|\mathrm{succ}_p(t_0)| \geq |\mathrm{succ}_{p_0}(t_0)| - 1 = (b_{k-1})^{(b_{k-1})^{\mathrm{norm}_{p_0}(t_0)}} - 1 > 0$$

so $\mathrm{succ}_p(t_0)$ is nonempty and for every $s \in \lim p$

$$\lim_{i\to\infty} \underline{\mathrm{norm}}_p(s \restriction i) \geq \lim_{i\to\infty} (\underline{\mathrm{norm}}_{p_0}(s \restriction i) - 1) = \infty.$$

This finishes the proof of Lemma 3.1. ■

To show that $(\star)$ holds in $V[G]$ we will use the following two propositions. The first of them is an easy modification of the Factor Theorem from [BaJu, Thm 1.5.10]. For the case of Sacks forcing this has been proved in [BaLa, Thm 2.5].

Proposition 3.2 *Let $\beta < \alpha \leq \omega_2$ and γ be such that $\beta + \gamma = \alpha$. If $\mathbb{P}^\star_\gamma$ is a $\mathbb{P}_\beta$-name for the iteration $\mathbb{P}_\gamma$ of $\mathbb{P}$ (as constructed in $V^{\mathbb{P}_\beta}$) then forcings $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta \star \mathbb{P}^\star_\gamma$ are equivalent.* ■

The analog of the next proposition for the iteration of Sacks forcing can be found in an implicit form in [Mi].

Proposition 3.3 *Suppose that $p \Vdash$ "$\tau \in 2^\omega \setminus V$" for some $p \in \mathbb{P}_{\omega_2}$. Then (in V) there exists a continuous function $f\colon 2^\omega \to 2^\omega$ with the property that*

- *for every $r \in 2^\omega$ there exists $q_r \geq p$ such that*

$$q_r \Vdash f(\tau) = r.$$

The proof of Proposition 3.3 will be postponed to the next section. The proof of $(\star)$ based on Proposition 3.3 and presented below is an elaboration of the proof from [Mi] that $(\star)$ holds in the iterated Sacks model.

First note (compare [Co]) that to prove $(\star)$ it is enough to show that

($\circ$) for every $X \subseteq 2^\omega$ of cardinality $\mathfrak{c}$ there exists a continuous function $f\colon 2^\omega \to 2^\omega$ such that $f[X] = 2^\omega$.

Indeed if $X \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and there is no zero-dimensional perfect set $P \subset \mathbb{R}$ such that $|X \cap P| = \mathfrak{c}$ then X is a $\mathfrak{c}$-Lusin subset of $\mathbb{R}$. Then there is $\mathfrak{c}$-Lusin subset of 2^ω as well, and such a set would contradict ($\circ$) since it cannot be mapped continuously onto $[0,1]$ (so onto 2^ω as well). (See e.g. [Mi, Sec. 2].)

So there are $a, b \in \mathbb{R}$ and a zero-dimensional perfect set $P \subset [a,b]$ with $|X \cap P| = \mathfrak{c}$. But P and 2^ω are homeomorphic. Therefore, by ($\circ$), there exists a continuous $f\colon P \to P \subset [a,b]$ such that $f[X \cap P] = P$. Then a continuous extension $F\colon \mathbb{R} \to [a,b]$ of f, which exists by Tietze Extension theorem, has a property that $F[X] \supseteq P$. Now if $g\colon \mathbb{R} \to [0,1]$ is continuous and such that $g[P] = [0,1]$ then $f = g \circ F$ satisfies $(\star)$.

We will prove $(\circ)$ in $V[G]$ by contraposition. So let $X \subseteq 2^\omega$ be such that $f[X] \neq 2^\omega$ for every continuous $f\colon 2^\omega \to 2^\omega$. Thus for any such f there exists an $F_0(f) \in 2^\omega$ such that $F_0(f) \notin f[X]$. We will prove that this implies $|X| < \mathfrak{c}$ by showing that $X \subseteq V[G_\alpha]$ for some $\alpha < \omega_2$. This is enough, since $V[G_\alpha]$ satisfies CH.

Now let $D = 2^{<\omega} \in V$. Since D is dense in 2^ω any continuous $f\colon 2^\omega \to 2^\omega$ is uniquely determined by $f \restriction D$. Let $F\colon (2^\omega)^D \to 2^\omega$, $F \in V[G]$, be such that $F(f \restriction D) = F_0(f)$ for every continuous $f\colon 2^\omega \to 2^\omega$. Thus

$$F(f \restriction D) \notin f[X]$$

for every continuous $f\colon 2^\omega \to 2^\omega$. We claim that there exists an $\alpha < \omega_2$ of cofinality ω_1 such that

$$F \restriction \left((2^\omega)^D \cap V[G_\alpha]\right) \in V[G_\alpha]. \tag{3}$$

To show (3) first recall that for every real number r and every $\alpha \leq \omega_2$ of uncountable cofinality if $r \in V[G_\alpha]$ then $r \in V[G_\beta]$ for some $\beta < \alpha$. This is a general property of a countable support iteration of forcings satisfying the axiom A (and, more generally, proper forcings). In particular

$$(2^\omega)^D \cap V[G_\alpha] = \bigcup_{\beta<\alpha} \left((2^\omega)^D \cap V[G_\beta]\right) \tag{4}$$

for every $\alpha < \omega_2$ of cofinality ω_1.

Now let $\langle f_\alpha \colon \alpha < \omega_2\rangle \in V[G]$ be a one-to-one enumeration of $(2^\omega)^D$, and put $y_\alpha = F(f_\alpha)$. Then there exists a sequence $S = \langle\langle \varphi_\alpha, \eta_\alpha\rangle \colon \alpha < \omega_2\rangle \in V$ such that φ_α and η_α are the $\mathbb{P}_{\omega_2}$-names for f_α and y_α, respectively. Moreover, since $\mathbb{P}_{\omega_2}$ is ω_2-cc in V, we can assume that for every $\alpha < \omega_2$ there is a $\delta(\alpha) < \omega_2$ such that φ_α and η_α are the $\mathbb{P}_{\delta(\alpha)}$-names. Also if we choose $\delta(\alpha)$ as the smallest number with this property, then function δ belongs to V, since it is definable from $S \in V$.

Note also that for every $\beta < \omega_2$ there is an $h_0(\beta) < \omega_2$ with the property that for every $f \in (2^\omega)^D \cap V[G_\beta]$ there is $\gamma < h_0(\beta)$ such that φ_γ is a name for f (with respect to G). Once again using the fact that $\mathbb{P}_{\omega_2}$ is ω_2-cc in V

we can find in V a function $h\colon \omega_2 \to \omega_2$ bounding $h_0 \in V[G]$, i.e., such that $h_0(\beta) \le h(\beta)$ for every $\beta < \omega_2$. Let

$$C = \{\alpha < \omega_2 \colon (\forall \gamma < \alpha)(\delta(\gamma), h(\gamma) < \alpha)\} \in V.$$

Then C is closed and unbounded in ω_2. Pick $\alpha \in C$ of cofinality ω_1. We claim that α satisfies (3).

To see it, note first that the definition of δ implies that every name in the sequence $\langle\langle \varphi_\gamma, \eta_\gamma\rangle \colon \gamma < \alpha\rangle$ is a $\mathbb{P}_\alpha$-name. So

$$F \restriction \{f_\gamma \colon \gamma < \alpha\} = \{\langle f_\gamma, y_\gamma\rangle \colon \gamma < \alpha\} \in V[G_\alpha].$$

Moreover clearly $\{f_\gamma \colon \gamma < \alpha\} \subseteq (2^\omega)^D \cap V[G_\alpha]$. However, by (4), for every $f \in (2^\omega)^D \cap V[G_\alpha]$ there exists $\beta < \alpha$ such that $f \in (2^\omega)^D \cap V[G_\beta]$. Thus, by the definition of h_0 and h, there exists $\gamma < h_0(\beta) \le h(\beta) < \alpha$ such that $f = f_\gamma$. So $\{f_\gamma \colon \gamma < \alpha\} = (2^\omega)^D \cap V[G_\alpha]$ and (3) has been proved.

Now take an $\alpha < \omega_2$ having property (3). For this α we will argue that $X \subseteq V[G_\alpha]$. But, by Proposition 3.2, $V[G]$ is a generic extension of $V[G_\alpha]$ via forcing $\mathbb{P}_{\omega_2}$ as defined in $V[G \cap \mathbb{P}_\alpha]$. Thus without loss of generality we can assume that $V[G_\alpha] = V$. In particular

$$F_1 = F \restriction \left((2^\omega)^D \cap V\right) \in V.$$

To see that $X \subseteq V$ take an arbitrary $z \in 2^\omega \setminus V$, and pick a $\mathbb{P}_{\omega_2}$-name τ for z. Let $p_0 \in G \subset \mathbb{P}_{\omega_2}$ be such that $p_0 \Vdash$ "$\tau \in 2^\omega \setminus V$" and fix an arbitrary $p_1 \ge p_0$. Working in V we will find a $p \in \mathbb{P}_{\omega_2}$ stronger than p_1 and a continuous function $f \in V$ from 2^ω to 2^ω such that

$$p \Vdash f(\tau) = F_1(f \restriction D). \tag{5}$$

To see it notice that by Proposition 3.3 there exists a continuous function $f\colon 2^\omega \to 2^\omega$ such that for every $r \in 2^\omega$ (from V) there exists $q_r \ge p_1$ with

$$q_r \Vdash f(\tau) = r.$$

Take $r = F_1(f \restriction D) \in V$. Then $p = q_r$ satisfies (5).

