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SMALL COMBINATORIAL CARDINAL CHARACTERISTICS AND THEOREMS OF EGOROV AND BLUMBERG

Abstract

We will show that the following set theoretical assumption

$\mathfrak{c} = \omega_2$, the dominating number \mathfrak{d} equals to ω_1 , and there exists an ω_1 -generated Ramsey ultrafilter on ω

(which is consistent with ZFC) implies that for an arbitrary sequence $f_n: \mathbb{R} \rightarrow \mathbb{R}$ of uniformly bounded functions there is a set $P \subset \mathbb{R}$ of cardinality continuum and an infinite $W \subset \omega$ such that $\{f_n \upharpoonright P: n \in W\}$ is a monotone uniformly convergent sequence of uniformly continuous functions. Moreover, if functions f_n are measurable or have the Baire property then P can be chosen as a perfect set.

We will also show that $\text{cof}(\mathcal{N}) = \omega_1$ implies existence of a magic set and of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright D$ is discontinuous for every $D \notin \mathcal{N} \cap \mathcal{M}$.

Our set theoretic terminology is standard and follows that of [8]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. We are using

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symbols \mathcal{N} and \mathcal{M} to denote the ideals of Lebesgue measure zero and meager subsets of \mathbb{R} , respectively. For the ideal $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ its *cofinality* is defined by $\text{cof}(\mathcal{I}) = \min\{|\mathcal{B}|: \mathcal{B} \subset \mathcal{I} \text{ generates } \mathcal{I}\}$. A set $L \subset \mathbb{R}$ is a κ -Luzin set if $|L| = \kappa$ but $|L \cap N| < \kappa$ for every nowhere dense subset N of \mathbb{R} . Recall that Martin's Axiom, MA, implies the existence of a \mathfrak{c} -Luzin set. The *dominating number* is defined as

$$\mathfrak{d} = \min\{|T|: T \subset \omega^\omega \ \& \ (\forall f \in \omega^\omega)(\exists g \in T)(\forall n < \omega) f(n) < g(n)\}.$$

It is well known that $\omega_1 \leq \mathfrak{d} \leq \text{cof}(\mathcal{N})$. (See e.g. [1].) In this paper we use term *Polish space* for a complete separable metric space without isolated points.

1 On a Convergence of Subsequences

This section can be viewed as an extension of the discussion around Egorov's theorem presented in [12, Ch. 9]. In 1932 Mazurkiewicz [13] proved the following variant of Egorov's theorem, where a sequence $\langle f_n \rangle_{n < \omega}$ of real-valued functions is *uniformly bounded* provided there exists an $r \in \mathbb{R}$ such that $\text{range}(f_n) \subset [-r, r]$ for every n .

Mazurkiewicz's Theorem *Every uniformly bounded sequence $\langle f_n \rangle_{n < \omega}$ of real-valued continuous functions defined on a Polish space X has a subsequence which is uniformly convergent on some perfect set P .*

Of course Mazurkiewicz' theorem cannot be proved if we do not assume some regularity of the functions f_n even if $X = \mathbb{R}$. But is it at least true that

(*) for every uniformly bounded sequence $\langle f_n: \mathbb{R} \rightarrow \mathbb{R} \rangle_{n < \omega}$ the conclusion of Mazurkiewicz' theorem holds for some $P \subset \mathbb{R}$ of cardinality \mathfrak{c} ?

The consistency of the negative answer follows from the next example, which is essentially due to Sierpiński [16].¹ (See [12, pp. 193-194], where it is proved under the assumption of the existence of ω_1 -Luzin set. The same proof works also for our more general statement.)

Example 1. *Assume that there exists a κ -Luzin set. Then for every Polish space X there exists a sequence $\langle f_n: X \rightarrow \{0, 1\} \rangle_{n < \omega}$ with the property that for every $W \in [\omega]^\omega$ the subsequence $\langle f_n \rangle_{n \in W}$ converges pointwise for less than κ -many points $x \in X$.*

In particular, under Martin's Axiom the above sequence exists for $\kappa = \mathfrak{c}$.

¹Sierpiński constructed this example under the assumption of the Continuum Hypothesis.

Note also that under MA the above example can hold only for $\kappa = \mathfrak{c}$, since MA implies that

for every set S of cardinality less than \mathfrak{c} every uniformly bounded sequence $\langle f_n : S \rightarrow \mathbb{R} \rangle_{n < \omega}$ has a pointwise convergent subsequence.

