“Generalized continuities”

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There are many different properties of functions that can be considered as generalized continuity notions. In this article we will describe mainly those that arose from the study of real functions, concentrating first on the classes of functions known under the common name of Darboux-like functions. (See e.g. survey articles [13, 14, 6, 3].)

For topological spaces X and Y, a function $f\colon X\to Y$ is **Darboux**, $f\in \mathrm{D}(X,Y)$, provided the image $f[C]$ of C under f is a *connected* subset of Y for every connected subset C of X. In particular, $f\colon \mathbb{R}\to\mathbb{R}$ is Darboux provided f maps intervals onto intervals, that is, when it has the *intermediate value property*. The name comes after G. Darboux who showed in 1875 that every derivative (of a function from $\mathbb{R}$ to $\mathbb{R}$) has the intermediate value property, while there are derivatives discontinuous on a *dense set*. (Some 19th century mathematicians thought that the intermediate value property could be taken as the definition of continuity. Some calculus teachers still think so.) One of the easiest examples of a discontinuous Darboux function is $f_0\colon\mathbb{R}\to\mathbb{R}$ given by $f_0(x)=\sin(1/x)$ for $x\neq 0$ and $f_0(0)=0$.

A function $f\colon X\to Y$ is **connectivity**, $f\in\mathrm{Conn}(X,Y)$, if the graph of the restriction $f\restriction Z$ of f to Z is connected in $X\times Y$ for every connected subset Z of X. It is easy to see that $f\colon\mathbb{R}\to\mathbb{R}$ is connectivity if and only if its graph is a connected subset of $\mathbb{R}^2$. However, there are functions $F\colon\mathbb{R}^2\to\mathbb{R}$ with a connected graph which are not connectivity functions. For example, this is the case if $F(x,y)=\sin(1/x)$ for $x\neq 0$, and $F(0,y)=h(y)$, where $h\colon\mathbb{R}\to[-1,1]$ is any function with a disconnected graph.

A function $f\colon X\to Y$ is **extendable**, $f\in\mathrm{Ext}(X,Y)$, provided there exists a connectivity function $F\colon X\times[0,1]\to Y$ such that $f(x)=F(x,0)$ for every $x\in X$. It is easy to see that

$$\mathrm{C}(X,Y)\subset\mathrm{Ext}(X,Y)\subset\mathrm{Conn}(X,Y)\subset\mathrm{D}(X,Y)$$

for arbitrary topological spaces, where $\mathrm{C}(X,Y)$ stands for the class of all *continuous functions* from X into Y.

A function $f\colon X\to Y$ is **almost continuous** (in the sense of Stallings), $f\in\mathrm{AC}(X,Y)$, provided each open subset of $X\times Y$ containing the graph of f also contains the graph of a continuous function from X to Y. This property was defined as a generalization of functions having the *fixed point property.* It is easy to see that if every function in $\mathrm{C}(X,X)$ has the fixed point property, then so does every $f\in\mathrm{AC}(X,X)$.

A function $f\colon X\to Y$ is **peripherally continuous**, $f\in\mathrm{PC}(X,Y)$, if for every $x\in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x\in W$ and $f[\mathrm{bd}(W)]\subset V$, where $\mathrm{bd}(W)$ is the *boundary* of W. For the functions $f\colon\mathbb{R}\to\mathbb{R}$ this means that f has the **Young property**, that is, for every $x\in\mathbb{R}$ there exist sequences $\{x_n\}_n$ and $\{y_n\}_n$ such that $x_n\nearrow x$, $y_n\searrow x$, and both $f(x_n)$ and $f(y_n)$ converge to $f(x)$. In 1907 J. Young showed that for the *Baire class 1* functions, the Darboux property and the Young property are equivalent.

We will discuss the above mentioned classes only when $Y = \mathbb{R}$. If $X = \mathbb{R}^n$ and $n > 1$ the relations between these classes are given by the following chart, where arrows $\longrightarrow$ denote strict inclusions. (See [6].)

$$\mathrm{C} \longrightarrow \mathrm{Ext} = \mathrm{Conn} = \mathrm{PC} \longrightarrow \mathrm{AC} \cap \mathrm{D} \begin{array}{l} \nearrow \mathrm{AC} \\ \searrow \mathrm{D} \end{array}$$

Chart 1: Darboux-like functions from $\mathbb{R}^n$, $n > 1$, into $\mathbb{R}$.

