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## TWO EXAMPLES CONCERNING EXTENDABLE AND ALMOST CONTINUOUS FUNCTIONS

### Abstract

The main purpose of this paper is to describe two examples. The first is that of an almost continuous, Baire class two, non-extendable function  $f: [0, 1] \rightarrow [0, 1]$  with a  $G_\delta$  graph. This answers a question of Gibson [15]. The second example is that of a connectivity function  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  with dense graph such that  $F^{-1}(0)$  is contained in a countable union of straight lines. This easily implies the existence of an extendable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with dense graph such that  $f^{-1}(0)$  is countable. We also give a sufficient condition for a Darboux function  $f: [0, 1] \rightarrow [0, 1]$  with a  $G_\delta$  graph whose closure is bilaterally dense in itself to be quasi-continuous and extendable.

### 1 Definitions and Notation

Our terminology is standard and follows [6]. We consider only real-valued functions of one or more real variables. No distinction is made between a function and its graph. A restriction of a function  $f: X \rightarrow Y$  to a set  $A \subset X$  is denoted by  $f \upharpoonright A$ . By  $\mathbb{R}$  and  $\mathbb{Q}$  we denote the set of all real and rational numbers, respectively, while  $I$  will stand for the interval  $[0, 1]$ . The closure of a set  $A \subset \mathbb{R}^n$  is denoted by  $\text{cl}(A)$ , its boundary by  $\text{bd}(A)$  and its diameter by

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Key Words: extendable functions, peripherally continuous functions.

Mathematical Reviews subject classification: Primary: 26A15; Secondary: 54A25.

Received by the editors March 19, 1999

\*The work of the first author was partially supported by NSF Cooperative Research Grant INT-9600548 with its Polish part financed by Polish Academy of Science PAN.

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$\text{diam}(A)$ . The first coordinate projection of a set  $A \subset \mathbb{R}^2$  will be denoted by  $\text{pr}(A)$ .

The ordinal numbers will be identified with the sets of all their predecessors and cardinals with the initial ordinals. In particular  $2 = \{0, 1\}$  and the first infinite ordinal  $\omega$  number is equal to the set of all natural numbers  $\{0, 1, 2, \dots\}$ .

We will also use the following terminology [16]. For  $X \subseteq \mathbb{R}^n$  a function  $f: X \rightarrow \mathbb{R}$  is:

- *Darboux* if  $f[K]$  is a connected subset of  $\mathbb{R}$  (i.e., an interval) for every connected subset  $K$  of  $X$ ;
- *almost continuous* (in the sense of Stallings) if each open subset of  $X \times \mathbb{R}$  containing  $f$  also contains a continuous function from  $X$  to  $\mathbb{R}$  [27];
- *connectivity* function if the graph of  $f \upharpoonright Z$  is connected in  $Z \times \mathbb{R}$  for any connected subset  $Z$  of  $X$ ;
- *extendable* function if there is a connectivity function  $F: X \times [0, 1] \rightarrow \mathbb{R}$  such that  $f(x) = F(x, 0)$  for every  $x \in X$ ;
- *peripherally continuous* if for every  $x \in X$  and for all pairs of open sets  $U$  and  $V$  containing  $x$  and  $f(x)$ , respectively, there exists an open subset  $W$  of  $U$  such that  $x \in W$  and  $f[\text{bd}(W)] \subset V$ .

The classes of these functions are denoted by D, AC, Conn, Ext and PC, respectively, where the space  $X$  will be always clear from the context. Recall also (see e.g. [16] or [8]) that for the functions from  $\mathbb{R}$  to  $\mathbb{R}$  and from  $I$  into  $I$  we have the following strict inclusions

$$\text{Ext} \subsetneq \text{AC} \subsetneq \text{Conn} \subsetneq \text{D} \subsetneq \text{PC} \quad (1)$$

while in the Baire class one all these classes coincide. (See Brown, Humke and Laczko [5].) On the other hand, for the classes of functions from  $\mathbb{R}^n$ , with  $n > 1$ , into  $\mathbb{R}$  we have

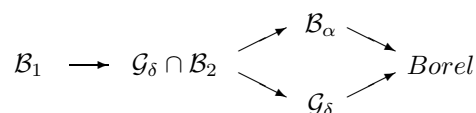
$$\text{Ext} = \text{Conn} = \text{PC} \longrightarrow \text{AC} \cap \text{D} \begin{array}{l} \nearrow \text{AC} \\ \searrow \text{D} \end{array}$$

where arrows denote strict inclusions. (Equation  $\text{Ext} = \text{Conn}$  on  $\mathbb{R}^n$  is proved in [10].)

## 2 Almost Continuous Non-Extendable Function with $G_\delta$ Graph

As noted above the classes listed in (1) cannot be distinguished within the family  $\mathcal{B}_1$  of Baire one functions. On the other hand, the classes of functions on  $\mathbb{R}$  or  $I$  listed in (1) can be distinguished within the family  $\mathcal{B}_2$  of Baire class two functions. (See e.g. [8, thm 1.2].) The next natural question is: for which classes  $\mathcal{G}$  containing  $\mathcal{B}_1$  but not  $\mathcal{B}_2$  will the inclusions remain strict? For several classes  $\mathcal{G}$  strictly between  $\mathcal{B}_1$  and  $\mathcal{B}_2$  this has been investigated by Brown [3] and Brown, Humke and Laczkovich [5].

In this section we will investigate the inclusions in (1) within the class  $\mathcal{G} = \mathcal{G}_\delta \cap \mathcal{B}_2$ , where  $\mathcal{G}_\delta$  is the class of functions with  $G_\delta$  graphs. Notice that  $\mathcal{B}_1 \subset \mathcal{G}_\delta$  since for every pointwise limit  $f$  of continuous functions  $f_n: I \rightarrow \mathbb{R}$  the graph of  $f$  is equal to  $\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty \{(x, y) : |f_i(x) - y| < 2^{-n}\}$ . Clearly also every function  $f$  from  $\mathcal{G}_\delta$  is Borel, since for every open  $U \subset \mathbb{R}$  the set  $f^{-1}(U) = \text{pr}(f \cap [I \times U]) = I \setminus \text{pr}(f \cap [I \times (\mathbb{R} \setminus U)])$  is simultaneously analytic and coanalytic, and hence Borel. (See e.g. [23, p. 489] or [19, p. 89].) On the other hand there are functions in  $\mathcal{G}_\delta$  of arbitrary high Borel class. This follows from the fact that every Borel set is a one-to-one continuous image of a closed subset of  $\mathbb{R} \setminus \mathbb{Q}$  (see e.g. [19, 15.3, p. 89]) which is clearly  $G_\delta$  in  $\mathbb{R}$ . Indeed, if  $A \subset I$  is of the class at least  $\alpha$  for some  $\alpha < \omega_1$ , take one-to-one continuous functions  $f_0$  from a closed subset of  $(0, 1) \setminus \mathbb{Q}$  onto  $A$  and  $f_1$  from a closed subset of  $(2, 3) \setminus \mathbb{Q}$  onto  $I \setminus A$ . Then  $f = f_0^{-1} \cup f_1^{-1}: I \rightarrow \mathbb{R}$  has a  $G_\delta$  graph and is of Borel class at least  $\alpha$ , since  $f^{-1}[(0, 1)] = A$ . Thus, if  $\mathcal{B}_\alpha$  stands for the Baire class  $\alpha$  functions, then we have the following relations for every  $2 \leq \alpha < \omega_1$ , where arrows denote strict inclusions.