Now (5) implies that the set

$$E = \{q \in \mathbb{P}_{\omega_2} \colon (\exists \text{ continuous } f\colon 2^\omega \to 2^\omega)(q \Vdash \text{“}f(\tau) = F_1(f \restriction D))\text{”}\} \in V$$

is dense above $p_0 \in G$. Therefore, there exist $q \in G \cap E$ and a continuous function $f\colon 2^\omega \to 2^\omega$ such that $q \Vdash$ "$f(\tau) = F_1(f \restriction D)$." In particular

$f(z) = F_1(f{\restriction}D) = F(f{\restriction}D) \notin f[X]$, implying that $z \notin X$. Since it is true for every $z \in 2^\omega \setminus V$, we conclude that $X \subset V$.

This finishes the proof of Theorem 1.1 modulo the proof of Proposition 3.3. ■

4 Proof of Proposition 3.3 — another reduction

In this short section we will prove Proposition 3.3 based on one more technical lemma. The proof of the lemma will be postponed to the next section.

To state the lemma and prove the proposition we need the following iteration version of the axiom A. For $\alpha \le \omega_2$, $F \in [\alpha]^{<\omega}$, and $n < \omega$ define a partial order relation $\le_{F,n}$ on $\mathbb{P}_\alpha$ by

$$q \ge_{F,n} p \;\Leftrightarrow\; q \ge p \;\&\; (\forall \xi \in F)(q{\restriction}\xi \Vdash q(\xi) \ge_n p(\xi)).$$

Note that if $\xi \notin \mathrm{dom}(p)$ for some $\xi \in F$ then it might be unclear what we mean by $p(\xi)$ in the above definition. However, in such a case we will identify p with its extension, for which we put $p(\xi) = \hat{T}^\star$, where $\hat{T}^\star$ is the standard $\mathbb{P}_\xi$-name for the weakest element $T^\star$ of $\mathbb{P}$. Recall also that if an increasing sequence $\langle F_n \colon n < \omega\rangle$ of finite subsets of α and $\langle p_n \in \mathbb{P}_\alpha \colon n < \omega\rangle$ are such that $p_{n+1} \ge_{F_n,n} p_n$ for every $n < \omega$ and $\bigcup_{n<\omega} \mathrm{dom}(p_n) = \bigcup_{n<\omega} F_n$ then there exists $q \in \mathbb{P}_\alpha$ extending each p_n. (See e.g. [BaJu, 7.1.3].)

Lemma 4.1 *Let $\alpha < \omega_2$, $p \in \mathbb{P}_\alpha$ and τ be a $\mathbb{P}_\alpha$-name such that for every $\gamma < \alpha$*

$$p \Vdash \tau \in 2^\omega \cap V[G_\alpha] \setminus V[G_\gamma].$$

Then there exists $q_0 \in \mathbb{P}_\alpha$ stronger than p such that for every $F \in [\alpha]^{<\omega}$, $n < \omega$, and $q \in \mathbb{P}_\alpha$ extending q_0 there exist (in V) an $m < \omega$, nonempty disjoint sets $B_0, B_1 \subset 2^m$ and $p_0, p_1 \ge_{F,n} q$ such that

$$p_j \Vdash \tau{\restriction}m \in B_j$$

for $j < 2$.

Basically Lemma 4.1 is true since p forces that τ is a new real number. However, its proof is quite technical and will be postponed for the next section.

Next we will show how Lemma 4.1 implies Proposition 3.3.

Proof of Proposition 3.3. Let $p \in \mathbb{P}_{\omega_2}$ and τ be a $\mathbb{P}_{\omega_2}$-name such that $p \Vdash$ "$\tau \in 2^\omega \setminus V$." Then, replacing p with some stronger condition if necessary, we can assume that there exists $\alpha < \omega_2$ such that for every $\gamma < \alpha$

$$p \Vdash \tau \in V[G_\alpha] \setminus V[G_\gamma].$$

In particular, since $p \Vdash$ "$\tau \in V[G_\alpha]$," we can assume that τ is a $\mathbb{P}_\alpha$-name. We can also find $p' \geq p \restriction \alpha$ such that

$$p' \Vdash \tau \in 2^\omega \cap V[G_\alpha] \setminus V[G_\gamma].$$

Thus it is enough to assume that $p \in \mathbb{P}_\alpha$ and find $f \in V$ and $q_r \in \mathbb{P}_\alpha$ satisfying Proposition 3.3. (Otherwise, we can replace q_r's with $q_r \cup p \restriction (\omega_2 \setminus \alpha)$.)

For $m < \omega$ and $B \subseteq 2^m$ let $[B] = \{x \in 2^\omega \colon x \restriction m \in B\}$. Thus $[B]$ is a clopen subset of 2^ω. For every $s \in 2^{<\omega}$ we will define $q_s \in \mathbb{P}_\alpha$, $m_s < \omega$ and $B_s \subseteq 2^{m_s}$. The construction will be done by induction on length $|s|$ of s. Simultaneously we will construct an increasing sequence $\langle F_n \in [\alpha]^{<\omega} \colon n < \omega \rangle$ such that the following conditions are satisfied for every $s \in 2^{<\omega}$ and $n = |s|$:

(I0) $\bigcup\{\mathrm{dom}(q_t) \colon t \in 2^{<\omega}\} = \bigcup_{n<\omega} F_n$;

(I1) $q_{s0}, q_{s1} \geq_{F_n,n} q_s$;

(I2) $B_{s0} \cap B_{s1} = \emptyset$, and $[B_{s0}] \cup [B_{s1}] \subseteq [B_s]$;

(I3) $q_{sk} \Vdash$ "$\tau \restriction m_s \in B_{sk}$" for every $k < 2$.

It is easy to fix an inductive schema of choice of F_n's which will force condition (I0) to be satisfied. Thus we will assume that we are using such a schema throughout the construction, without specifying its details.

Now let q_0 be as in Lemma 4.1. This will be our $q_\emptyset$. Moreover if q_s is already defined for some $s \in 2^{<\omega}$ then we choose m_s, q_{s0}, q_{s1}, B_{s0}, and B_{s1} by using Lemma 4.1 for $q = q_s \geq q_0$, $n = |s|$ and $F = F_n$. This finishes the inductive construction.

Next for $n < \omega$ let $m_n = \max\{m_s \colon s \in 2^{\leq n}\}$ and for $s \in 2^n$ and $k < 2$ put $B^\star_{sk} = \{t \in 2^{m_n} \colon t \restriction m_s \in B_{sk}\}$. Then $p_{sk} \Vdash$ "$\tau \restriction m_n \in B^\star_{sk}$" for every $k < 2$.

Thus, replacing sets B_{sk} with $B^\star_{sk}$ if necessary, we can assume that $m_s = m_n$ for every $s \in 2^n$.

Note also that $\lim_{n\to\infty} m_n = \infty$. This follows easily from (I3) and (I2). Let

$$P = \bigcap_{n<\omega} \bigcup_{s\in 2^n} [B_s].$$

Then P is perfect subset of 2^ω. Define function $f_0\colon P \to 2^\omega$ by putting

$$f_0(x) = r \quad \text{if and only if} \quad x \in [B_{r\restriction n}] \text{ for every } n < \omega.$$

It is easy to see that f_0 is continuous. Thus, by Tietze Extension theorem, we can find a continuous extension $f\colon 2^\omega \to 2^\omega$ of f_0. We will show that f satisfies the requirements of Proposition 3.3.

Indeed take $r \in 2^\omega$ and let $q_n = q_{r\restriction n}$. Then, by (I1), $q_{n+1} \geq_{F_n,n} q_n$ for every $n < \omega$. Moreover, by (I0), $\bigcup_{n<\omega} \mathrm{dom}(q_n) \subseteq \bigcup_{n<\omega} F_n$. In addition, we can assume that the equation holds, upon the identification described in the definition of $\geq_{F,n}$. Thus there exists a $q_r \in \mathbb{P}_\alpha$ extending each q_n. But for every $n < \omega$

$$q_{n+1} \Vdash \tau \restriction m_n \in B_{r\restriction n+1}$$

so that

$$q_r \Vdash f_0([\{\tau\restriction m_n\}] \cap P) \in f_0([B_{r\restriction n+1}] \cap P) \subseteq [\{r\restriction n+1\}].$$

Therefore, by the continuity of f,

$$q_r \Vdash f(\tau) = r.$$

This finishes the proof of Proposition 3.3.

5 Proof of Lemma 4.1

We will start this section with the following property that will be used several times in the sequel.

Lemma 5.1 *Forcing $\mathbb{P}$ has the property* **B** *from [BaJu, p. 330]. That is, for every $p \in \mathbb{P}$, a $\mathbb{P}$-name μ, and $k < \omega$, if $p \Vdash$ "$\mu \in \omega$" then there exist $m < \omega$ and $p' \geq_k p$ such that $p' \Vdash$ "$\mu \leq m$."*

Proof. This follows immediately from Corollary 2.3 applied to a maximal antichain in the set $D = \{q \geq p \colon (\exists m < \omega)(q \Vdash \mu = m)\}$. ■

Recall also the following result concerning the property **B**. (See [BaJu, Lemma 7.2.11].)

Corollary 5.2 *Let $\alpha \leq \omega_2$. If $p \in \mathbb{P}_\alpha$, $n \in \omega$, $F \in [\omega_2]^{<\omega}$ and $p \Vdash$ "$\mu \in \omega$" then there exist $m < \omega$ and $p' \geq_{F,n} p$ such that $p' \Vdash$ "$\mu < m$."*

The difficulty of the proof of Lemma 4.1 comes mainly from the fact that we have to find "real" sets B_0 and B_1 using for this only $\mathbb{P}_\alpha$-name τ, and $p \in \mathbb{P}_\alpha$, which is also formed mainly from different names. For this we will have to describe how to recover "real pieces of information" form τ and p. We will start this with the following lemma.