(See [12, p. 195].) Sharper results concerning the above two facts were recently obtained by Fuchino and Plewik [11], in which they relate them to the splitting number \mathfrak{s} . (For the definition of \mathfrak{s} see e.g. [1]. For us it is only important that $\omega_1 \leq \mathfrak{s} \leq \mathfrak{d}$.) More precisely, the authors show there that: *For any $X \subset [\mathbb{R}]^{<\mathfrak{s}}$ any sequence $\langle f_n : X \rightarrow [-\infty, \infty] \rangle_{n < \omega}$ has a subsequence convergent pointwise on X ; however for any $X \subset [\mathbb{R}]^{\mathfrak{s}}$ there exists a sequence $\langle f_n : X \rightarrow [0, 1] \rangle_{n < \omega}$ with no pointwise convergent subsequence.*

Our main goal of this section is to prove that (*) is consistent with (so, by the example, also independent from) the usual axioms of set theory ZFC. To state this precisely we need the following terminology and facts.

A maximal non-principal filter \mathcal{F} on ω is said to be *Ramsey* provided for every $B \in \mathcal{F}$ and $h : [B]^2 \rightarrow \{0, 1\}$ there exist $i < 2$ and $A \in \mathcal{F}$ such that $A \subset B$ and $h[[A]^2] = \{i\}$. We say that a family $\mathcal{W} \subset \mathcal{F}$ *generates* filter \mathcal{F} provided for every $F \in \mathcal{F}$ there exists a $W \in \mathcal{W}$ such that $W \subset F$.

Theorem 2. *Assume that $\mathfrak{d} = \omega_1$ and there exists a Ramsey ultrafilter \mathcal{F} on ω generated by a family $\mathcal{W} \subset \mathcal{F}$ of cardinality ω_1 .*

Let X be an arbitrary set and $\langle f_n : X \rightarrow \mathbb{R} \rangle_{n < \omega}$ be a sequence of functions such that the set $\{f_n(x) : n < \omega\}$ is bounded for every $x \in X$. Then there are sequences: $\langle P_\xi : \xi < \omega_1 \rangle$ of subsets of X and $\langle W_\xi \in \mathcal{F} : \xi < \omega_1 \rangle$ such that $X = \bigcup_{\xi < \omega_1} P_\xi$ and for every $\xi < \omega_1$:

the sequence $\langle f_n \upharpoonright P_\xi \rangle_{n \in W_\xi}$ is monotone and uniformly convergent.

The conclusion of Theorem 2 is obvious for sets X with cardinality $\leq \omega_1$, since sets P_ξ can be chosen just as singletons. Thus, we will be interested in the theorem only for the sets X of cardinality greater than ω_1 . If X is a Polish space this leads to $\mathfrak{c} = |X| > \omega_1$. Luckily, the assumptions of Theorem 2 are consistent with ZFC+“ $\mathfrak{c} = \omega_2$ ”. This holds in the iterated perfect set model. More precisely, the fact that in this model we have $\mathfrak{c} = \omega_2$ and $\text{cof}(\mathcal{N}) = \omega_1$ can be found in [1, p. 339]. The fact that in this model there exists a desired Ramsey ultrafilter has been proved in Baumgartner, Laver [2]. (They proved there that there exists a selective ω_1 -generated ultrafilter on ω . But it is well known that an ultrafilter on ω is selective if and only if it is Ramsey.) All these facts follow also from the axiom CPA, which is a subject of a forthcoming monograph [9]. (Some of the results proved here may also be included in [9] as the examples of interesting consequences of CPA.)

In particular, we get the following corollary which, under additional set theoretical assumptions, generalizes Mazurkiewicz' theorem and implies (*).

Corollary 3. *It is consistent with ZFC+“ $\mathfrak{c} = \omega_2$ ” that for each Polish space X and each uniformly bounded sequence $\langle f_n: X \rightarrow \mathbb{R} \rangle_{n < \omega}$ there exist sequences: $\langle P_\xi: \xi < \omega_1 \rangle$ of subsets of X and $\langle W_\xi \in [\omega]^\omega: \xi < \omega_1 \rangle$ such that $X = \bigcup_{\xi < \omega_1} P_\xi$ and for every $\xi < \omega_1$:*

the sequence $\langle f_n \upharpoonright P_\xi \rangle_{n \in W_\xi}$ is monotone and uniformly convergent.