The properness of inclusions in (1) within the class  $\mathcal{G}_\delta$  has been summarized by Brown in [4]. (See also [16, sec. 3].) In particular, he noticed that the inclusions  $\text{AC} \subset \text{Conn} \subset \text{D} \subset \text{PC}$  remain proper within this class  $\mathcal{G}_\delta$  leaving open the problem of properness of inclusion  $\text{Ext} \subset \text{AC}$  in the  $\mathcal{G}_\delta$  class. (This problem is stated explicitly by Gibson in [15, Question 4].) In fact the examples exhibiting properness of the inclusions  $\text{AC} \subset \text{Conn} \subset \text{D} \subset \text{PC}$  within the class  $\mathcal{G}_\delta \cap \mathcal{B}_2$  can be found in the literature. For  $\text{AC} \subset \text{Conn}$  Jones and Thomas [18] (also see Thomas [28]) constructed a non almost continuous function  $f: I \rightarrow I$  with a connected  $G_\delta$  graph. This  $f$  is also  $\mathcal{B}_2$ , but it is not pointed out in

the paper. Also Brown [3, remark 3] gives a bit stronger example and states explicitly that it is in  $\mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{Conn}$  but not in AC. The properness of the inclusion  $\text{Conn} \subset \text{D}$  within  $\mathcal{G}_\delta \cap \mathcal{B}_2$  is stated explicitly by Brown in [3, example 1 and remark 3], using the example constructed by him in [1, example 2]. (Also, in [24] Miller gives an example of a Darboux function  $f: I \rightarrow I$  having a  $G_\delta$  graph but no fixed point and so the graph of  $f$  is not connected. This function is also  $\mathcal{B}_2$ , but it is not pointed out in the paper.) Concerning the properness of the inclusion  $\text{D} \subset \text{PC}$  in  $\mathcal{G}_\delta \cap \mathcal{B}_2$  Brown [4] states that there is one (not mentioning  $\mathcal{B}_2$ ) and Miller [24] describes a peripherally continuous non Darboux function with  $G_\delta$  graph, which turns out to be also in  $\mathcal{B}_2$ .

The main goal of this section is to describe an example justifying properness of  $\text{Ext} \subset \text{AC}$  within  $\mathcal{G}_\delta \cap \mathcal{B}_2$ , thus answering a question of Gibson in [15, Question 4]. However, we will notice also that the small modifications of this example justify also properness of the remaining inclusions of (1) within  $\mathcal{G}_\delta \cap \mathcal{B}_2$ .

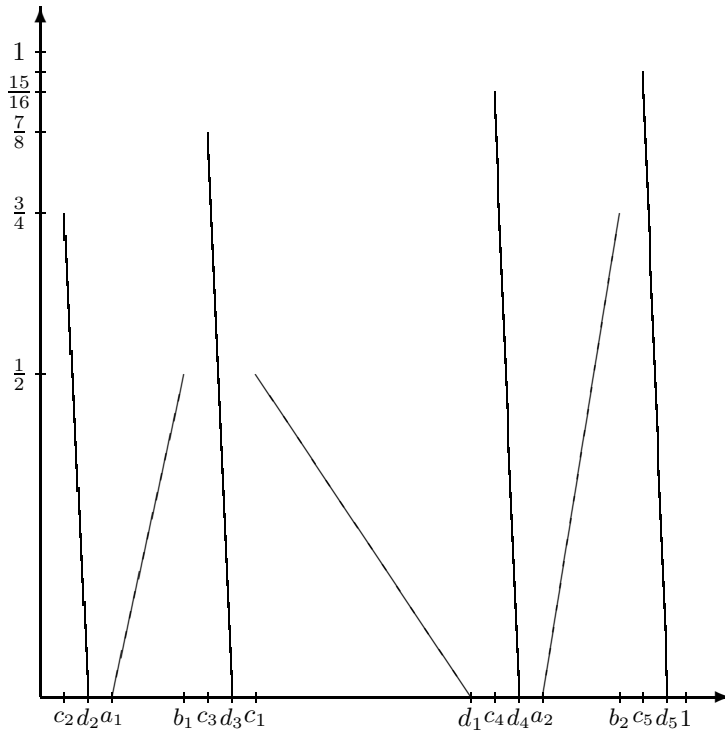


Figure 1: function  $f \in \mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{AC} \setminus \text{Ext}$  from Example 2.1

**Example 2.1.** *There exists an almost continuous, non-extendable, Baire class two function  $f: I \rightarrow I$  with  $G_\delta$  graph.*

PROOF. The example is a slight modification of a function constructed by Ciesielski and Jastrzębski in [8, thm 3.1]. It also slightly simplifies the argument given there.

Let  $C$  be the Cantor ternary set in  $I$  and let  $\mathcal{J}$  be the family of all component intervals of  $I \setminus C$ . For  $i \in \{0, 1\}$  let  $\mathcal{J}_i$  be the family of all intervals  $J \in \mathcal{J}$  of length  $3^{-2n+i}$  for some integer  $n$ . Thus,  $\{\mathcal{J}_0, \mathcal{J}_1\}$  is a partition of  $\mathcal{J}$ . Let  $\{(a_n, b_n): 0 < n < \omega\}$  and  $\{(c_n, d_n): 0 < n < \omega\}$  be the enumerations of  $\mathcal{J}_0$  and  $\mathcal{J}_1$ , respectively. Define function  $f: [0, 1] \rightarrow [0, 1]$  in the following way. (See Figure 1.)

- For every  $0 < n < \omega$  we put  $f(a_n) = f(d_n) = 0$ ,  $f(b_n) = f(c_n) = 1 - 2^{-n}$  and extend it linearly on each interval  $[a_n, b_n]$  and  $[c_n, d_n]$ . We refer to  $b_n$  and  $c_n$  as *upper endpoints for  $f$*  of  $[a_n, b_n]$  and  $[c_n, d_n]$ , respectively.
- For all other  $x$ 's we put  $f(x) = 0$ .

(Ciesielski and Jastrzębski define their function by assigning to  $f(b_n)$  and  $f(c_n)$  value 1 instead of  $1 - 2^{-n}$ . It does not have  $G_\delta$  graph since  $f \cap (I \times \{1\}) = \bigcup_{n=1}^\infty \{b_n, c_n\} \times \{1\}$  is not  $G_\delta$  being countable dense in  $C \times \{1\}$ .) First note that the graph of  $f$  is a  $G_\delta$  set as a union of three  $G_\delta$  sets:

- $f \upharpoonright (C \setminus \bigcup_{n=1}^\infty \{b_n, c_n\}) = (C \setminus \bigcup_{n=1}^\infty \{b_n, c_n\}) \times \{0\}$ ,
- $f \upharpoonright (I \setminus C) = \bigcap_{n=1}^\infty \{(x, y): x \in I \setminus C \ \& \ |f(x) - y| < 2^{-n}\}$  and
- $f \upharpoonright \bigcup_{n=1}^\infty \{b_n, c_n\} = \bigcup_{n=1}^\infty \{\langle b_n, 1 - 2^{-n} \rangle, \langle c_n, 1 - 2^{-n} \rangle\}$  which is discrete.