Lemma 5.3 *Let $p \in \mathbb{P}$ and τ be a $\mathbb{P}$-name such that $p \Vdash$ "$\tau \in 2^\omega$." Then for every $n, m < \omega$ there exist $q \geq_n p$, $i < \omega$, and a family $\{x_s \in 2^m \colon s \in q \cap \omega^i\}$ such that for any $s \in q \cap \omega^i$*

$$q^s \Vdash \tau \restriction m = x_s.$$

Proof. Let $D = \{p \in \mathbb{P} \colon (\exists x \in 2^m)(p \Vdash \tau \restriction m = x)\}$ and let $j < \omega$ be such that $\underline{\mathrm{norm}}_p(j) \geq n+1$. By Lemma 2.2 for every $t \in p \cap \omega^j$ there exist $p_t \geq_n p^t$ and a finite set $A_t \subseteq p_t$ such that $p_t = \bigcup_{s \in A_t} (p_t)^s$ and $(p_t)^s \in D$ for every $s \in A_t$. Put $q = \bigcup_{t \in p \cap \omega^j} p_t$ and let $i < \omega$ be such that $\bigcup\{A_t \colon t \in p \cap \omega^j\} \subseteq \omega^{\leq i}$. Then q and i satisfy the requirements. ■

Let $p \in \mathbb{P}$ and τ be a $\mathbb{P}$-name such that $p \Vdash$ "$\tau \in 2^\omega$." We will say that p *reads τ continuously* if for every $m < \omega$ there exist $i_m < \omega$ and a family $\{x_s \in 2^m \colon s \in p \cap \omega^{i_m+1}\}$ such that for any $s \in p \cap \omega^{i_m+1}$

$$p^s \Vdash \tau \restriction m = x_s.$$

Lemma 5.4 *Let $p \in \mathbb{P}$ and τ be a $\mathbb{P}$-name such that $p \Vdash$ "$\tau \in 2^\omega$." Then for every $n < \omega$ there exists $q \geq_n p$ such that q reads τ continuously.*

Proof. By Lemma 5.3 we can define inductively a sequence $\langle q_m \colon m < \omega \rangle$ such that $q_0 = p$ and for every $m < \omega$

- $q_{m+1} \geq_{n+m} q_m$; and,

- there exist $i_m < \omega$ and a family $\{x_s \in 2^m \colon s \in q \cap \omega^{i_m+1}\}$ such that for any $s \in q \cap \omega^{i_m+1}$

$$q^s \Vdash \tau \restriction m = x_s.$$

Then the fusion $q = \bigcap_{m<\omega} q_m$ of all q_m's has the desired properties. ■

The next lemma is an important step in our proof of Lemma 4.1. It also implies it quite easily for $\mathbb{P}_\alpha = \mathbb{P}$. (See Corollary 5.6.)

Lemma 5.5 *Let $p \in \mathbb{P}$ and τ be a $\mathbb{P}$-name such that*

$$p \Vdash \tau \in 2^\omega \setminus V$$

and p reads τ continuously with the sequence $\langle i_m \colon m < \omega\rangle$ witnessing it. Then for every $n, k < \omega$ with $\underline{\mathrm{norm}}_p(k) \geq n+1 \geq 2$ there exist an arbitrarily large number $m < \omega$ and $q \geq_n p$ which can be represented as

$$q = \bigcup_{t \in A} p_t, \tag{6}$$

where $A \subseteq p \cap \left(\omega^{\leq i_m} \setminus \omega^{<k}\right)$ and the elements of A are pairwise incompatible (as functions). Moreover for every $t \in A$ we have $p_t \geq p^t$, and there exists a $\underline{\text{one-to-one}}$ mapping $\mathrm{succ}_q(t) \ni s \longmapsto x_s \in 2^m$ such that

$$q^s \Vdash \tau \restriction m = x_s$$

for every $s \in \mathrm{succ}_q(t)$.

Proof. Fix p, τ, n, and k as in the lemma. For every $u \in p \cap \omega^k$ and $m < \omega$ with $i = i_m > k$ consider the following *trimming procedure.*

For every $s \in p^u \cap \omega^{i+1}$ let $x_s \in 2^m$ be such that

$$p^s \Vdash \tau \restriction m = x_s,$$

put $q_{i+1} = p^u$, and assign to every $s \in p^u \cap \omega^{i+1}$ a tag "constant x_s." By induction we define a sequence

$$q_{i+1} \leq q_i \leq q_{i-1} \leq q_{i-2} \leq \ldots \leq q_k$$

of elements of $\mathbb{P}$ such that for $k \leq j \leq i$ every $t \in q_j \cap \omega^j$ has a tag of either "one-to-one" or a "constant x_t" with $x_t \in 2^m$.

If for some $j \geq k$ the tree q_{j+1} is already defined then for every $t \in q_{j+1} \cap \omega^j$ choose $U_t \subseteq \mathrm{succ}_{q_{j+1}}(t)$ of cardinality $\geq .5\, |\mathrm{succ}_{q_{j+1}}(t)| \geq |\mathrm{succ}_{q_{j+1}}(t)|^{1/2}$ such that either every $s \in \mathrm{succ}_{q_{j+1}}(t) \subseteq q_{j+1}$ has the tag "one-to-one" or every such an s has a tag "constant x_s." In the first case put $V_t = U_t$ and tag t as "one-to-one." In the second case we can find a subset V_t of U_t of size at least $|U_t|^{1/2} \geq |\mathrm{succ}_{q_{j+1}}(t)|^{1/4}$ such that the mapping $V_t \ni s \longmapsto x_s \in 2^m$ is either one-to-one or constant equal to x_t. We tag t accordingly and define

$$q_j = \bigcup \{(q_{j+1})^s \colon (\exists t \in q_{j+1} \cap \omega^j)(s \in V_t)\}.$$

This finishes the "trimming" construction.

Note that by the construction for every $k \leq j \leq i$ and $t \in q_j \cap \omega^j$:

- $q_j \cap \omega^j = q_{j+1} \cap \omega^j$;
- $\mathrm{norm}_{q_j}(s) = \mathrm{norm}_{q_{j+1}}(s)$ for every $s \in q_j \setminus \omega^j$;
- $\mathrm{norm}_{q_j}(t) = \log_{b_j} \log_{b_j} |V_t| \geq \log_{b_j} \log_{b_j} |\mathrm{succ}_{q_{j+1}}(t)|^{\frac{1}{4}} = \mathrm{norm}_{q_{j+1}}(t) + \log_{b_j} \frac{1}{4} \geq \mathrm{norm}_p(t) - 1$;
- if t has a tag "constant x_t" then every $s \in (q_j)^t \cap \left(\bigcup_{j \leq l \leq i} \omega^l\right)$ has also the tag "constant x_t."

In particular $\underline{\mathrm{norm}}_{q_k}(u) \geq \underline{\mathrm{norm}}_p(u) - 1 \geq \underline{\mathrm{norm}}_p(k) - 1 \geq n$. Thus if we put $q_{m,u} = q_k$ then $\underline{\mathrm{norm}}_{q_{m,u}}(u) \geq n$ and either u has a tag "one-to-one" or "constant x_u." Moreover in the second case all $s \in q_{m,u} \cap \omega^i$ have the same tag "constant x_u."

Now if for some $m < \omega$ every $u \in p \cap \omega^k$ is tagged in $q_{m,u}$ as "one-to-one" then it is easy to see that

$$q = \bigcup \{q_{m,u} \colon u \in p \cap \omega^k\}$$

has a representation as in (6). Indeed, for every $s \in q \cap \omega^i$ let j_s be the largest $j \leq i$ such that $s \restriction j$ is tagged "one-to-one" in $q_{m,u}$. Let $A = \{s \restriction j_s \colon s \in q \cap \omega^i\}$. Then $\bigcup_{s \in A} q^s$ is the required representation.

Thus it is enough to prove that there exist an arbitrarily large m such that all $u \in p \cap \omega^k$ have a tag "one-to-one" in $q_{m,u}$.

By way of contradiction assume that this is not the case. Then there exist an infinite set $X_0 \subseteq \omega$ and $u \in p \cap \omega^k$ such that for every $m \in X_0$ there exists $x_m \in 2^m$ with u having a tag "constant x_m" in $q_{m,u}$. In particular,

$$q_{m,u} \Vdash \tau \restriction m = x_m.$$

By induction choose an infinite sequence $X_0 \supset X_1 \supset X_2 \supset \cdots$ of infinite sets such that for every $i < \omega$ there exist $y_i \in 2^i$ and $T_i \subseteq \omega^{\leq i}$ with the property that $q_{m,u} \cap \omega^{\leq i} = T_i$ and $x_m \restriction i = y_i$ for every $m \in X_i$.

Choose an infinite set $X = \{m_i < \omega \colon i < \omega\}$ such that $m_i \in X_i$ for every $i < \omega$ and let $q' = \lim_{i \to \infty} q_{m_i,u} = \bigcup_{i<\omega} T_i$. Then for every $t \in q' \cap T_i$ we have $\mathrm{norm}_{q'}(t) = \mathrm{norm}_{q_{m_i,u}}(t) \geq \mathrm{norm}_p(t) - 1$. Thus $q \in \mathbb{P}$ and $q \geq p$, as $q_{m_i,u} \geq p$ for every $i < \omega$. So it is enough to prove that

$$q' \Vdash \text{“}\tau \restriction j = y_j\text{”} \qquad \text{for every } j < \omega \tag{7}$$

since then $y = \bigcup_{j<\omega} y_j \in 2^\omega \cap V$ and $q' \Vdash$ "$\tau = y \in V$," contradicting the fact that $p \Vdash$ "$\tau \notin V$."

