

# On the cofinalities of Boolean algebras and the ideal of null sets

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## Abstract

We will show that if the cofinality of the ideal of Lebesgue measure zero sets is equal to  $\omega_1$  then there exists a Boolean algebra  $B$  of cardinality  $\omega_1$  which is not a union of strictly increasing  $\omega$ -sequence of its subalgebras. This generalizes a result of Just and Koszmider who showed that it is consistent with  $ZFC+\neg CH$  that such an algebra exists.

## 1 Preliminaries

For an infinite Boolean algebra  $B$  its *cofinality*  $\text{cof}(B)$  is defined as the least infinite cardinal number  $\kappa$  such that  $B$  is a union of strictly increasing se-

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\*AMS classification numbers: Primary 06E25; Secondary 03E17, 03G05.

Key words and phrases: cofinality, null sets.

quence of type  $\kappa$  of subalgebras of  $B$ ; its *homomorphism type*  $h(B)$  is the least cardinality of an infinite homomorphic image of  $B$ .

In [6] Koppelberg proved that

- (a)  $\omega \leq \text{cof}(B) \leq h(B) \leq \mathfrak{c}$ , and
- (b) if Martin's Axiom holds then  $\text{cof}(B) = \omega$  for every Boolean algebra  $B$  with  $|B| < \mathfrak{c}$ ; in particular  $h(B) \in \{\omega, \mathfrak{c}\}$  for every Boolean algebra  $B$ .

(See also [5] and [4].)

In [5] Just and Koszmider examined a question whether in (b) the assumption of Martin's Axiom is important. They gave a positive answer to it by showing that there exists a model of ZFC (obtained by adding Sacks reals side-by-side) in which there is a Boolean algebra  $B$  such that  $|B| = \text{cof}(B) = \omega_1 < \mathfrak{c}$ . Clearly for this algebra we have also  $h(B) = \omega_1 \notin \{\omega, \mathfrak{c}\}$  since  $\text{cof}(B) \leq h(B) \leq |B|$ .

The goal of this paper is to prove the following theorem, in which  $\mathcal{N}$  denotes the ideal of Lebesgue measure subset of  $\mathbb{R}$  and  $\text{cof}(\mathcal{N})$  its cofinality, that is,

$$\text{cof}(\mathcal{N}) = \min\{|\mathcal{B}|: \mathcal{B} \subset \mathcal{N} \text{ generates } \mathcal{N}\}.$$

**Theorem 1**  $\text{cof}(\mathcal{N}) = \omega_1$  implies that there exists a Boolean algebra  $B$  of cardinality  $\omega_1$  such that  $\text{cof}(B) = \omega_1$ .

This generalizes the result of Just and Koszmider since in the model they worked with  $\text{cof}(\mathcal{N}) = \omega_1$  holds. However,  $\text{cof}(\mathcal{N}) = \omega_1$  holds in many other models as well. (For example, it follows from the Covering Property Axiom CPA as shown by the authors in [3].) Moreover, the argument presented here is considerably simpler than the one from [5].

Our set theoretic terminology is standard and follows that of [2]. In particular,  $|X|$  stands for the cardinality of a set  $X$  and  $\mathfrak{c} = |\mathbb{R}|$ . In what follows we will use the following characterization of  $\text{cof}(\mathcal{N})$ , in which  $\mathcal{C}_H$  stands for the family of all subsets  $\prod_{n < \omega} T_n$  of  $\omega^\omega$  such that  $T_n \in [\omega]^{\leq n+1}$  for all  $n < \omega$ .

**Proposition 2** (Bartoszyński [1, thm. 2.3.9])

$$\text{cof}(\mathcal{N}) = \min \left\{ |\mathcal{F}|: \mathcal{F} \subset \mathcal{C}_H \ \& \ \bigcup \mathcal{F} = \omega^\omega \right\}.$$

## 2 The proof

The proof of Theorem 1 will be based on the following lemma.

**Lemma 3** *If  $\text{cof}(\mathcal{N}) = \omega_1$  then for every infinite countable Boolean algebra  $\mathcal{A}$  there exists a family  $\{a_n^\xi \in \mathcal{A} : n < \omega \ \& \ \xi < \omega_1\}$  with the following properties.*

- (i)  $a_n^\xi \wedge a_m^\xi = \mathbf{0}$  for every  $n < m < \omega$  and  $\xi < \omega_1$ .
- (ii) For every increasing sequence  $\langle \mathcal{A}_n : n < \omega \rangle$  of proper subalgebras of  $\mathcal{A}$  with  $\mathcal{A} = \bigcup_{n < \omega} \mathcal{A}_n$  there exists a  $\xi < \omega_1$  such that  $a_n^\xi \notin \mathcal{A}_n$  for all  $n < \omega$ .