Also, since the first two restrictions are continuous, it is easy to see that the preimage  $f^{-1}(U)$  of any open set  $U \subset I$  is a union of a  $G_\delta$ -set and an  $F_\sigma$ -set; so  $f$  is of Baire class two.

Next we will show that  $f$  is not extendable. By way of contradiction, assume that  $f$  is extendable; that is, that there is a connectivity function  $F: I^2 \rightarrow I$  with  $F(x, 0) = f(x)$  for all  $x \in I$ . Thus  $F$  is peripherally continuous. We will deduce from this that there exists an  $x \in I$  such that  $F(x, 0) = 1$ , which implies  $f(x) = 1$ , contradicting our definition of  $f$ . For this we define inductively the sequences  $\langle p_1, p_2, p_3, \dots \rangle$  of upper endpoints for  $f$  and  $\langle B_0, B_1, B_2, \dots \rangle$  of closed connected subsets of  $I^2$  such that for each  $n < \omega$  we have

$$F[B_n] \subset (1 - 2^{-n+1}, 1], \quad B_n \cap B_{n+1} \neq \emptyset, \quad \text{diam}(B_n) \leq 2^{-n},$$

and

$$\langle p_{n+1}, 0 \rangle \in B_n, \quad F(p_{n+1}, 0) = f(p_{n+1}) \geq 1 - 2^{-(n+1)}.$$

We will start with  $B_0 = [c_2, d_2] \times \{0\}$  and  $p_1 = c_2$ . Clearly, the above conditions are satisfied. So assume that  $p_n$  and  $B_{n-1}$  satisfying the above are already constructed for some  $n > 0$ . Then  $p_n$  is the upper endpoint for  $f$  of some interval  $J_0 \in \mathcal{J}$ . We have to find  $B_n$  and  $p_{n+1}$ . So, choose an  $\varepsilon > 0$  less than the length of  $J_0$  such that  $\varepsilon < \min\{2^{-(n+1)}, p_n, 1 - p_n, \text{diam}(B_{n-1})\}$  and

(\*) if  $J \in \mathcal{J} \setminus \{J_0\}$  is closer to  $p_n$  than  $\varepsilon$ , then it has the length at most  $2^{-(n+1)}$  and  $f(p) \geq 1 - 2^{-(n+1)}$  for the upper endpoint  $p$  for  $f$  of  $J$ .

Since  $F(p_n, 0) \geq 1 - 2^{-n}$  and  $F$  is peripherally continuous we can find an open neighborhood  $W_n \subset [0, 1]^2$  of  $\langle p_n, 0 \rangle$  with diameter less than  $\varepsilon$  and such that  $F[\text{bd}(W_n)] \subset (1 - 2^{-n+1}, 1]$ . Without loss of generality we can also assume that  $\text{bd}(W_n)$  is connected. (This is a standard argument. See e.g. [27].) Note that the choice of  $\varepsilon$  guarantees that  $\text{bd}(W_n) \cap B_{n-1} \neq \emptyset \neq \text{bd}(W_n) \cap (I_0 \times \{0\})$ , where  $I_0$  is a component of  $I \setminus J_0$  containing  $p_n$ . This is so, since  $\text{bd}(W_n)$  disconnects  $[0, 1]^2$ , while  $B_{n-1}$  and  $I_0 \times \{0\}$  are connected, containing  $\langle p_n, 0 \rangle \in W_n$  and of diameter greater than  $\varepsilon \geq \text{diam}(\text{bd}(W_n))$ . Let  $z \in I_0$  be such that  $\langle z, 0 \rangle \in \text{bd}(W_n) \cap (I_0 \times \{0\})$ . Since  $F(z, 0) \in F[\text{bd}(W_n)] \subset (1 - 2^{-n+1}, 1] \subset (0, 1]$ , there exists a  $J \in \mathcal{J}$  such that  $z \in \text{cl}(J)$ . Let  $p_{n+1}$  be the upper endpoint for  $f$  of  $J$ . Then, by (\*),  $f(p_{n+1}) \geq 1 - 2^{-(n+1)}$  and the distance of  $\langle p_{n+1}, 0 \rangle$  from  $\langle z, 0 \rangle \in \text{bd}(W_n)$  is at most  $2^{-(n+1)}$ . The set  $B_n$  is defined as a union of  $\text{bd}(W_n)$  and a closed segment joining  $\langle z, 0 \rangle$  and  $\langle p_{n+1}, 0 \rangle$ . It is easy to see that the inductive conditions are satisfied. This finishes the construction.

Now, to finish the argument notice that for every  $n < \omega$  the set  $B^n = \bigcup_{k=n}^{\infty} B_k$  is connected, has diameter at most  $\sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}$  and contains  $\langle p_{n+1}, 0 \rangle$ . In particular, there exists  $p \in I$  with  $p = \lim_n p_n$ . We claim that  $F(p, 0) = 1$ . Indeed, if this is not the case, then there exists an  $n < \omega$  such that  $|F(p, 0) - 1| > 2^{-n+2}$ . Take  $\varepsilon > 0$  less than the diameter of  $B^n$  and let  $U$  be an open neighborhood of  $\langle p, 0 \rangle$  of diameter less than  $\varepsilon$  and such that  $F[\text{bd}(U)] \subset (F(p, 0) - 2^{-n+1}, F(p, 0) + 2^{-n+1})$ . Then there exists a point  $w \in B^n \cap \text{bd}(U)$ , since  $U \cap B^n \neq \emptyset$ ,  $U$  has diameter less than  $\text{diam}(B^n)$  and  $B^n$  is connected. But then  $F(w)$  belongs to both  $(F(p, 0) - 2^{-n+1}, F(p, 0) + 2^{-n+1})$  and  $F[B^n] \subset (1 - 2^{-n+1}, 1]$ , which is impossible, since these two sets are disjoint. So,  $F(p, 0) = 1$ . But this is impossible as well since then we would have  $f(p) = F(p, 0) = 1$  and  $f$  does not attain 1. This contradiction finishes the proof that  $f$  is not extendable.

To show that  $f$  is almost continuous let  $G$  be an open subset of  $I^2$  containing the graph of  $f$ . Notice that for every  $x \in [0, 1]$  there exists an interval  $(r_x, s_x)$  (for  $x = 0$  and  $x = 1$  we consider intervals  $[r_x, s_x)$  and  $(r_x, s_x]$ , respectively) such that

- $x \in (r_x, s_x)$ ,  $f(r_x) = f(s_x) = 0$  and
- there is a continuous function  $g_x: [0, 1] \rightarrow \mathbb{R}$  with  $g_x \upharpoonright [r_x, s_x] \subset G$  and such that  $g_x(t) = 0$  for  $t \notin (r_x, s_x)$ .