To see (7) fix a $j < \omega$ and let $l < \omega$ be such that $l \geq j$ and $l > i_j$. Take an $m \in X_l \subseteq X_j$ such that $m \geq j$. Then $q_{m,u} \cap \omega^{\leq l} = T_l = q' \cap \omega^{\leq l}$.

Fix an arbitrary $s \in q_{m,u} \cap \omega^l = q' \cap \omega^l$. Then $q^s_{m,u} \Vdash$ "$\tau \restriction m = x_m$" while $x_m \restriction j = y_j$, since $m \in X_j$. Thus

$$q^s_{m,u} \Vdash \tau \restriction j = y_j.$$

But $s \in q_{m,u} \cap \omega^l \subseteq p \cap \omega^{>i_j}$. So there exists an $x_s \in 2^j$ with the property that $p^s \Vdash$ "$\tau \restriction j = x_s$." Since $q^s_{m,u} \geq p^s$ we conclude that $q^s_{m,u}$ forces the same thing and so $x_s = y_j$. Thus, $p^s \Vdash$ "$\tau \restriction j = y_j$." But $(q')^s \geq p^s$. So

$$(q')^s \Vdash \tau \restriction j = y_j$$

as well. Since it happens for every $s \in q' \cap \omega^l$ and $j < \omega$ was arbitrary, we conclude (7). ∎

The next corollary is equivalent of Lemma 4.1 for $\alpha = 1$. It will not be used in a sequel. However the same approach will be used in the proof of Lemma 4.1 in its general form, and the proof presented here can shed some light on what follows.

Corollary 5.6 *Let $p \in \mathbb{P}$ and τ be a $\mathbb{P}$-name such that*

$$p \Vdash \tau \in 2^\omega \setminus V$$

and p reads τ continuously. Then for every $n < \omega$ there exist an $m < \omega$, nonempty disjoint sets $B_0, B_1 \subset 2^m$, and $p_0, p_1 \geq_n p$ such that

$$p_i \Vdash \tau \restriction m \in B_i$$

for $i < 2$.

Proof. Let $k < \omega$ be such that $\underline{\mathrm{norm}}_p(k) \geq n + 5$. Then, by Lemma 5.5, there exist $m, i_m < \omega$, and $q \geq_{n+4} p$ such that

$$q = \bigcup_{t \in A} p_t,$$

where $A \subseteq p \cap \left(\omega^{\leq i_m} \setminus \omega^{<k}\right)$, the elements of A are pairwise incompatible, $p_t \geq p^t$ for every $t \in A$, and for every $t \in A$ there exists a one-to-one mapping $h_t \colon \mathrm{succ}_q(t) \to 2^m$ such that

$$q^s \Vdash \tau \restriction m = h_t(s)$$

for every $s \in \mathrm{succ}_q(t)$.

Let $\{t_j \colon j < M\}$ be a one-to-one enumeration of A such that $|t_j| \leq |t_{j+1}|$ for every $j < M - 1$. By induction on $j < M$ we will choose a sequence $\langle C^i_j \colon i < 2 \ \&\ j < M\rangle$ such that for every $i < 2$ and $j < M$

- $C^i_j \in [\mathrm{succ}_q(t_j)]^{(b_l)^{(b_l)^n}}$, where $l = |t_j|$, and
- the sets $\{h_{t_j}[C^i_j] \subset 2^m \colon i < 2 \ \&\ j < M\}$ are pairwise disjoint.

Given $\langle C^i_r \colon i < 2 \ \&\ r < j\rangle$ the choice of C^0_j and C^1_j is possible since for $l = |t_j|$

$$\left| \bigcup_{i<2,\, r<j} C^i_r \right| \leq 2j \ (b_l)^{(b_l)^n} \leq 2\left|T^\star \cap \omega^{\leq l}\right| \ (b_l)^{(b_l)^n} \leq (b_l)^{(b_l)^{n+2}}$$

and $|\mathrm{succ}_q(t_j)| \geq (b_l)^{(b_l)^{n+4}}$ so we can choose disjoint $C^0_j, C^1_j \in [\mathrm{succ}_q(t_j)]^{(b_l)^{(b_l)^n}}$ with

$$h_{t_j}[C^0_j \cup C^1_j] \cap \left(\bigcup_{i<2,\, r<j} h_{t_r}[C^i_r] \right) = \emptyset.$$

For $i < 2$ define $p_i = \bigcup\{q^s \colon s \in \bigcup_{j<M} C^i_j\}$ and $B_i = \bigcup_{j<M} h_{t_j}[C^i_j]$. It is easy to see that they have the required properties. ∎

Let us also note the following easy fact.

Lemma 5.7 *Let $\mathbb{Q}$ be an arbitrary forcing, $q \in \mathbb{Q}$, and let τ be a $\mathbb{Q}$-name such that*

$$q \Vdash \tau \in 2^\omega \setminus V.$$

Then for every $N < \omega$ there exists an $m_0 < \omega$ with the following property. If $m_0 \leq m < \omega$ then there exist $\{q_n \geq q \colon n < N\}$, and a one-to-one sequence $\langle z_n \in 2^m \colon n < N\rangle$ such that

$$q_n \Vdash \tau \restriction m = z_n$$

for every $n < N$.

Proof. By induction on $n < N$ define infinite sequences $\{x_i^n \in 2^i \colon i < \omega\}$ and $q \leq q_0^n \leq q_1^n \leq q_2^n, \ldots$ such that for every $i < \omega$

$$q_i^n \Vdash \tau \restriction i = x_i^n.$$

Moreover if $x^n = \bigcup_{i<\omega} x_i^n \in 2^\omega \cap V$, then the construction will be done making sure that $x^n \notin \{x^k \colon k < n\}$. It is possible, since $\{x^k \colon k < n\} \in V$, while q forces that τ is not in V.

Now choose $m_0 < \omega$ such that all restrictions $\{x^n \restriction m_0 \colon n < N\}$ are different. Then for $m_0 \leq m < \omega$ define $z_n = x^n \restriction m$ and $q_n = p_m^n$ for every $n < N$. Clearly they have the desired properties. ■

Remark 5.8 In the text that follows (including the next lemma) we will often identify forcing $\mathbb{P}_\alpha$ with $\mathbb{P}_\beta \star \mathbb{P}_\gamma^\star$, where $\beta + \gamma = \alpha$ and $\mathbb{P}_\gamma^\star$ is a $\mathbb{P}_\beta$-name for $\mathbb{P}_\gamma$, via mapping $\mathbb{P}_\alpha \ni p \mapsto \langle p \restriction \beta, p \restriction \alpha \setminus \beta\rangle \in \mathbb{P}_\beta \star \mathbb{P}_\gamma^\star$. However, although this mapping is an order embedding onto a dense subset of $\mathbb{P}_\beta \star \mathbb{P}_\gamma^\star$, it is not onto. Thus, each time we will be identifying an element $\langle p', q'\rangle \in \mathbb{P}_\beta \star \mathbb{P}_\gamma^\star$ with a $q \in \mathbb{P}_\alpha$, in reality we will be defining q as such an element of $\mathbb{P}_\alpha$ such that $q \restriction \beta \geq_{F,n} p'$ and $q \restriction \beta \Vdash$ "$q \restriction \alpha \setminus \beta = q'$" for the current values of F and n. To define such a q first find $q \restriction \beta \in \mathbb{P}_\beta$ and a countable set $A \subseteq \alpha$ such that $q \restriction \beta \geq_{F,n} p'$ and $q \restriction \beta$ forces that the domain of q' is a subset of A. (See [Sh, Lemma 1.6, p. 81]. Compare also [BaLa, Lemma 2.3(iii)].) Then it is enough to extend $q \restriction \beta$ to $q \in \mathbb{P}_\alpha$ with the domain equal to $A \cup \mathrm{dom}(q \restriction \beta)$ in such a way that $q \restriction \xi \Vdash$ "$q(\xi) = q'(\xi)$" for every $\xi \in A$.

Using Lemma 5.7 we can obtain the following modification of Lemma 5.5. In its statement we will use the symbol $p|s$ associated with $p \in \mathbb{P}_\delta$ and $s \in p(0)$ to denote an element of $\mathbb{P}_\delta$ such that $\mathrm{dom}(p|s) = \mathrm{dom}(p)$, $(p|s)(0) = [p(0)]^s$, and $(p|s) \restriction (\delta \setminus \{0\}) = p \restriction (\delta \setminus \{0\})$.

Lemma 5.9 *Let $1 < \delta < \omega_2$, $p \in \mathbb{P}_\delta$, and τ be a $\mathbb{P}_\delta$-name such that*

$$p \Vdash \tau \in 2^\omega \setminus V[G_1].$$

Then for every $n, k < \omega$ with $\underline{\mathrm{norm}}_{p(0)}(k) \geq n$ there exist an arbitrarily large number $m < \omega$, $q \geq_{\{0\},n} p$, and a $\mathbb{P}_1$-name φ such that for every $t \in q(0) \cap \omega^k$

$$q|t \Vdash \varphi \text{ is a one-to-one function from } \mathrm{succ}_{q(0)}(t) \text{ into } 2^m$$

and

$$q|s \Vdash \tau{\restriction}m = \varphi(s)$$

for every $s \in \mathrm{succ}_{q(0)}(t)$.

Proof. Identify $\mathbb{P}_\delta$ with $\mathbb{P}_1 \star \mathbb{Q}$ and p with $\langle p(0), \bar{p}\rangle$, where $\mathbb{Q}$ is a $\mathbb{P}_1$-name for $\mathbb{P}_\gamma$ and $1 + \gamma = \delta$. Let $S = T^\star \cap \omega^{k+1}$ and $N = |S|$.