PROOF. In the argument that follows every sequence  $\bar{\mathcal{A}} = \langle \mathcal{A}_n : n < \omega \rangle$  as in (ii) will be identified with a function  $f_{\bar{\mathcal{A}}} = f \in \omega^{\mathcal{A}}$  for which  $f^{-1}(n) = \mathcal{A}_n \setminus \bigcup_{i < n} \mathcal{A}_i$ . We will denote the set of all such functions by  $X$ . Also, let  $\{b_n : n < \omega\}$  be an enumeration of  $\mathcal{A}$  and for each  $n < \omega$  let  $B_n$  be a finite algebra generated by  $\{b_i : i < n\}$ . Thus,  $\mathcal{A} = \bigcup_{i < \omega} B_i$ .

Since  $\text{cof}(\mathcal{N}) = \omega_1$ , the dominating number

$$\mathfrak{d} = \min \{|\mathcal{K}| : \mathcal{K} \subset \omega^\omega \ \& \ (\forall f \in \omega^\omega)(\exists g \in \mathcal{K})(\forall n < \omega) f(n) < g(n)\}$$

is equal to  $\omega_1$ . (See e.g. [1].) So, there exists a dominating family  $\mathcal{K} \subset \omega^\omega$  of cardinality  $\omega_1$ . We can also assume that the sequences in  $\mathcal{K}$  are strictly increasing and that for every  $g \in \mathcal{K}$  function  $\bar{g}$  defined by  $\bar{g}(n) = \sum_{i \leq n} g(i)$  also belongs to  $\mathcal{K}$ .

Next note that for every  $f \in X$  there are  $\bar{d} = \langle d_k : k < \omega \rangle \in \mathcal{K}$  and  $\bar{r} = \langle r_k : k < \omega \rangle \in \mathcal{K}$  such that for every  $k < \omega$

- (a)  $f(b) < r_k$  for all  $b \in B_k$ ; and
- (b) there are disjoint  $b_0, \dots, b_{2k} \in B_{d_k}$  with  $r_{d_{k-1}} < f(b_0) < \dots < f(b_{2k})$ .

Indeed, the existence of  $\bar{r}$  satisfying (a) follows directly from the definition of a dominating family. Moreover, since all algebras  $\mathcal{A}_n = f^{-1}(\{0, \dots, n\})$  are proper, for every number  $d < \omega$  there exist disjoint  $b_0, \dots, b_{2d} \in \mathcal{A}$  such that  $r_d < f(b_0) < \dots < f(b_{2d})$ . Let  $h \in \omega^\omega$  be such that  $b_0, \dots, b_{2d} \in B_{d+h(d)}$  for every  $d < \omega$  and let  $g \in \mathcal{K}$  be a function dominating  $h$ . Then  $\bar{d} = \bar{g}$  is as required.

The above implies, in particular, that for every  $f \in X$  there are  $\bar{d}, \bar{r} \in \mathcal{K}$  such that  $f$  satisfies (b) and the sequence  $f^{\bar{r}} = \langle f \upharpoonright (B_{d_k} \setminus B_{d_{k-1}}) : k < \omega \rangle$  belongs to

$$X(\bar{d}, \bar{r}) = \prod_{k < \omega} (r_{d_k})^{B_{d_k} \setminus B_{d_{k-1}}}.$$

Now, since  $\text{cof}(\mathcal{N}) = \omega_1$ , by Proposition 2 (applied to  $\prod_{k < \omega} \omega^{B_{d_k} \setminus B_{d_{k-1}}}$  in place of  $\omega^\omega$ ) we can find an  $\omega_1$ -covering of  $X(\bar{d}, \bar{r})$  by sets  $T$  of the form  $\prod_{k < \omega} T_k$ , where  $T_k \in \left[ \omega^{B_{d_k} \setminus B_{d_{k-1}}} \right]^{\leq k+1}$  for all  $k < \omega$ . Since the total number of these sets  $T$  (for different  $\bar{d}, \bar{r} \in \mathcal{K}$ ) is equal to  $\omega_1$ , to finish the proof it is enough to show that for any such  $T$  there is one sequence  $\langle a_n : n < \omega \rangle$  satisfying (i) and such that (ii) holds for every  $\bar{\mathcal{A}} = \langle \mathcal{A}_n : n < \omega \rangle$  for which  $f_{\bar{\mathcal{A}}}^{\bar{r}}$  belongs to  $T$  and  $f_{\bar{\mathcal{A}}}$  satisfies (b).