Indeed, if  $f(x) = 0$ , then it is easy to find  $r_x < s_x$  for which  $g_x \equiv 0$  works. If  $f(x) \neq 0$ , then  $x \in \text{cl}(J)$  for some  $J \in \mathcal{J}$ . For the sake of simplicity assume that  $f$  is increasing on  $J$ ; that is, that  $J \in \mathcal{J}_0$ , the other case being similar. Let  $J = (a, b)$ . Then  $f(b) > 0$ . Let  $U \subset G$  be an open circular neighborhood of  $\langle b, f(b) \rangle$ . Then there exists an interval  $J' = (c, d) \in \mathcal{J}_1$  with  $b < c$  such that  $f \upharpoonright (c, d)$  intersects  $U$ . Let  $u \in (c, d)$  be such that  $\langle u, f(u) \rangle \in U$ . Put  $r_x = a$ ,  $s_x = d$ , define  $g_x(b) = f(b)$ ,  $g_x(u) = f(u)$ ,  $g_x(0) = g_x(r_x) = g_x(s_x) = g_x(1) = 0$  and extend  $g_x$  linearly on each interval with these endpoints. Thus  $g_x$  has a “hat” shape. We have  $g_x \upharpoonright [r_x, s_x] \subset G$  since  $g_x \upharpoonright [b, u] \subset U \subset G$  and  $g_x \upharpoonright ([a, b] \cup [u, d]) = f \upharpoonright ([a, b] \cup [u, d]) \subset G$ . Now choose a finite subcover  $\{[r_0, s_0], (r_1, s_1]\} \cup \{(r_{x_i}, s_{x_i}) : i \leq n\}$ , of the cover  $\{[r_0, s_0], (r_1, s_1]\} \cup \{(r_x, s_x) : x \in (0, 1)\}$  of the interval  $[0, 1]$ . Then

$$g(x) = \max\{g_0(x), g_1(x), g_{x_0}(x), g_{x_1}(x), \dots, g_{x_n}(x)\}$$

is continuous and  $g \subset G$ . This ends the proof that  $f$  is almost continuous.  $\square$

Note that as in [8, Proposition 3.4] one can show the following.

**Remark 2.1.** *If  $f$  is from Example 2.1, then there exists a characteristic function  $\chi_B$  of a meager  $F_\sigma$  subset  $B$  of  $C$  such that  $f_0 = f + \chi_B$  is extendable.*

The next example is a slight modification of the function from Example 2.1 and is a variant of examples of Jastrzębski [17] and Kellum [21]. In what follows we will use the notation from Example 2.1.

**Example 2.2.** *There exists a connectivity, non almost continuous, Baire class two function  $f: I \rightarrow I$  with  $G_\delta$  graph.*

PROOF. Define  $f$  as follows. (See Figure 2.)

- For every  $0 < n < \omega$  we put  $f(a_n) = f(c_n) = 0$ ,  $f(b_n) = f(d_n) = 1 - 2^{-n}$  and extend it linearly on each interval  $[a_n, b_n]$  and  $[c_n, d_n]$ . (Of course we could now reduce our enumeration of  $\mathcal{J}$  to just one sequence, instead of two.)
- For all other  $x$ 's we put  $f(x) = 0$ .

The proof that such  $f$  is in  $\mathcal{G}_\delta \cap \mathcal{B}_2$  is identical to that for  $f$  from Example 2.1. The proof that  $f$  is connectivity is easy and is essentially the same as for the examples in [17, 21]. The fact that  $f \notin \text{AC}$  follows from [21, Lemma 1].  $\square$

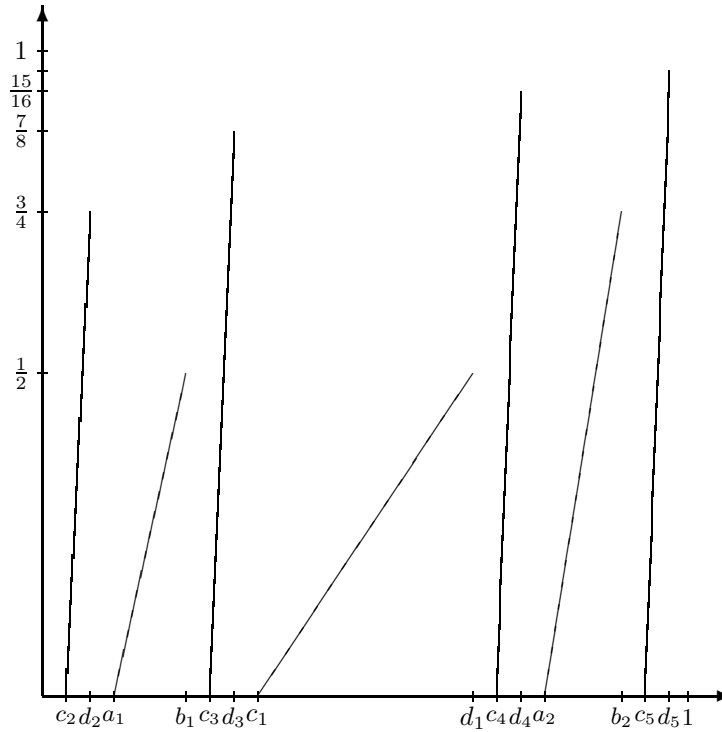


Figure 2: function  $f \in \mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{Conn} \setminus \text{AC}$  from Example 2.2

**Example 2.3.** *There exists a Darboux, non connectivity, Baire class two function  $f: I \rightarrow I$  with  $G_\delta$  graph.*

PROOF. This is a slight modification of function  $k: I \rightarrow I$  constructed by Ciesielski and Kellum in [9]. Let  $C$  denote the Cantor middle two-fifths set:

$$C = \left\{ \sum_{n=1}^{\infty} \frac{i_n}{5^n} : i_n \in \{0, 2, 4\} \text{ for every } n \right\}.$$

Geometrically,  $C$  is obtained from  $I$  by first removing the pair of intervals  $(1/5, 2/5)$  and  $(3/5, 4/5)$  from  $I$ ; then by removing similar pairs of intervals (the middle two fifths) from each of  $[0, 1/5]$ ,  $[2/5, 3/5]$  and  $[4/5, 1]$ , etc. Let  $\{(a_n, b_n), (c_n, d_n)\} : n < \omega$  be an enumeration of all such pairs (with  $b_n < c_n$ ). Also, put  $C^\circ = I \setminus \bigcup_{n < \omega} [a_n, b_n] \cup [c_n, d_n]$  and let  $\Delta$  denote the diagonal  $\{(x, x) : x \in I\}$  in  $I^2$ . Define  $f$  as follows.

- For  $n < \omega$  put  $f(c_n) = 0$ , choose  $f(b_n) > d_n > b_n$  from  $(1 - 2^{-n}, 1)$ , pick  $a_n < f(a_n) < f(d_n) < d_n$  and extend  $f$  linearly on the intervals  $[a_n, b_n]$  and  $[c_n, d_n]$ . Note that  $f[[a_n, b_n] \cup [c_n, d_n]] = [0, f(b_n)]$  and that  $f \upharpoonright \bigcup_{n < \omega} [a_n, b_n] \cup [c_n, d_n]$  is disjoint from  $\Delta$ .
- Put  $f(0) = 1/2$  and  $f(x) = 0$  for  $x \in C^\circ \setminus \{0\}$ .