In particular, there exists a $\xi < \omega_1$ such that $|P_\xi| = \mathfrak{c}$.

Moreover, if functions f_n are continuous then we can additionally require that all sets P_ξ are closed in X .

PROOF. The main part follows immediately from the discussion above and the Pigeon Hole Principle. To see the additional part it is enough to note that for continuous functions sets P_ξ can be replaced by their closures, since for any sequence $\langle f_n: P \rightarrow \mathbb{R} \rangle_{n < \omega}$ of continuous functions if $\langle f_n \upharpoonright D \rangle_{n < \omega}$ is monotone and uniformly convergent for some dense subset D of P then so is $\langle f_n \rangle_{n < \omega}$. \square

PROOF OF THEOREM 2. For every $x \in X$ define $h_x: [\omega]^2 \rightarrow \{0, 1\}$ by putting for every $n < m < \omega$

$$h_x(n, m) = 1 \text{ if and only if } f_n(x) \leq f_m(x).$$

Since \mathcal{F} is Ramsey and \mathcal{W} generates \mathcal{F} we can find a $W_x \in \mathcal{W}$ and an $i_x < 2$ such that $h_x[[W_x]^2] = \{i_x\}$. Thus, the sequence $S_x = \langle f_n(x) \rangle_{n \in W_x}$ is monotone. It is increasing when $i_x = 1$ and it is decreasing for $i_x = 0$.

For $W \in \mathcal{W}$ and $i < 2$ let $P_W^i = \{x \in X: W_x = W \ \& \ i_x = i\}$. Then $\{P_W^i: W \in \mathcal{W} \ \& \ i < 2\}$ is a partition of X and for every $W \in \mathcal{W}$ and $i < 2$ the sequence $\langle f_n \upharpoonright P_W^i \rangle_{n \in W}$ is monotone and pointwise convergent to some function $f: P_W^i \rightarrow \mathbb{R}$.

To get uniform convergence note that for every $x \in P_W^i$ there exists an $s_x \in \omega^\omega$ such that

$$(\forall k < \omega) (\forall n \in W \setminus s_x(k)) |f_n(x) - f(x)| < 2^{-k}.$$

Since $\mathfrak{d} = \omega_1$, there exists a $T \in [\omega^\omega]^{\omega_1}$ dominating ω^ω . In particular, for every $x \in P_W^i$ there exists a $t_x \in T$ such that $s_x(n) \leq t_x(n)$ for all $n < \omega$. For $t \in T$ let

$$P_W^i(t) = \{x \in P_W^i: t_x = t\}.$$

Since every sequence $\langle f_k \upharpoonright P_W^i(t) \rangle_{k \in W}$ is monotone and uniformly convergent, $\{P_W^i(t): i < 2, W \in \mathcal{W}, t \in T\}$ is the desired covering $\{P_\xi: \xi < \omega_1\}$ of X . \square

2 $\text{cof}(\mathcal{N}) = \omega_1$, **Blumberg Theorem, and Magic Set**

In this section we will show two consequences of $\text{cof}(\mathcal{N}) = \omega_1$.

In 1922 Blumberg [4] proved that for every $f: \mathbb{R} \rightarrow \mathbb{R}$ there exists a dense subset D of \mathbb{R} such that $f \upharpoonright D$ is continuous. This theorem sparked a lot of discussion and generalizations, see e.g. [7, pp. 147–150]. In particular, Shelah [15] showed that there is a model of ZFC in which for every $f: \mathbb{R} \rightarrow \mathbb{R}$ there is a nowhere meager subset D of \mathbb{R} such that $f \upharpoonright D$ is continuous. The dual measure result, that is the consistency of a statement for every $f: \mathbb{R} \rightarrow \mathbb{R}$ there is a subset D of \mathbb{R} of positive outer Lebesgue measure such that $f \upharpoonright D$ is continuous, has been also recently established by Rosłanowski and Shelah [14]. Below we note that each of these properties contradicts $\text{cof}(\mathcal{N}) = \omega_1$. (We use here the well known inequality $\text{cof}(\mathcal{M}) \leq \text{cof}(\mathcal{N})$. See e.g. [1].)