So, let  $T$  be as above and let  $T^*$  be the set of all functions  $f_{\bar{\mathcal{A}}}$  satisfying (b) for which  $f_{\bar{\mathcal{A}}}^{\bar{r}} \in T$ . By induction on  $k < \omega$  we will construct a sequence  $\langle c_k \in B_{d_k} \setminus B_{d_{k-1}} : k < \omega \rangle$  such that

$$(*) \quad f(c_k) > r_{d_k} \geq k \text{ for every } k < \omega \text{ and } f \in T^*.$$

So fix a  $k < \omega$  and let  $\{f_i : i < k\}$  be such that

$$\{f_i \upharpoonright B_{d_k} \setminus B_{d_{k-1}} : i < k\} = \{f \upharpoonright B_{d_k} \setminus B_{d_{k-1}} : f \in T^*\} \subset T_k.$$

We show inductively that for every  $m < k$

$$\text{there is a } c \in B_{d_k} \text{ such that } f_j(c) > r_{d_k} \text{ for all } j \leq m. \quad (1)$$

So, fix an  $m < k$  and let  $a \in B_{d_k}$  such that  $f_j(a^c) = f_j(a) > r_{d_k}$  for all  $j < m$ . If  $f_m(a) > r_{d_k}$  then  $c = a$  satisfies (1). Thus, assume that  $f_m(a^c) = f_m(a) \leq r_{d_k}$ . By (b) we can find  $b_0, \dots, b_{2k} \in B_{d_k}$  such that  $r_{d_{k-1}} < f_m(b_0) < \dots < f_m(b_{2k})$ . By Pigeon Hole Principle we can find an  $I \in [\{0, \dots, 2k\}]^{k+1}$  and a  $b \in \{a, a^c\}$  such that  $f_m(b \wedge b_i) = f_m(b_i)$  for all  $i \in I$ . Without loss of generality we can assume that  $I = \{0, \dots, k\}$  and  $b \wedge b_i = b_i$  for all  $i \leq k$ . Then

$$f_m(b^c \vee b_i) > r_{d_k} \text{ for all } i \leq k.$$

Moreover, for every  $j < m$  there is at most one  $i_j \leq k$  for which

$$f_j(b^c \vee b_{i_j}) \leq r_{d_k}$$

since for different  $i, i' \leq k$  we have  $f_j((b^c \vee b_i) \wedge (b^c \vee b_{i'})) = f_j(b^c) > r_{d_k}$ . Thus, by Pigeon Hole Principle, there is an  $i \leq k$  such that  $c = b^c \vee b_i$  satisfies (1). This finishes the proof of (\*).

Clearly the sequence  $\langle c_k \in B_{d_k} \setminus B_{d_{k-1}} : k < \omega \rangle$  satisfies (ii) for every  $\bar{\mathcal{A}}$  with  $f_{\bar{\mathcal{A}}} \in T^*$ . Thus, we need only to modify it to get also the condition (i).

To do it, use the fact that

$$r_{d_k} < f(c_k) < r_{d_{k+1}} \text{ for every } k < \omega \text{ and } f \in T^*$$

to construct the sequences:  $\omega = I_0 \supset I_1 \supset \dots$  of infinite subsets of  $\omega$ , increasing  $\langle k_j \in I_j : j < \omega \rangle$ , and  $\langle c_{k_j}^* \in \{c_{k_j}, c_{k_j}^c\} : j < \omega \rangle$  such that for every  $j < \omega$

- $f(\bar{a}_{k_j} \wedge c_l) > r_{d_l}$  for every  $f \in T^*$  and  $l > k_j$  with  $l \in I_j$ ,

where  $\bar{a}_{k_j} = c_{k_0}^* \wedge \dots \wedge c_{k_j}^*$ . Then the sequence  $\langle \bar{a}_{k_j} : j < \omega \rangle$  is a strictly decreasing sequence satisfying (ii) and it is now easy to see that by putting  $a_j = \bar{a}_{k_j} \wedge \bar{a}_{k_{j+1}}^c$  we obtain the desired sequence. ■

**PROOF OF THEOREM 1.** Algebra  $B$  we construct will be a subalgebra of the algebra  $\mathcal{P}(\omega)$  of all subsets of  $\omega$ . First, let  $\mathcal{K} \subset \omega^\omega$  be a dominating family with  $|\mathcal{K}| = \omega_1$  and fix a partition  $\{D_k : k < \omega\}$  of  $\omega$  into infinite subsets.