The proof that such an  $f$  is in  $\mathcal{G}_\delta \cap \mathcal{B}_2$  is identical to that used in Example 2.1. The function  $f$  is not connectivity, since  $\Delta$  separates its graph. It is Darboux, since  $f[J] = [0, 1)$  for every interval  $J$  intersecting  $C^\circ$ . □

**Example 2.4.** *There exists a peripherally continuous, non Darboux, Baire class two function  $f: I \rightarrow I$  with  $G_\delta$  graph.*

PROOF. We use here the notation from Example 2.3. Define  $f$  as follows.

- For  $n < \omega$  put  $f(a_n) = 0$ ,  $f(b_n) = \frac{1}{2} - 2^{-n-3}$ ,  $f(c_n) = \frac{1}{2} + 2^{-n-3}$ ,  $f(d_n) = 1 - 2^{-n-3}$  and extend  $f$  linearly on intervals  $[a_n, b_n]$  and  $[c_n, d_n]$ .
- Put  $f(x) = 0$  for all  $x \in C^\circ$ .

The proof that such an  $f$  is in  $\mathcal{G}_\delta \cap \mathcal{B}_2$  is essentially identical to that used in Example 2.1. The function  $f$  is in  $PC \setminus D$  since  $f[J] = [0, 1) \setminus \{\frac{1}{2}\}$  for every interval  $J$  intersecting  $C^\circ$ . □

**Corollary 2.1.** *The inclusions*

$$\text{Ext} \subset \text{AC} \subset \text{Conn} \subset \text{D} \subset \text{PC}$$

*remain strict for the Baire class two functions  $f: I \rightarrow I$  with  $G_\delta$  graphs.*

### 3 An Extendable Function with Dense Graph and Countable Zero Level

In this section we concentrate on functions from  $\mathbb{R}$  to  $\mathbb{R}$ , though essentially all presented results remain the same for functions from  $I$  to  $I$ . Our main result here is the following.

**Theorem 3.1.** *There exists an extendable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with dense graph for which  $f^{-1}(0)$  is countable.*

The motivation for searching for such an example comes from several directions. First of all in 1970 Brown [2] proved that if  $\mathcal{F}$  is the class of connectivity functions from  $\mathbb{R}$  to  $\mathbb{R}$ , then

- (A) if  $f \in \mathcal{F}$  has a dense graph, then every nowhere dense set  $N$  is  $f$ -negligible for  $\mathcal{F}$ ; that is, any function  $g: \mathbb{R} \rightarrow \mathbb{R}$  equal to  $f$  outside of  $N$  remains in  $\mathcal{F}$ ;
- (B) part (A) is false for countable sets  $N$ ; that is, there exists an  $f \in \mathcal{F}$  with a dense graph and a countable dense set  $D \subset \mathbb{R}$  which is not  $f$ -negligible for  $\mathcal{F}$ .

The similar result for the class  $\mathcal{F}$  of almost continuous functions has been proved in 1982 by Kellum [20]. Also for the class  $\mathcal{F}$  of extendable functions in 1994 Rosen [25] proved part (A). However, so far there has been no example justifying (B) for the class Ext. The function  $f$  from Theorem 3.1 clearly justifies (B) for  $\mathcal{F} = \text{Ext}$ , since the countable zero level  $D = f^{-1}(0)$  cannot be  $f$ -negligible for Ext. (Modifying  $f$  on  $D$  we can get a non Darboux function.)

The other motivation comes for the following theorem of Ciesielski and Rosłanowski [12, thm. 3.1], (which has been used in constructing an additive, almost continuous, non extendable function with the SCIVP property).

**Proposition 3.2.** *If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is extendable and has a dense graph, then  $f$  has the following (super SCIVP) property*

*For every  $x, y \in \mathbb{R}$  and for each perfect  $K$  between  $f(x)$  and  $f(y)$  there is a perfect  $C$  between  $x$  and  $y$  such that  $f[C] \subset K$  and  $f \upharpoonright C$  is continuous strictly monotone.*

It seems natural to ask whether in the above the phrase “strictly monotone” can be replaced with “constant” or, more generally, the perfect set  $K$  be replaced by a singleton. We do not know the answer to the first of these questions. However the example from Theorem 3.1 is a counterexample for the second conjecture.

Theorem 3.1 is also related to the following open problem [16, Question 9.31]. *Is it true that  $g: \mathbb{R} \rightarrow \mathbb{R}$  is a product of two extendable functions if and only if  $g$  has a zero in each subinterval in which it changes sign?* (This characterization is true if the class of extendable functions is replaced by the class of Darboux, connectivity, or almost continuous functions.) It gives the positive answer in the following particular case.

**Corollary 3.3.** *If  $g: \mathbb{R} \rightarrow \mathbb{R}$  is such that  $g^{-1}(0)$  is dense in  $\mathbb{R}$ , then  $g$  is a product of two extendable functions.*

PROOF. The proof is a slight modification of the fact that every function  $g: \mathbb{R} \rightarrow \mathbb{R}$  is a sum of two extendable functions. (See [25] or [11].)

Let  $f$  be from Theorem 3.1. Then there exists a dense  $G_\delta$  subset  $G$  of  $\mathbb{R}$  which is  $f$ -negligible for Ext. (See [25, Theorem 1].) Also there exists a

homeomorphism  $h_0$  of  $\mathbb{R}$  such that  $G \cup h_0[G] = \mathbb{R}$ . Moreover,  $f_0 = f \circ h_0^{-1}$  is extendable and  $h_0[G]$  is  $f_0$ -negligible for Ext.

Let  $h$  be a homeomorphism of  $\mathbb{R}$  such that  $h[f^{-1}(0) \cup f_0^{-1}(0)] \subset g^{-1}(0)$  and put  $f_1 = f \circ h^{-1}$  and  $f_2 = f_0 \circ h^{-1} = f \circ h_0^{-1} \circ h^{-1}$ . Then,  $f_1$  and  $f_2$  are extendable and the sets  $G_1 = h[G]$  and  $G_2 = h[h_0[G]]$  are  $f_1$ - and  $f_2$ -negligible, respectively. Also,  $Z = f_1^{-1}(0) \cup f_2^{-1}(0) \subset g^{-1}(0)$ . Define

$$\hat{f}_1(x) = \begin{cases} \frac{g(x)}{f_2(x)} & \text{for } x \in G_1 \setminus Z \\ f_1(x) & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{f}_2(x) = \begin{cases} f_2(x) & \text{for } x \in G_1 \cup Z \\ \frac{g(x)}{f_1(x)} & \text{otherwise.} \end{cases}$$

Then  $\hat{f}_1$  and  $\hat{f}_2$  are well defined since  $f_1$  and  $f_2$  are not 0 outside  $Z$ . They are extendable, since they are respective modifications of  $f_1$  and  $f_2$  on negligible sets  $G_1 \setminus Z \subset G_1$  and  $\mathbb{R} \setminus (G_1 \cup Z) \subset G_2$ . Also,

- for  $x \in Z$  we have  $\hat{f}_1(x)\hat{f}_2(x) = 0 = g(x)$ ,
- for  $x \in G_1 \setminus Z$  we have  $\hat{f}_1(x)\hat{f}_2(x) = \frac{g(x)}{f_2(x)} f_2(x) = g(x)$  and
- for  $x \in \mathbb{R} \setminus (G_1 \cup Z)$  we have  $\hat{f}_1(x)\hat{f}_2(x) = f_1(x) \frac{g(x)}{f_1(x)} = g(x)$ .