Take a V-generic filter H in $\mathbb{P}_1$ such that $p(0) \in H$. For a moment we will work in the model $V[H]$. In this model let $\tilde{\mathbb{Q}}$ and $\tilde{p}$ be the H-interpretations of $\mathbb{Q}$ and $\bar{p}$, respectively. Moreover let $\tilde{\tau} \in V[H]$ be a $\tilde{\mathbb{Q}}$-name such that $\tilde{p}$ forces that $\tilde{\tau} = \tau$. Then $\tilde{p}$ forces that $\tilde{\tau} \in 2^\omega \setminus V[H]$. Thus, by Lemma 5.7 used in $V[H]$ to $\tilde{\tau}$, there exists an $m_0 < \omega$ such that for every $m \geq m_0$ there are $\{q_s \geq \tilde{p} \colon s \in S\}$, and a one-to-one function $f \colon S \to 2^m$ such that

$$q_s \Vdash \tilde{\tau}{\restriction}m = \tau{\restriction}m = f(s)$$

for every $s \in S$.

Let μ be a $\mathbb{P}_1$-name for m_0. Then, by Lemma 5.1, there exists $p' \in \mathbb{P}_1$ and an arbitrarily large $m < \omega$ such that $p' \geq_n p(0)$ and $p' \Vdash$ "$\mu \leq m$."

Now let $\{q_s^\star \geq q \colon s \in S\}$ and φ be the $\mathbb{P}_1$-names for $\{q_s \geq q \colon s \in S\}$ and $f \colon S \to 2^m$, respectively, such that p' forces the above properties about them. Moreover let q' be a $\mathbb{P}_1$-name for an element of $\mathbb{Q}$ such that

$$[p(0)]^s \Vdash q' = q_s^\star$$

for every $s \in p' \cap \omega^{k+1}$. Put $q = \langle p', q'\rangle$. It is easy to see that m, q and φ have the desired properties. ∎

Lemmas 5.5 and 5.9 can be combined together in the following corollary. Its form is a bit awkward, but it will allow us to combine two separate cases into one case in the proof of Lemma 4.1.

Corollary 5.10 *Let $1 \le \delta < \omega_2$, $p \in \mathbb{P}_\delta$, and τ be a $\mathbb{P}_\delta$-name such that for every $\gamma < \delta$*

$$p \Vdash \tau \in 2^\omega \setminus V[G_\gamma].$$

Moreover if $\delta = 1$ assume additionally that p reads τ continuously. Then for every $n, k < \omega$ with $\underline{\mathrm{norm}}_{p(0)}(k) \ge n+1 \ge 2$ there exist an arbitrarily large number $m < \omega$, $i < \omega$ with $i > k$, and $q \ge_{\{0\},n} p$ such that $q(0)$ can be represented as

$$q(0) = \bigcup_{t \in A} p_t, \tag{8}$$

where $A \subseteq p(0) \cap \big(\omega^{\le i} \setminus \omega^{<k}\big)$ and the elements of A are pairwise incompatible (as functions). Moreover for every $t \in A$ we have $p_t \ge [p(0)]^t$, and there exists a $\mathbb{P}_1$-name φ_t such that

$$q|t \Vdash \varphi_t \text{ is a one-to-one mapping from } \mathrm{succ}_q(t) \text{ into } 2^m$$

and

$$q|s \Vdash \tau \restriction m = \varphi(s)$$

for every $s \in \mathrm{succ}_{q(0)}(t)$.

Proof. For $\delta > 1$ use Lemma 5.9 with $i = k+1$ and put $A = q(0) \cap \omega^k$.
For $\delta = 1$ use Lemma 5.5 taking as φ_t the standard names for the maps $\mathrm{succ}_q(t) \ni s \longmapsto x_s \in 2^m$. ■

Next we will consider several properties of the iteration of forcing $\mathbb{P}$.
For $p \in \mathbb{P}_\alpha$, where $\alpha \le \omega_2$, and $\sigma\colon F \to \prod_{i<k} n_i \subset \omega^k$, where $k < \omega$ and $F \in [\alpha]^{<\omega}$, define a function $p|\sigma$ as follows. The domain of $p|\sigma$ is equal to $\mathrm{dom}(p)$, and $(p|\sigma)\restriction(\mathrm{dom}(p) \cap \beta)$ is defined by induction on $\beta \le \alpha$:

- $(p|\sigma)\restriction(\mathrm{dom}(p) \cap \beta) = \bigcup_{\gamma<\beta} (p|\sigma)\restriction(\mathrm{dom}(p) \cap \gamma)$ if β is a limit ordinal;
- if $\beta = \gamma + 1$ we put $(p|\sigma)\restriction(\mathrm{dom}(p) \cap \beta) = (p|\sigma)\restriction(\mathrm{dom}(p) \cap \gamma)$ provided $\gamma \notin \mathrm{dom}(p)$;
- if $\beta = \gamma + 1$ and $\gamma \in \mathrm{dom}(p)$ we define $(p|\sigma)(\gamma)$ as follows:

  (A) if $(p|\sigma)\restriction(\mathrm{dom}(p) \cap \gamma) \notin \mathbb{P}_\gamma$ we define $(p|\sigma)(\gamma)$ arbitrarily;

(B) if $(p|\sigma)\restriction(\mathrm{dom}(p)\cap\gamma)\in\mathbb{P}_\gamma$ then we put $(p|\sigma)(\gamma)=\tau$ where τ is a $\mathbb{P}_\gamma$-name such that

$$(p\restriction\gamma)|(\sigma\restriction\gamma)\Vdash \text{“}\tau=[p(\gamma)]^{\sigma(\gamma)}\text{”}$$

if $\gamma\in F$, and $\tau=p(\gamma)$ if $\tau\notin F$.

We say that σ is *consistent* with p if $p|\sigma$ belongs to $\mathbb{P}_\alpha$, i.e., when case (A) was never used in the above definition. We will be interested in function $p|\sigma$ only when σ is consistent with p. In this case intuitively $p|\sigma$ represents a condition $q\in\mathbb{P}_\alpha$ with the same domain that p such that $q(\gamma)=p(\gamma)$ for every $\gamma\notin F$ and $q(\gamma)=[p(\gamma)]^{\sigma(\gamma)}$ for $\gamma\in F$. We will use a symbol $\mathrm{con}(p,F,k)$ to denote the set of all $\sigma\colon F\to\omega^k$ consistent with p.

Note that if $s\in p(0)$ then function $p|s$ used in Proposition 3.3 is equal to $p|\sigma$, where $\mathrm{dom}(\sigma)=\{0\}$ and $\sigma(0)=s$. Also such $p|s$ belongs to $\mathbb{P}_\alpha$ if and only if $s\in p(0)$. Thus we will identify $\mathrm{con}(p,\{0\},k)$ with $p(0)\cap\omega^k$.

For $F\in[\alpha]^{<\omega}$ and $k<\omega$ we say that $p\in\mathbb{P}_\alpha$ is $\langle F,k\rangle$-*determined* if for every $\beta\in F\cap\mathrm{dom}(p)$ and $\sigma\colon F\cap\beta\to\omega^k$ consistent with p the condition $(p\restriction\beta)|\sigma$ decides already the value of $p(\beta)\cap\omega^k$, that is, if for every $s\in\omega^k$

$$\text{either}\quad (p\restriction\beta)|\sigma\Vdash\text{“}s\in p(\beta)\text{”}\quad\text{or}\quad (p\restriction\beta)|\sigma\Vdash\text{“}s\notin p(\beta).\text{”}$$

Note that each $p\in\mathbb{P}_\alpha$ is $\langle\{0\},k\rangle$-determined. Notice also that for every $p\in\mathbb{P}_\alpha$, $k<\omega$, and $F\in[\alpha]^{<\omega}$ if p is $\langle F,k\rangle$-determined then

$$\{p|\sigma\colon\sigma\in\mathrm{con}(p,F,k)\}\quad\text{is a maximal antichain above } p. \tag{9}$$

This can be easily proved by induction on $|F|$. In the same setting we also have

$$\mathrm{con}(p,F\cap\beta,k)=\mathrm{con}(p\restriction\beta,F\cap\beta,k)=\{\sigma\restriction\beta\colon\sigma\in\mathrm{con}(p,F,k)\}$$

and

$$(q\restriction\beta)|\sigma=(q\restriction\beta)|(\sigma\restriction\beta)$$

for every $\beta\leq\alpha$ and $\sigma\in\mathrm{con}(p,F,k)$.

Lemma 5.11 *Let $\alpha\leq\omega_2$, τ be a $\mathbb{P}_\alpha$-name, $X\in V$ be finite, and $p\in\mathbb{P}_\alpha$ be such that*

$$p\Vdash\tau\in X.$$

If $i < \omega$ is such that $|X| \leq (b_i)^2$, $t \in p(0) \cap \omega^i$, and $n < \omega$ is such that $\underline{\mathrm{norm}}_{p(0)}(t) \geq n \geq 1$ then there exist $p_t \in \mathbb{P}_\alpha$ extending $p|t$ and $x \in X$ such that $\underline{\mathrm{norm}}_{p_t(0)}(t) \geq n-2$ and

$$p_t \Vdash \tau = x.$$

Proof. Let

$$D = \{T \in \mathbb{P} \colon (\exists q \geq p|t)(\exists x \in X)(T = q(0) \ \&\ q \Vdash \text{“}\tau = x\text{”})\}.$$

Clearly D is dense above $[p(0)]^t$. We will prove the lemma by induction on $r^n_D(t)$, as defined on page 6.

If $r^n_D(t) = 0$ then it is obvious.