Theorem 4. *Let $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$. If $\text{cof}(\mathcal{I}) = \omega_1$ then there exists an $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright D$ is discontinuous for every $D \in \mathcal{P}(\mathbb{R}) \setminus \mathcal{I}$.*

PROOF. We will assume that $\mathcal{I} = \mathcal{N}$, the proof for $\mathcal{I} = \mathcal{M}$ being essentially identical.

Let $\{N_\xi \subset \mathbb{R}^2: \xi < \omega_1\}$ be a family cofinal in the ideal of null subsets of \mathbb{R}^2 and for each $\xi < \omega_1$ let

$$N_\xi^* = \{x \in \mathbb{R}: (N_\xi)_x \notin \mathcal{N}\},$$

where $(N_\xi)_x = \{y \in \mathbb{R}: \langle x, y \rangle \in N_\xi\}$. By Fubini’s theorem each N_ξ^* is null. For each $x \in N_\xi^* \setminus \bigcup_{\zeta < \xi} N_\zeta^*$ we choose $f(x)$ so that

$$f(x) \notin \bigcup_{\zeta < \xi} (N_\zeta)_x.$$

Then function f is as desired.

Indeed, if $f \upharpoonright D$ is continuous for some $D \subset \mathbb{R}$ then $f \upharpoonright D$ is null in \mathbb{R}^2 . In particular, there exists a $\xi < \omega_1$ such that $f \upharpoonright D \subset N_\xi$. But this means that $D \subset \bigcup_{\zeta \leq \xi} N_\zeta^*$. □

Note that essentially the same proof works if we assume only that $\text{cof}(\mathcal{I})$ is equal to the additivity number $\text{add}(\mathcal{I})$ of \mathcal{I} .

Corollary 5. *Assume $\text{cof}(\mathcal{N}) = \omega_1$. Then there exists an $f: \mathbb{R} \rightarrow \mathbb{R}$ such that if $f \upharpoonright D$ is continuous then $D \in \mathcal{N} \cap \mathcal{M}$.*

PROOF. Let $f_{\mathcal{N}}$ and $f_{\mathcal{M}}$ be from Theorem 4 constructed for the ideals \mathcal{N} and \mathcal{M} , respectively. Let $G \subset \mathbb{R}$ be a dense G_δ of measure zero and put $f = [f_{\mathcal{M}} \upharpoonright G] \cup [f_{\mathcal{N}} \upharpoonright (\mathbb{R} \setminus G)]$. Then this f is as desired. □

Recall that a set $M \subset \mathbb{R}$ is a *magic set* (or *set of range uniqueness*) if for every different nowhere constant functions $f, g \in C(\mathbb{R})$ we have $f[M] \neq g[M]$. It has been proved by Berarducci and Dikranjan [3, thm. 8.5] that a magic set exists under CH. We like to note here that the same is implied by a much weaker assumption that $\text{cof}(\mathcal{M}) = \omega_1$. However, the existence of a magic set is independent of ZFC, as proved by Ciesielski and Shelah in [10].

Proposition 6. *If $\text{cof}(\mathcal{M}) = \omega_1$ then there exists a magic set.*

PROOF. An uncountable set $L \subset \mathbb{R}$ is a *2-Luzin set* provided for every disjoint subsets $\{x_\xi: \xi < \omega_1\}$ and $\{y_\xi: \xi < \omega_1\}$ of L , where the enumerations are one-to-one, the set of pairs $\{(x_\xi, y_\xi): \xi < \omega_1\}$ is not a meager subset of \mathbb{R}^2 . In [5, prop. 4.8] it was noticed that every ω_1 -dense 2-Luzin set is a magic set. It is also a standard and easy diagonal argument that $\text{cof}(\mathcal{M}) = \omega_1$ implies the existence of a ω_1 -dense 2-Luzin set. (The proof presented in [17, prop. 6.0] works also under the assumption $\text{cof}(\mathcal{M}) = \omega_1$.) So, $\text{cof}(\mathcal{M}) = \omega_1$ implies that there is a magic set. \square

Recall also that the existence of a magic set for the class D^1 of all differentiable functions can be proved in ZFC. This follows from [6, thm. 3.1], since every function from D^1 belongs to the class (T_2) . (Compare also [6, cor. 3.3 and 3.4].)

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²Preprints marked by * are available in electronic form from *Set Theoretic Analysis Web Page*: <http://www.math.wvu.edu/~kcies/STA/STA.html>

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