Thus,  $g = \hat{f}_1\hat{f}_2$ . □

Theorem 3.1 will be concluded from the following theorem. The construction is a modification of Ciesielski-Reclaw’s construction from [11, Theorem 3.3]. However, the triangles in the triangulations are not equilateral, as in the generalization of [11, Theorem 3.3] presented in [13, prop. 2.3].

**Theorem 3.4.** *There exists a connectivity function  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  with graph dense in  $\mathbb{R}^3$  such that  $F^{-1}(0)$  is contained in a countable union of straight lines in  $\mathbb{R}^2$ .*

Before we prove Theorem 3.4 let us notice how it implies Theorem 3.1.

PROOF OF THEOREM 3.1. Let  $F$  be as in Theorem 3.4. Note that by rotating the domain  $\mathbb{R}^2$  of  $F$  we obtain the same kind of example. Thus, without loss of generality we can assume the countable family  $\mathcal{L}$  of lines with  $F^{-1}(0) \subset \bigcup \mathcal{L}$  does not contain a line parallel to the  $x$ -axis. Then every line  $L_y = \mathbb{R} \times \{y\}$  intersects  $F^{-1}(0)$  in a countable set. Moreover, by examining the function  $F$  constructed below (or by an argument similar to that of Rosen from [25, Theorem 1]) there exists a dense  $G_\delta$  subset  $G$  of  $\mathbb{R}^2$  such that  $G$  is  $F$ -negligible for the class Conn. Thus, by the Kuratowski-Ulam theorem (a category analog of the Fubini theorem), there is  $y \in \mathbb{R}$  such that  $G \cap L_y$  is a dense  $G_\delta$  subset of  $L_y$ . This implies that the graph of  $F \upharpoonright L_y$  is dense in  $L_y \times \mathbb{R}$ . It is easy to see that  $f$  defined by  $f(x) = F(x, y)$  has all the desired properties. □

PROOF OF THEOREM 3.4. The constructed function  $F$  will be peripherally continuous, so connectivity. (See e.g. [16, Theorem 8.1] or [7, p. 171].)

**Basic Idea:** By induction on  $n < \omega$  we will construct a sequence  $\langle S_n : n < \omega \rangle$  of triangular “grids,” that is, triangulations of the plane. We will begin with  $S_0$  formed with equilateral triangles of side length 1 as in Figure 3. Then, at the stage  $n + 1$ , we will subdivide each triangle from  $S_n$  into finitely many pieces making sure that their sizes tend to zero. The grid  $S_n$  will be identified with the points on the edges of triangles forming it and we will assume that  $S_n \subseteq S_{n+1}$  for all  $n < \omega$ . With each grid  $S_n$  we will associate a continuous function  $f_n : S_n \rightarrow \mathbb{R}$  which is linear on each side of a triangle from  $S_n$ . Moreover, each  $f_{n+1}$  will be an extension of  $f_n$ . The function  $F$  will be defined as an extension of  $\bigcup_{n < \omega} f_n$  such that  $F^{-1}(0) \subset \bigcup_{n < \omega} S_n$ . Thus,  $F^{-1}(0)$  will be contained in a countable union of straight lines; those that contain sides of triangles from  $S_n$ 's.

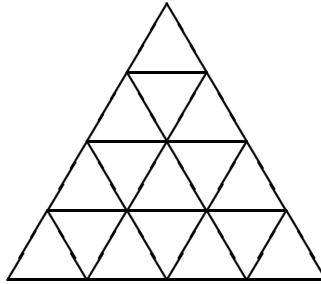


Figure 3: grid  $S_0$

**Terminology:** In what follows a *triangle* will be identified with the set of points of its interior or its boundary. For a grid  $S$  we say that a *triangle*  $T$  is *from*  $S$  if the interior of  $T$  is equal to a component of  $\mathbb{R}^2 \setminus S$ .

For a triangle  $T$  its *basic partition* will be its split into ten triangles as in Figure 4. The central triangle  $\hat{T}$  of Figure 4 will be referred to as *the middle quarter of*  $T$ . It is similar to  $T$  and its sides are parallel to the sides of  $T$  and of  $1/4$  of their respective lengths. Also,  $T$  and  $\hat{T}$  have the same center and the vertices on the sides of  $T$  are at their centers. It is easy to see that the diameter of each of triangle from the basic partition of  $T$  is at most half of the diameter of  $T$ . Also  $\hat{T} \cap \text{bd}(T) = \emptyset$ .

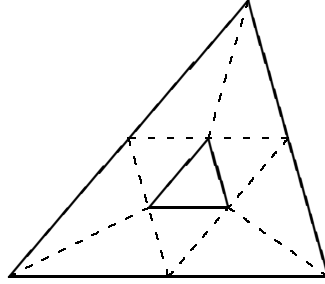


Figure 4: the middle quarter  $\hat{T}$  of a triangle  $T$  and the basic partition of  $T$

If a function  $f$  is defined on the three vertices of a triangle  $T$ , its *basic extension* is defined as the unique function  $\hat{f}: T \rightarrow \mathbb{R}$  extending  $f$  whose graph is a subset of a plane in  $\mathbb{R}^3$ . Notice, that  $\hat{f}$  is linear on each side of the triangle  $T$  and that  $\hat{f}$  extends  $f$  even if the function  $f$  has already been defined on some side of  $T$  as long as  $f$  is linear on this side. In particular, if  $f$  is defined on a grid  $S$  and is linear on each side of every triangle from  $S$ , then  $\hat{f}: \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined in a natural way.

**Inductive construction:** We will inductively define two increasing sequences:  $\langle S_n : n < \omega \rangle$  of triangular grids and  $\langle f_n \in \mathbb{R}^{S_n} : n < \omega \rangle$  of continuous functions such that the following inductive conditions are satisfied for every  $n < \omega$ .

- (i) For each triangle  $T$  from  $S_n$  function  $f_n$  on  $T$  is linear not identically 0 and  $f_n[T]$  is a subset of one of the intervals:  $[-2^n, -1]$ ,  $[-1, 0]$ ,  $[0, 1]$ , or  $[1, 2^n]$ .
- (ii) The side length of each triangle from  $S_n$  is at most  $1/2^n$ .
- (iii) The variation of  $f_n$  on each triangle from  $S_n$  is at most  $1/2^n$ .
- (iv) If  $n > 0$ ,  $k \in \{-1, 1\}$ , then for every triangle  $T$  from  $S_{n-1}$  for which  $(k \cdot f_n)[T] \subset [0, \infty)$  the following holds.
  - (a) With every non-zero dyadic number  $i/2^n \in [-2^n, 2^n]$ , where  $i$  belongs to  $D_n = \{i \in \mathbb{Z} : -4^n \leq i \leq 4^n \ \& \ i \neq 0\}$ , we associate a triangle  $T_i \subseteq \hat{T}$  from  $S_n$  such that all triangles  $T_i$  are disjoint,  $\hat{f}_n \left[ \hat{T} \setminus \bigcup_{i \in D_n} T_i \right] = \{k\}$  and  $\hat{f}_n [T_i] = \{i/2^n\}$  for every  $i \in D_n$ .
  - (b) For every  $x \in T \setminus \hat{T}$  either  $\hat{f}_{n-1}(x) \leq \hat{f}_n(x) \leq k$  or  $\hat{f}_{n-1}(x) \geq \hat{f}_n(x) \geq k$ .