If $r^n_D(t) = \alpha > 0$ choose $U \in [\mathrm{succ}_{p(0)}(t)]^{\geq (b_i)^{(b_i)^{n-1}}}$ from the definition of $r^n_D(t)$. By the inductive assumption for every $s \in U$ there exists $T_s \in D$ extending $[p(0)]^t$ such that $\underline{\mathrm{norm}}_{T_s}(s) \geq n-2$. Choose q_s and x_s witnessing $T_s \in D$, i.e., such that $q_s \geq p|t$, $q_s(0) = T_s$ and

$$q_s \Vdash \tau = x_s.$$

Since $|X| \leq (b_i)^2$, we can find an $x \in X$ and $V \subseteq U$ of cardinality greater than or equal to $|U|/|X| \geq (b_i)^{(b_i)^{n-1}}/(b_i)^2 = (b_i)^{(b_i)^{n-2}}$ such that $x_s = x$ for every $s \in V$.

Let $S = \bigcup\{q_s(0) \colon s \in V\}$. Then $\underline{\mathrm{norm}}_S(t) = \log_{b_i}|V| \geq n-2$. Take $p_t \geq p$ such that $\mathrm{dom}(p_t) = \bigcup_{s \in V} \mathrm{dom}(q_s)$, $p_t(0) = S$, and for $\beta \neq 0$

$$(p_t \restriction \beta)|s \Vdash \text{“}p_t(\beta) = q_s(\beta)\text{”}$$

for every $s \in V$. Then p_t satisfies the lemma. ∎

Lemma 5.12 *Let $\alpha \leq \omega_2$, $p \in \mathbb{P}_\alpha$, $k \leq i < \omega$, $\langle X_l \colon k \leq l \leq i\rangle$ be a sequence of finite subsets from V, and $\langle \tau_l \colon k \leq l \leq i\rangle$ a sequence of $\mathbb{P}_\alpha$-names. Assume that for every $k \leq l \leq i$*

$$p \Vdash \tau_l \in X_l$$

and $|X_l| \leq b_l$. If $n < \omega$ is such that $\underline{\mathrm{norm}}_{p(0)}(k) \geq n+2 \geq 3$ then there exist a family $\left\{x_t \in \bigcup_{k \leq l \leq i} X_l \colon t \in p(0) \cap \left(\bigcup_{k \leq l \leq i} \omega^l\right)\right\}$ and $q \geq p$ with the property that $p(0) \cap \omega^k = q(0) \cap \omega^k$, $\underline{\mathrm{norm}}_{q(0)}(k) \geq n$, and

$$q|t \Vdash \tau_{|t|} = x_t$$

for every $t \in p(0) \cap \left(\bigcup_{k \leq l \leq i} \omega^l\right)$.

Proof. For every $k \leq l \leq i$ let $Y_l = \prod_{k \leq j \leq l} X_j$ and notice that

$$|Y_l| \leq \prod_{k \leq j \leq l} b_j \leq \left(\prod_{k \leq j < l} n_j \right) \cdot b_l \leq n_{l-1}! \cdot b_l \leq (b_l)^2.$$

So, by Lemma 5.11, for every $t \in p(0) \cap \omega^i$ there exist $p_t \in \mathbb{P}_\alpha$ extending $p|t$ and $y_t \in Y_i$ such that $\underline{\mathrm{norm}}_{p_t(0)}(t) \geq n$ and

$$p_t \Vdash \tau_l = y_t(l)$$

for every $k \leq l \leq i$. We can also assume that all conditions p_t have the same domain D.

Now let $S_i = p(0) \cap \omega^{\leq i}$. We will construct inductively a sequence of trees $S_i \supset S_{i-1} \supset \cdots \supset S_k$, such that for every $k \leq l < i$

(a) $S_l \cap \omega^l = S_{l+1} \cap \omega^l$;

(b) $\mathrm{succ}_{S_l}(t) = \mathrm{succ}_{S_{l+1}}(t)$ for every $t \in S_l$ with $|t| > l$;

(c) $|\mathrm{succ}_{S_l}(t)| \geq (b_l)^{(b_l)^n}$ for every $t \in S_l \cap \omega^l$; and,

(d) for every $s \in S_l \cap \omega^l$ there exists $y_s \in Y_l$ with the property that

$$y_t \restriction (l+1) = y_s \quad \text{for every } t \in S_l \cap \omega^i \text{ with } \ s \subseteq t.$$

To make an inductive step take an $l < i$, $i \geq k$, for which S_{l+1} is already defined. For each $s \in S_{l+1} \cap \omega^l$ choose $y_s \in Y_l$ and $L_s \in [\mathrm{succ}_{S_{l+1}}(s)]^{\geq (b_l)^{(b_l)^n}}$ such that

$$y_s = y_t \restriction (l+1) \quad \text{for every } t \in L_s.$$

Such a choice can be made, since $|\mathrm{succ}_{S_{l+1}}(s)| = |\mathrm{succ}_{S_i}(s)| \geq (b_l)^{(b_l)^{n+2}}$ (by the assumption that $\mathrm{norm}_{p(0)}(l) \geq \underline{\mathrm{norm}}_{p(0)}(k) \geq n+2$) while $|Y_l| \leq (b_l)^2$. Define $L = \bigcup\{L_s \colon s \in S_{l+1} \cap \omega^l\}$ and

$$S_l = \{s \in S_{l+1} \colon \text{ either } |s| \leq l \text{ or } t \subseteq s \text{ for some } t \in L\}.$$

This finishes the inductive construction.

Now put $T = \bigcup\{[p(0)]^t \colon t \in S_k \cap \omega^i\}$, and for every $t \in S_k \cap \left(\bigcup_{k \leq l \leq i} \omega^l\right)$ define $x_t = y_t(|t|)$. Let $q \in \mathbb{P}_\alpha$ be such that $\mathrm{dom}(q) = D$, $q(0) = T$, and $(q \restriction \beta)|t \Vdash$ "$q(\beta) = p_t(\beta)$" for every $\beta \in D$, $\beta > 0$, and $t \in S_k \cap \omega^i$. It is easy to see that q and all x_t's satisfy the requirements. ■

Lemma 5.13 *Let $\alpha \leq \omega_2$, $k, n < \omega$, $0 \in F \in [\omega_2]^{<\omega}$, and $p \in \mathbb{P}_\alpha$ be such that*

$$p \restriction \beta \Vdash \underline{\mathrm{norm}}_{p(\beta)}(k) \geq n+2 \geq 3$$

for every $\beta \in F$. Moreover assume that $k \leq i < \omega$, $\langle X_l \colon k \leq l \leq i \rangle$ is a sequence of finite subsets from V, and $\langle \tau_l \colon k \leq l \leq i \rangle$ a sequence of $\mathbb{P}_\alpha$-names with the properties that for every $k \leq l \leq i$

$$p \Vdash \tau_l \in X_l,$$

$|X_l| \geq 2$, and $|X_l|^{(n_{l-1}!)^{2|F|}} \leq b_l$. Then there exists $q \geq_{F,n} p$ with the following properties. For every $k \leq l \leq i$

- *q is $\langle F, l\rangle$-determined; and,*
- *there exists a family $\{x_s \in X_l \colon s \in (\omega^l)^F$ & s is consistent with $q\}$ such that*

  $$q|s \Vdash \tau_l = x_s,$$

  for every $s \in (\omega^l)^F$ consistent with q.

Proof. The proof will be by induction on $m = |F|$.

If $m = |F| = 1$ then $F = \{0\}$ and the conclusion follows from Lemma 5.12. (Every $p \in \mathbb{P}_\alpha$ is $\langle\{0\}, l\rangle$-determined.)

So assume that $m = |F| > 1$ and let $\beta = \max F$. Then $0 < \beta < \alpha$ and $\mathbb{P}_\alpha$ is equivalent to $\mathbb{P}_\beta \star \mathbb{P}^\star_\gamma$ where $\beta + \gamma = \alpha$ and $\mathbb{P}^\star_\gamma$ is a $\mathbb{P}_\beta$-name for $\mathbb{P}_\gamma$. Let $p_0 = \langle p\restriction\beta, \pi_1\rangle \in \mathbb{P}_\beta \star \mathbb{P}^\star_\gamma$ be such that $\langle p\restriction\beta, \pi_1\rangle$ is stronger then p and $p\restriction\beta \Vdash$ "$p(\beta) = \pi_1(0)$." Then $p_0 \geq_{F,n} p$. Thus we can replace p with p_0.

To make an inductive step, for every $l \leq i$, $l \geq k$, define

$$X'_l = \bigcup\left\{(X_l)^T \colon T \subseteq \prod_{j<l} n_j \subset \omega^l\right\}.$$

Then

$$|X'_l| \leq 2^{\left|\Pi_{j<l} n_j\right|} \cdot |X_l|^{\left|\Pi_{j<l} n_j\right|} \leq 2^{n_{l-1}!}|X_l|^{n_{l-1}!} = (2|X_l|)^{n_{l-1}!} \leq |X_l|^{(n_{l-1}!)^2}.$$

In particular

$$|X'_l|^{(n_{l-1}!)^{2|F\cap\beta|}} \leq \left(|X_l|^{(n_{l-1}!)^2}\right)^{(n_{l-1}!)^{2(|F|-1)}} = |X_l|^{(n_{l-1}!)^{2|F|}} \leq b_l.$$

So the sequence $\langle X'_l \colon k \leq l \leq i\rangle$ and $F \cap \beta$ satisfy the size requirements of the inductive assumptions.