For every sequence  $\bar{a} = \langle a_n : n < \omega \rangle$  of pairwise disjoint subsets of  $\omega$  and  $k < \omega$  put  $a_k^* = \bigcup \{a_n : n \in D_k\}$ . In addition, for every  $h \in \mathcal{K}$  we put

$$a^h = \bigcup \{a_{n_h(k)} : k < \omega\}$$

where  $n_h(k) = \min\{n \in D_k : n > \max\{h(k), k\}\}$ . We also put

$$F(\bar{a}) = \{a_k^* : k < \omega\} \cup \{a^h : h \in \mathcal{K}\} \in [\mathcal{P}(\omega)]^{\leq \omega_1}.$$

Next, we will construct an increasing sequence  $\langle B_\xi \in [\mathcal{P}(\omega)]^{\omega_1} : \xi \leq \omega_1 \rangle$  of subalgebras of  $\mathcal{P}(\omega)$  aiming for  $B = B_{\omega_1}$ . Thus, we choose  $B_0$  as an arbitrary subalgebra of  $\mathcal{P}(\omega)$  with  $|B_0| = \omega_1$  and for limit ordinal numbers  $\lambda \leq \omega_1$  we put  $B_\lambda = \bigcup_{\xi < \lambda} B_\xi$ . The algebra  $B_{\xi+1}$  is formed from  $B_\xi$  in the following way.

Let  $\{b_\eta : \eta < \omega_1\}$  be an enumeration of  $B_\xi$  and for  $\eta < \omega$  let  $\mathcal{A}_\eta^\xi$  be a subalgebra of  $B_\xi$  generated by  $\{b_\zeta : \zeta < \eta\}$ . For each such algebra we apply Lemma 3 to find the sequences  $\bar{a}^\gamma = \langle a_n^\gamma : n < \omega \rangle$ ,  $\gamma < \omega_1$ , satisfying (i) and (ii) and let

$$G(\mathcal{A}_\eta^\xi) = \bigcup_{\gamma < \omega_1} F(\bar{a}^\gamma).$$

$B_{\xi+1}$  is defined as the algebra generated by  $B_\xi \cup \bigcup_{\eta < \omega_1} G(\mathcal{A}_\eta^\xi)$ . This finishes the construction of  $B$ .

Clearly,  $|B| = \omega_1$ . To prove that  $\text{cof}(B) = \omega_1$  it is enough to show that  $B$  is not a union of an increasing sequence  $\bar{\mathcal{B}} = \langle B_n : n < \omega \rangle$  of proper subalgebras. So, by way of contradiction, assume that such a sequence  $\bar{\mathcal{B}}$  exists. For every  $n < \omega$  choose  $b_n \in B \setminus B_n$  and find  $\xi, \eta < \omega_1$  such that  $\{b_n : n < \omega\} \subset \mathcal{A}_\eta^\xi$ . Then the algebras  $\mathcal{A}_n = B_n \cap \mathcal{A}_\eta^\xi$  form an increasing sequence of proper subalgebras of  $\mathcal{A} = \mathcal{A}_\eta^\xi$ . Thus, one of the sequences  $\bar{a}^\gamma$  satisfies (ii) for  $\bar{\mathcal{A}}$ . So, if we put  $\bar{a}^\gamma = \bar{a} = \langle a_n : n < \omega \rangle$  we conclude that  $\{a_k^* : k < \omega\} \cup \{a^h : h \in \mathcal{K}\} \subset B$ . Let  $f(k) = \min\{n < \omega : a_k^* \in B_n\}$  and let  $h \in \mathcal{K}$  be such that  $f(k) < h(k)$  for all  $k < \omega$ .

The final contradiction is obtained by noticing that  $a^h$  cannot belong to any  $B_k$ . Indeed, if  $a^h \in B_k$  for some  $k$ , then  $a^h \cap a_k^* = a_{n_h(k)}$  belongs to  $B_{\max\{f(k), k\}}$ , since  $a_k^* \in B_{f(k)}$ . But  $\max\{f(k), k\} \leq \max\{h(k), k\} < n_h(k)$  so we get  $a_{n_h(k)} \in B_{n_h(k)}$  contradicting the fact that  $a_{n_h(k)}$  belongs to  $\bar{\mathcal{A}} \setminus \mathcal{A}_{n_h(k)}$ , which is disjoint with  $B_{n_h(k)}$ . ■

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