To start the induction define grid  $S_0$  as in Figure 3 with all sides of length 1 and let  $f_0: S_0 \rightarrow \mathbb{R}$  be identically 1. It is easy to see that conditions (i)–(iv) are satisfied for such a choice.

Next, assume that for some  $n > 0$  we already have  $S_{n-1}$  and  $f_{n-1}$  satisfying (i)–(iv). We will define  $S_n$  and extend  $f_{n-1}$  to  $f_n: S_n \rightarrow \mathbb{R}$  such that (i)–(iv) will still hold. Let  $T$  be a triangle from  $S_{n-1}$  and put  $k = 1$  if  $\hat{f}_{n-1}[T] \subset [0, \infty)$  and  $k = -1$  for  $\hat{f}_{n-1}[T] \subset (-\infty, 0]$ . The  $S_n$ -triangulation of  $T$  will be a refinement of the basic triangulation of  $T$ . This will guarantee (ii).

Clearly  $f_n$  is already defined on  $\text{bd}(T)$ . We define  $f_n$  on each vertex of the middle quarter  $\hat{T}$  of  $T$  by assigning it the value  $k$ . Next, on each triangle  $T'$  from the basic partition of  $T$  except for  $\hat{T}$  (see Figure 4) we extend  $f_n$  to  $\hat{f}_n$  linearly. Such an extension is unique since  $f_n$  is already defined on each vertex of  $T'$ . Note that this and the choice of  $k$  guarantee (iv)(b). This also gives us (i) for any triangle from a subtriangulation of  $T'$ .

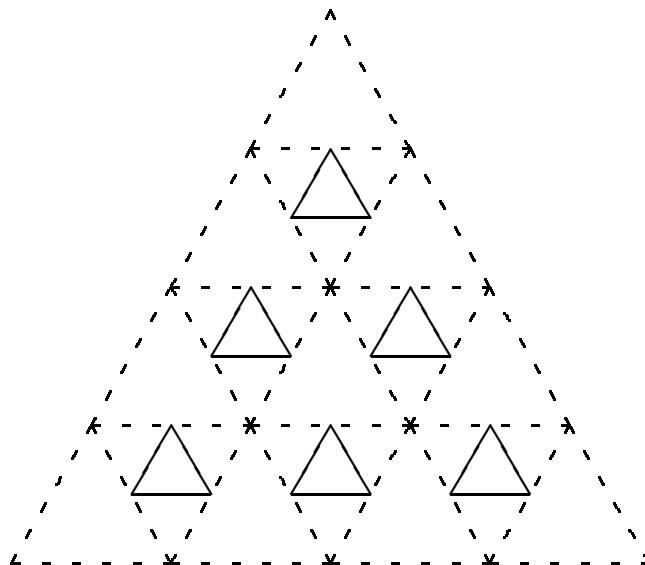


Figure 5: some triangles  $T_i$  of the grid of  $\hat{T}$

Thus,  $\hat{f}_n$  is defined so far on  $T \setminus \hat{T}$ . To extend  $\hat{f}_n$  to  $\hat{T}$  we proceed as follows. Partition  $\hat{T}$  into a grid  $S$  such that  $S$  contains  $2 \cdot 4^n$  disjoint triangles  $\{T_i: i \in D_n\}$ . (See Figure 5.) We extend  $\hat{f}_n$  to  $\hat{T} \setminus \bigcup_{i \in D_n} T_i$  by assigning it the constant value  $k$ , while on each  $\hat{T}_i$  with  $i \in D_n$ ,  $\hat{f}_n$  is identically  $i/2^n$ . This guarantees the remainder of condition (iv) as well as (i) for the triangles

considered so far. Finally,  $\hat{f}_n$  is extended linearly on each triangle from the basic partition of  $T_i$  using the fact that it is already defined on each of its vertices. This finishes the construction of  $\hat{f}_n$ .

Now, to make sure that (iii) and the remaining part of (i) are satisfied we will refine the triangulations defined so far to the grid  $S_n$ . This will lead to the definition of  $f_n$  as  $\hat{f}_n \upharpoonright S_n$ .

For (i) we proceed as follows. If  $k$  and  $i \in D_n$  have the same sign, then  $\hat{f}_n \upharpoonright T_i$  does not attain value 0 and (i) is already satisfied by the triangles from the basic partition of  $T_i$ . In this case  $S_n$  on  $T_i$  will be formed from the triangles considered so far. On the other hand, if  $k$  and  $i \in D_n$  have different signs, then  $(\hat{f}_n \upharpoonright T_i)^{-1}(0)$  is a boundary of a triangle  $T'$  similar to  $T_i$ . Then we subtriangulate the basic partition of  $T_i$  in such a way that every edge of  $T'$  is covered by the edges of this subtriangulation and that each triangle from the subtriangulation intersects at most one edge (excluding the end points) of  $T'$ . If, in addition,  $(\hat{f}_n \upharpoonright T_i)^{-1}(-k)$  is nonempty, then it is a boundary of a triangle (or the entire triangle) and we will make sure that this boundary is a subset of our subtriangulation.

Condition (iii) is easily guaranteed by refining the triangulation described so far. This finishes the inductive construction.

Now,  $F$  is defined on  $S = \bigcup_{n < \omega} S_n$  as  $f = \bigcup_{n < \omega} f_n$ . We will extend it to  $\mathbb{R}^2$  making sure that  $F(x) \neq 0$  for every  $x \in \mathbb{R}^2 \setminus S$ . This will guarantee that  $F^{-1}(0) \subset S$ . Note also that, by (iv)(a), the graph of  $f$  is already dense in  $\mathbb{R}^3$ . So any extension  $F$  of  $f$  will have this property as well.

To extend  $F$  to  $\mathbb{R}^2$  pick an  $x \in \mathbb{R}^2 \setminus S$ . Note that, by (i),  $\hat{f}_n(x) \neq 0$  for every  $n < \omega$ . For every  $n < \omega$  let  $T_n^x$  be the triangle from  $S_n$  containing  $x$  and let  $N = \{n < \omega : x \in \hat{T}_n^x\}$ . We consider two cases.

Case 1. The set  $N$  is infinite. If  $N^k = \{n \in N : \hat{f}_n[\hat{T}_n^x] = \{k\}\}$  for  $k \in \{-1, 1\}$ , then  $N = N^{-1} \cup N^1$  and so at least one of the sets  $N^{-1}$  and  $N^1$  is infinite. Let  $k = 1$  if  $N^1$  is infinite and put  $k = -1$  otherwise. In this case we define  $F(x) = k$ . Notice that this guarantees that  $F$  is peripherally continuous at  $x$  since for any  $n \in N^k$

$$F[\text{bd}(\hat{T}_n^x)] = f_n[\text{bd}(\hat{T}_n^x)] = \{k\} = \{F(x)\},$$

$x$  belongs to the interior of  $\hat{T}_n^x$  and the diameter of  $\hat{T}_n^x$  is at most  $1/2^n$ .