Now, for a moment, we will work in a model $V[H_\beta]$, where H_β is a V-generic filter in $\mathbb{P}_\beta$ containing $p\restriction\beta$. Let p_1 be the valuation of π_1 in $V[H_\beta]$. By Lemma 5.12 there exist $p' \in \mathbb{P}_\gamma$ extending p_1 with $p'(0)\cap\omega^k = p_1(0)\cap\omega^k$ and $\underline{\mathrm{norm}}_{p_1(0)}(k) \geq n$, and for every $l \leq i$, $l \geq k$, a function $f_l \colon p'(0) \cap \omega^l \to X_l$ such that

$$p'|t \Vdash \tau_l = f_l(t)$$

for every $t \in p'(0) \cap \omega^l$. Note that, $f_l \in X'_l$.

Let φ_l and π be the $\mathbb{P}_\beta$-names for f_l and p', respectively, such that $p\restriction\beta$ forces all the above facts about them. In particular $p\restriction\beta \Vdash$ "$\varphi_l \in X_l'$" for all appropriate l's, so, by the inductive assumption, there exist $q_0 \in \mathbb{P}_\beta$ and for every $k \leq l \leq i$ a family

$$\left\{ f_s \in X_l \colon s \in \left(\omega^l\right)^{F\cap\beta} \ \& \ s \text{ is consistent with } p\restriction\beta \right\}$$

such that q_0 is $\langle F\cap\beta, l\rangle$-determined, $q_0 \geq_{F\cap\beta,n} p\restriction\beta$, and

$$q_0|s \Vdash \varphi_l = f_s$$

for every $s \in (\omega^l)^{F\cap\beta}$ consistent with $p\restriction\beta$. In particular every $q_0|s$ decides the value of $\pi(0)\cap\omega^l$, since it is equal to the domain of φ_l, and forces that $\underline{\mathrm{norm}}_{\pi(0)}(k) \geq n$.

Let $q = \langle q_0, \pi\rangle$ and for every $s \in (\omega^l)^F$ consistent with p define

$$x_s = f_{s\restriction\beta}(s(\beta)).$$

It is not difficult to see that it has the required properties. ■

Proof of Lemma 4.1. Let $\alpha \geq 1$, p and τ be as in the lemma.

Now for arbitrary $\beta < \alpha$, $\beta \geq 1$, let $\delta \leq \alpha$ be such that $\beta + \delta = \alpha$. We will identify $\mathbb{P}_\alpha$ with $\mathbb{P}_\beta \star \mathbb{P}^\star_\delta$, where $\mathbb{P}^\star_\delta$ is a $\mathbb{P}_\beta$-name for $\mathbb{P}_\delta$. We will also identify p with $\langle p\restriction\beta, \pi\rangle$. Upon such identification, we can find a $\mathbb{P}_\beta$-name $\tau^\star$ such that

$$p\restriction\beta \Vdash \tau^\star \text{ is a name for the same object that } \tau \text{ is.}$$

In particular $p\restriction\beta \Vdash$ "$\pi \Vdash \tau^\star = \tau$."

Now if α is a successor ordinal number put $\alpha = \beta + 1$. In this case $p\restriction\beta$ forces that π and $\tau^\star$ satisfy the assumptions of the Lemma 5.4, so there exists a $\mathbb{P}_\beta$-name π_0 such that

$$p\restriction\beta \Vdash \ \pi_0 \geq_n \pi \text{ and } \pi \text{ reads } \tau^\star \text{ continuously.}$$

We put $q_0 = \langle p\restriction\beta, \pi_0\rangle$ and additionally assume that $\beta \in F$.

If α is a limit ordinal, we put $\pi_0 = \pi$ and $q_0 = p$.

Now without loss of generality we can assume that $0 \in F$ and $n \geq 1$. We also put $\beta = \max F$ and fix $q \geq q_0$.

By an easy inductive application of Corollary 5.2 $|F|$-many times we can find $k < \omega$ and $p' \geq_{F,n} q$ such that

$$p' \restriction \gamma \Vdash \underline{\mathrm{norm}}_{p(\gamma)}(k) \geq n + 9$$

for every $\gamma \in F$. We can also increase k, if necessary, to guarantee that

$$2|F| + 2 \leq k. \tag{10}$$

Also since

$$p' \restriction \beta \Vdash \text{“}\pi_0 \Vdash \tau^\star = \tau,\text{”}$$

$p' \restriction \beta$ forces that the assumptions of Corollary 5.10 are satisfied. Thus, applying it to π_0, $\tau^\star$, and k defined above, we can find $\mathbb{P}_\beta$-names μ, ρ, π', $\mathcal{A}$, and ψ for m, i, q, A and mapping $A \ni t \mapsto p_t$ respectively, such that $p' \restriction \beta$ forces

$$\mu, \rho < \omega \quad \& \quad \pi' \geq_{\{0\},n+8} \pi_0 \quad \& \quad \pi'(0) = \bigcup_{t \in \mathcal{A}} \varphi(t) \text{ is a representation as in (8).}$$

Also, by Corollary 5.2, replacing π' with an $\geq_{\{0\}n+8}$-stronger condition, if necessary, we can assume that there are $m, i < \omega$ such that

$$p' \restriction \beta \Vdash \mu < m \quad \& \quad \rho < i.$$

Increasing i and m, if necessary, we can also assume that $m \geq 2$ and

$$|2^m|^{(n_{i-1}!)^{2|F|}} < b_i. \tag{11}$$

Now notice that we can use Lemma 5.13 to $p' \restriction \beta \in \mathbb{P}_\beta$, and the sequences

$$\langle \tau_l \colon k \leq l \leq i \rangle = \left\langle \pi'(0) \cap \omega^l \colon k \leq l \leq i \right\rangle$$

and

$$\langle X_l \colon k \leq l \leq i \rangle = \left\langle \mathcal{P}\left(T^\star \cap \omega^{\leq l}\right) \colon k \leq l \leq i \right\rangle$$

since $2|F| + 2 \leq k$ implies that for every $k \leq l \leq i$

$$|X_l|^{(n_{l-1}!)^{2|F|}} \leq \left(2^{(n_{l-1}!)^2}\right)^{(n_{l-1}!)^{2|F|}} \leq (l+2)^{(n_{l-1}!)^{2|F|+2}} \leq b_l,$$

where the first inequality is justified by the fact that $|X_l| \le 2^{(n_{l-1}!)^2}$, which follows from the following estimation

$$\left|T^\star \cap \omega^{\le l}\right| \le \sum_{j<l} n_j! \le \prod_{j<l} n_j! \le n_{l-1}! \prod_{j<l-1} n_j! \le n_{l-1}! \prod_{j<l-1} n_{j+1} \le (n_{l-1}!)^2.$$

So we can find $p'' \in \mathbb{P}_\beta$ which is $\langle F \cap \beta, l\rangle$-determined for each $k \le l \le i$, such that $p'' \ge_{F\cap\beta,n+8} p' \restriction \beta$, and that $p''|s$ determines the value of $\pi'(0) \cap \omega^l$ for every $s \in \left(\omega^l\right)^{F\cap\beta}$ consistent with p''.

Next notice also that $A \cap \omega^{\le l} \subseteq T^\star \cap \omega^{\le l}$. Thus, the above calculation shows that we can also use Lemma 5.13 to $p'' \in \mathbb{P}_\beta$, and the sequences

$$\langle \tau_l \colon k \le l \le m\rangle = \langle \mathcal{A} \cap \omega^{\le l} \colon k \le l \le i\rangle$$

and

$$\langle X_l \colon k \le l \le i\rangle = \langle \mathcal{P}(T^\star \cap \omega^{\le l}) \colon k \le l \le i\rangle.$$

So we can find $p''' \in \mathbb{P}_\beta$ such that $p''' \ge_{F\cap\beta,n+6} p''$, and that $p'''|s$ determines the value of $\mathcal{A} \cap \omega^{\le l}$ for every $s \in \left(\omega^l\right)^{F\cap\beta}$ consistent with p'''.

Now let $q^1 = \langle p''', \pi'\rangle \in \mathbb{P}_\alpha$. Then $q^1 \ge_{F,n+6} q$,

$$q^1 \restriction \gamma \Vdash \underline{\mathrm{norm}}_{q^1(\gamma)}(k) \ge n + 6$$

for every $\gamma \in F$, and q^1 is $\langle F, l\rangle$-determined for each $k \le l \le i$. Hence, by the condition (11), the assumptions of Lemma 5.13 are satisfied by q^1, and the sequences $\langle \tau_l \colon k \le l \le m\rangle$ and $\langle X_l \colon k \le l \le i\rangle$, where $X_i = 2^m$, τ_i is the restriction to m of the term τ from the assumptions of Lemma 4.1, while for $k \le l < i$ we put $X_l = 2$ and τ_l a standard name for 0. So, we can find $q^2 \ge_{F,n+4} q^1$, which is still $\langle F, l\rangle$-determined for each $k \le l \le i$, and a family $\{x_s \in 2^m \colon s \in (\omega^i)^F \text{ \& } s \text{ is consistent with } q^2\}$ such that

$$q^2|s \Vdash \tau \restriction m = x_s$$

for every $s \in (\omega^i)^F$ consistent with q^2. Identify q^2 with $\langle q^2 \restriction \beta, \pi^2\rangle$ and note that $q^2 \restriction \beta$ still forces that $\pi^2(0)$ has a representation as in (8) and it "determines" a big part of this representation in the sense defined above. Our final step will be to "trim" q^2 (of which we will think as of $\mathrm{con}(q^2, F, i)$) to q^3 (identified with $\mathrm{con}(q^3, F, i)$) for which we will be able to repeat the construction from Corollary 5.6.