Case 2. The set  $N$  is finite. Fix an  $m < \omega$  such that  $N \subset \{0, \dots, m-2\}$ . Then  $x \in T_n^x \setminus \hat{T}_n^x$  for every  $n \geq m$ . So, if  $k \in \{-1, 1\}$  is such that  $(k \cdot f_m)[T_m^x] \subset [0, \infty)$ , then, by (iv)(b), either  $\hat{f}_m(x) \leq \hat{f}_{m+1}(x) \leq \hat{f}_{m+2}(x) \leq \dots \leq k$  or  $\hat{f}_m(x) \geq \hat{f}_{m+1}(x) \geq \hat{f}_{m+2}(x) \geq \dots \geq k$ . So the limit  $L = \lim_{m \rightarrow \infty} \hat{f}_m(x)$  is well defined and not equal to 0. We put  $F(x) = L$ . This, together with (ii) and (iii), guarantees that  $F$  is peripherally continuous at  $x$ .

This finishes the construction of function  $F$  and the argument that  $F$  is peripherally continuous on  $\mathbb{R}^2 \setminus S$ . To see that  $F$  is peripherally continuous on  $S$  take an  $x \in S$ . Then, there exists a  $k < \omega$  such that  $x \in S_n$  for every  $n \geq k$ . For any such  $n$  let  $\mathcal{T}_n$  be the set of all triangles from  $S_n$  to which  $x$  belongs. Then  $x$  belongs to the interior of the polygon  $P_n = \bigcup \mathcal{T}_n$ . Moreover, by (ii) and (iii), the variation on the boundary of  $P_n$  and the diameter of  $P_n$  are at most  $2/2^n$ . So, the sequence  $\langle P_n \rangle$  guarantees that  $F$  is peripherally continuous at  $x$ . This finishes the proof of Theorem 3.4.  $\square$

## 4 Quasi-Continuous Extendable Functions

To state the results of this section we need the following additional terminology and facts.

For  $x \in I$  let  $l_x = \{x\} \times I$  and for function  $f: I \rightarrow I$  let  $C(f)$  stand for the set of points of continuity of  $f$ . Recall that for a Darboux function  $f: I \rightarrow I$  the set  $\text{cl}(f) \cap l_x$  is connected for every  $x \in I$  and that  $C(f)$  is a dense  $G_\delta$  provided  $f$  has a  $G_\delta$  graph. (See e.g. [18].) The function  $f: I \rightarrow I$  is *quasi-continuous* if  $f \upharpoonright C(f)$  is dense in the graph of  $f$ . A function  $f: I \rightarrow I$  is said to have a closure that is *bilaterally dense in itself* if  $\text{cl}(f \upharpoonright (0, x)) \cap l_x = \text{cl}(f \upharpoonright (x, 1)) \cap l_x$  for each  $x \in (0, 1)$ .

Croft's function  $f: I \rightarrow I$  from [14] is Darboux, upper semicontinuous (hence of Baire class one) and 0 almost everywhere, but is not identically 0. It follows that the closure of  $f$  is bilaterally dense in itself. However, since it is not quasi-continuous, Croft's function does not satisfy the second category condition in the following result. But a function like Kellum and Garrett's [22, Example 1] does satisfy it and the rest of the hypothesis.

**Theorem 4.1.** *Suppose  $f: I \rightarrow I$  is a Darboux function with a  $G_\delta$  graph whose closure is bilaterally dense in itself. Also suppose for every point  $\langle x, f(x) \rangle$  of  $f$  there exists an open neighborhood  $W \subset I^2$  of  $\langle x, f(x) \rangle$  such that if  $B$  is closed and nowhere dense in  $\text{pr}(f \cap W)$ , then  $\text{pr}^{-1}(B) \cap (f \cap W)$  is nowhere dense in  $f \cap W$ . Then  $f$  is quasi-continuous and extendable.*

PROOF. Let  $A = \{x \in I: \text{cl}(f) \cap l_x \text{ is nondegenerate}\}$ . Assume  $f$  is not quasi-continuous at some point  $x_0 \in I$ . Then there exists a rectangular open neighborhood  $W = I_1 \times I_2 \subset I^2$  of  $\langle x_0, f(x_0) \rangle$  that obeys the second category condition of the hypothesis and that contains no point of  $f \upharpoonright C(f)$ . Note that  $\text{pr}(f \cap W) = \{x \in A \cap I_1: \langle x, f(x) \rangle \in W\}$ . According to the Alexandroff theorem, the  $G_\delta$  subset  $f \cap W$  of  $I^2$  is homeomorphic to a complete metric space. Therefore, by hypothesis,  $\text{pr}(f \cap W)$  is of second category in itself.

Observe that if  $x \in \text{pr}(f \cap W)$ , then  $\text{cl}(f) \cap l_x$  is a nondegenerate interval meeting  $I^2 \setminus W$  and so  $\text{cl}(f \cap W) \cap l_x$  contains a nondegenerate interval contain-

ing  $\langle x, f(x) \rangle$ . To show  $f \cap W$  is nowhere dense in  $\text{pr}(f \cap W) \times I$  let  $[a, b] \times [c, d]$  be a rectangle in  $I^2$  that meets  $\text{pr}(f \cap W) \times I$  and let  $G_1 \supset G_2 \supset \dots$  be open subsets of  $I^2$  such that  $f \cap W = (\bigcap_{n=1}^{\infty} G_n) \cap (\text{pr}(f \cap W) \times I)$ . For all positive integers  $n$  and rational numbers  $r$  and  $s$  with  $c \leq r < s \leq d$  let  $H(n, r, s)$  be the set of all  $x \in \text{pr}(f \cap W) \cap [a, b]$  for which some component of  $l_x \setminus G_n$  meets both  $I \times \{r\}$  and  $I \times \{s\}$ . Each  $H(n, r, s)$  is closed in  $\text{pr}(f \cap W) \cap [a, b]$  and each point of  $\text{pr}(f \cap W) \cap [a, b]$  belongs to some  $H(n, r, s)$ . Therefore there exist  $n, r, s$  and an interval  $(u, v)$  such that  $H(n, r, s) \supset \text{pr}(f \cap W) \cap (u, v) \neq \emptyset$ . Since the sets  $f \cap W$  and  $[\text{pr}(f \cap W) \cap (u, v)] \times (r, s)$  are disjoint,  $f \cap W$  is nowhere dense in  $\text{pr}(f \cap W) \times I$ .

For each positive integer  $n$  let  $Q_n$  be the set of all  $x \in \text{pr}(f \cap W)$  for which  $\text{cl}(f \cap W) \cap l_x$  contains an interval containing  $\langle x, f(x) \rangle$  of length at least  $\frac{1}{n}$ . Each  $Q_n$  is closed in  $\text{pr}(f \cap W)$ . Since  $f \cap W$  is nowhere dense in  $\text{pr}(f \cap W) \times I$ , each  $Q_n$  is nowhere dense in  $\text{pr}(f \cap W)$ . Because  $\text{pr}(f \cap W)$  is of second category and  $\text{pr}(f \cap W) = \bigcup_{n=1}^{\infty} Q_n$ , this a contradiction. By [26, Theorem 1] a Darboux quasi-continuous function  $f$  whose closure is bilaterally dense in itself is extendable.  $\square$

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