For this first note that for every $C \subseteq \mathrm{con}(q^2, F, i)$ there exists a condition $q^2|C$ associated with q^2 in a similar way that the condition $q^2|\sigma$ is associated to $\sigma \in \mathrm{con}(q^2, F, i)$. Also we will consider the elements of $\mathrm{con}(q^2, F, i)$ as functions from $i \times F$, where we treat $i \times F$ as ordered lexicographically by $\leq_{lex}$, and for $\langle l, \gamma \rangle \in i \times F$ we define

$$O(l, \gamma) = \{\langle j, \delta \rangle \in i \times F \colon \langle j, \delta \rangle \leq_{lex} \langle l, \gamma \rangle\}.$$

Put $C_0 = \mathrm{con}(q^2, F, i)$ and let $\{\langle l_j, \gamma_j \rangle \colon j \leq r\}$ be a decreasing enumeration of $(i \setminus k) \times F$ with respect to $\leq_{lex}$. Note that for every $s \in C_0$ we can associate a tag "constant x_s" for which $q^2|s \Vdash \tau \restriction m = x_s$. We will construct by induction on $j \leq r$ a sequence $C_0 \supset C_1 \supset \cdots \supset C_r$ such that for every $j \leq r$ and $s \in C_j$ the node $\langle l_j, \gamma_j \rangle$ of s is either tagged "one-to-one" (in a sense defined below) or "constant $x_{s,j}$" in which case

$$(q^2|C_j)|s[j] \Vdash \tau \restriction m = x_{s,j},$$

where $s[j] = s \restriction O(l_j, \gamma_j)$. The above requirement is clearly satisfied for $j = 0$, since $q^2|C_0 = q^2$, $s[0] = s$, and so every $s \in C_0$ is tagged by some constant. Thus the tag "one-to-one" does not appear for $j = 0$. For $j > 0$ we will use the tag "one-to-one" to $s \in C_j$ if for $W = \{t \in C_j \colon s[j] \subset t\}$ either

$$\text{the node } \langle l_{j-1}, \gamma_{j-1} \rangle \text{ is tagged "one-to-one" for every } t \in W$$

or for every $t \in W$ the node $\langle l_{j-1}, \gamma_{j-1} \rangle$ of t is tagged as a "constant $x_{t,j-1}$" and for every $s, t \in W$ if $s[j-1] \neq t[j-1]$ then $x_{s,j-1} \neq x_{t,j-1}$. Thus if we think of C_j as of tree $T(C_j)$ being formed from all $\leq_{lex}$ initial segments of elements of C_j, then the mapping $\mathrm{succ}_{T(C_j)}(s[j]) \ni t[j-1] \mapsto x_{t,j-1} \in 2^m$ is one-to-one.

So assume that for some $0 < j \leq r$ the set C_{j-1} is already constructed. To construct C_j consider first the set $D = \{s \restriction [(l_{j-1}+1) \times (\gamma_{j-1}+1)] \colon s \in C_{j-1}\}$ and note that $D = \mathrm{con}(q^2|C_{j-1}, F \cap (\gamma_{j-1}+1), l_{j-1}+1)$. Define

$$D_0 = \{s[j] \restriction [(l_{j-1}+1) \times (\gamma_{j-1}+1)] \colon s \in C_{j-1}\}.$$

Since also $D_0 = \{s \restriction \mathrm{dom}(s) \setminus \{\langle l_{j-1}, \gamma_{j-1} \rangle\} \colon s \in D\}$ the elements of D_0 are predecessors of those from D in a natural sense. Now, for every $s_0 \in D_0$ let D_{s_0} be the set of all successors of s_0 which belong to D, that is, $D_{s_0} = \{s \in D \colon s_0 \subset s\}$. In what follows we will describe the method of a choice of subsets E_{s_0} of D_{s_0}. Then we will define C_j by

$$C_j = \{s \in C_{j-1} \colon s \restriction [(l_{j-1}+1) \times (\gamma_{j-1}+1)] \in E_{s_0} \text{ for some } s_0 \in D_0\}.$$

Note that by this definition the norms of $q^2|C_j$ and $q^2|C_{j-1}$ are the same at every node of a level $\langle l,\gamma\rangle$ except for $\langle l,\gamma\rangle=\langle l_{j-1}-1,\gamma_{j-1}\rangle\}$, in which case the norm is controlled by the choice of E_{s_0}.

Now to choose sets $E_{s_0}\subset D_{s_0}$ fix an $s_0\in D_0$. We would like to look at the tags of elements from D_{s_0} and use the procedure from Corollary 5.6 to trim D_{s_0}. However the elements of D_{s_0} do not need to have tags. Thus we will modify this idea in the following way. Let $Z_{s_0}=\{s[j]\colon s_0\subset s\in C_{j-1}\}$ and notice that the elements of Z_{s_0} are differed from s_0 only by a "tail" defined on some pairs $\langle l,\gamma\rangle$ with $l<l_{j-1}$. Since the possible values of these "tails" are already determined by $q^2|s_0$ we have

$$|Z_{s_0}|\leq\left|\mathbb{P}\cap\omega^{l_{j-1}}\right|^{|F|}\leq\left(n_{l_{j-1}}!\right)^{|F|}.$$

For $t\in Z_{s_0}$ and $E\subset D_{s_0}$ let $E[t]=\{s[j-1]\colon s\restriction[(l_{j-1}+1)\times(\gamma_{j-1}+1)]\in E\}$. Then every element of $E[t]$ has a tag, and we can choose a subset $E'[t]$ of $E[t]$ of size $\geq|E[t]|^{1/4}$ with either all elements of $E'[t]$ having the same tag, or all having the tag "constant" with different constant values. Then $E''[t]=\{s\restriction[(l_{j-1}+1)\times(\gamma_{j-1}+1)]\colon s\in E'[t]\}$ is an $\langle E,t\rangle$-approximation for E_{s_0}. The actual construction of the set E_{s_0} is obtained by using the above described operation to all elements $t_1,\ldots,t_p$ of Z_{s_0} one at a time. More precisely, we put $E_0=D_{s_0}$ and define E_ν for $1\leq\nu\leq p$ as $E''_{\nu-1}[t_\nu]$. Then we put $E_{s_0}=E_p$ and note that

$$|E_{s_0}|\geq|D_{s_0}|^{4^{-|Z_{s_0}|}}\geq|D_{s_0}|^{4^{-\left(n_{l_{j-1}}!\right)^{|F|}}}\geq|D_{s_0}|^{\left(b_{l_{j-1}}\right)^{-1}}.$$

This finishes the inductive construction.

Now define $q^3=q^2|C_r$ and notice that $q^3\geq_{F,n+3}q^2$. Indeed, this follows from the norm preservation remark above and the fact that

$$|E_{s_0}|\geq|D_{s_0}|^{\left(b_{l_{j-1}}\right)^{-1}}\geq\left((b_{l_{j-1}})^{(b_{l_{j-1}})^{n+4}}\right)^{\left(b_{l_{j-1}}\right)^{-1}}=(b_{l_{j-1}})^{(b_{l_{j-1}})^{n+3}}.$$

By the above construction for every $s\in C_r$ every node $s[j]$ of s from level $\langle l_j,\gamma_j\rangle$ has a tag in q^3. Moreover, although $s=s[0]$ has a tag "constant," all this tags cannot be "constant." Indeed, if l is such that $(q^2\restriction\beta)|(s\restriction i\times\beta)$ forces that the node $q^2(\beta)(l)$ is tagged "one-to-one" while its successors are tagged as constants, then it is easy to see that the same node (more precisely, the node from level $\langle\max(F\cap\beta),l+1\rangle$) will remain tagged "one-to-one" in our recent tagging procedure. In particular, for every $s \in C_r$ there exists a maximal number $j_s < r$ for which $s[j_s]$ is marked "one-to-one."

To make the final step let $T_1 = T(C_r)$ be the tree as defined above and let $\{t_j\colon j < M\}$ be a one-to-one enumeration of $\{s[j_s]\colon s \in C_r\}$ such that $|\mathrm{succ}_{T_1}(t_j)| \leq |\mathrm{succ}_{T_1}(t_{j+1})|$ for every $j < M-1$. We will proceed as in Corollary 5.6. By induction on $j < M$ we will choose a sequence $\langle C^u_j\colon u < 2 \;\&\; j < M\rangle$ such that for every $u < 2$ and $j < M$

- if $|\mathrm{succ}_{T_1}(t_j)| = (b_l)^{(b_l)^{n+3}}$ then $C^i_j \in [\mathrm{succ}_q(t_j)]^{(b_l)^{(b_l)^n}}$; and,
- the sets $\{h[C^u_j] \subset 2^m\colon u < 2 \;\&\; j < M\}$ are pairwise disjoint.

Given $\langle C^u_r\colon u < 2 \;\&\; r < j\rangle$ we can choose C^0_j and C^1_j since for $l = |t_j|$

$$\left|\bigcup_{u<2,\, r<j} C^u_r\right| \leq 2j\ (b_l)^{(b_l)^n} \leq 2\left|T^\star \cap \omega^{\leq l}\right|^{|F|}\ (b_l)^{(b_l)^n} \leq (b_l)^{(b_l)^{n+2}}$$

and $|\mathrm{succ}_{T_1}(t_j)| = (b_l)^{(b_l)^{n+3}}$ therefore it is possible to choose disjoint sets $C^0_j, C^1_j \in [\mathrm{succ}_{T_1}(t_j)]^{(b_l)^{(b_l)^n}}$ with

$$h[C^0_j \cup C^1_j] \cap \left(\bigcup_{u<2,\, r<j} h[C^u_r]\right) = \emptyset.$$

For $u < 2$ define $C_u = \bigcup\{(T_1)^s\colon s \in \bigcup_{j<M} C^u_j\}$ and $B_u = \bigcup_{j<M} h[C^u_j]$. It is easy to see that $p_u = q^2|C_u$ and B_u have the required properties. ■

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