The inclusion Conn $\subset$ Ext was proved by K. Ciesielski, T. Natkaniec, and J. Wojciechowski [8]. The containment Conn $\subset$ PC was proved by O.H. Hamilton and J. Stallings, and the inclusion PC $\subset$ Conn by M.R. Hagan. The relation Conn $\subset$ AC was proved by J. Stallings. It is important to notice that Chart 1 remains unchanged if we consider only Baire class 1 functions [6].

Classes presented in Chart 1 were also studied within the class $\mathrm{Add}(\mathbb{R}^n)$ of **additive functions**, that is, functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$. In this case Chart 1 transforms to

$$\mathrm{C} = \mathrm{Ext} = \mathrm{Conn} = \mathrm{PC} = \mathrm{AC} \cap \mathrm{D} \begin{array}{l} \nearrow \mathrm{AC} \\ \searrow \mathrm{D} \end{array}$$

Chart 2: Additive Darboux-like functions from $\mathbb{R}^n$, $n > 1$, into $\mathbb{R}$.

The inclusion AC $\cap$ D $\subset$ C was proved by K. Ciesielski and J. Jastrzębski [6].

The Darboux-like functions were most intensively studied when $X = Y = \mathbb{R}$. In this setting, more classes are considered Darboux-like. Thus, a function $f\colon \mathbb{R} \to \mathbb{R}$ has: the **Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ and for each *perfect set* K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$; the **strong Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ and for each perfect set K between $f(x)$ and $f(y)$ there is a perfect set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous; the **weak Cantor intermediate value property** if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set C between x and y such that $f[C] \subset (f(x), f(y))$; the **perfect road** if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral (i.e., two sided) limit point for which $f \restriction P$ is continuous at x. The classes of these functions are denoted by CIVP, SCIVP, WCIVP, and PR, respectively. The relations between them are as follows [13, 6].

$$\begin{array}{ccccccccc} & & & \nearrow & \mathrm{AC} & \longrightarrow & \mathrm{Conn} & \longrightarrow & \mathrm{D} \searrow \\ \mathrm{C} & \longrightarrow & \mathrm{Ext} & & & & & & \quad \mathrm{PC} \\ & & & \searrow & \mathrm{SCIVP} & \longrightarrow & \mathrm{CIVP} & \longrightarrow & \mathrm{PR} \nearrow \\ & & & & & & & \searrow & \mathrm{WCIVP} \end{array}$$

Chart 3: Darboux-like functions from $\mathbb{R}$ into $\mathbb{R}$.

The inclusions Ext $\subset$ AC $\subset$ Conn were proved by J. Stallings while the containment Ext $\subset$ SCIVP was proved by H. Rosen, R.G. Gibson, and F. Roush.

The main interest in Darboux-like functions comes from the fact that the class Δ' of the derivatives from $\mathbb{R}$ into $\mathbb{R}$ is contained in all these classes, that is, $\mathrm{C} \subsetneq \Delta' \subsetneq \mathrm{Ext}$. This follows from the fact that every derivative is Darboux Baire class 1 (see e.g. [2]) while within the Baire class 1 Chart 3 reduces to

$$\mathrm{C} \longrightarrow \mathrm{Ext} = \mathrm{AC} = \mathrm{Conn} = \mathrm{D} = \mathrm{PC} = \mathrm{SCIVP} = \mathrm{CIVP} = \mathrm{PR} \longrightarrow \mathrm{WCIVP}.$$

The proof that every peripherally continuous Baire class 1 function $f\colon \mathbb{R} \to \mathbb{R}$ is extendable is due to J. Brown, P. Humke, and M. Laczkovich [1]. In fact, most of the properties used to define Darboux-like functions were introduced as characterizations of Darboux functions within the Baire class 1, in a form "a Baire class 1 function f is *Darboux* if and only if f satisfies the *given property*." But these properties make sense without the Baire class 1 restriction, so it was natural to study these various conditions on their own, and to find the interrelations. A number of mathematicians did just that in the latter part of the 20th century.

It is interesting that within the Baire class 2 Chart 3 has yet another, quite different form [6]:

$$\mathrm{C} \longrightarrow \mathrm{Ext} \longrightarrow \mathrm{AC} \longrightarrow \mathrm{Conn} \longrightarrow \mathrm{D} \longrightarrow \mathrm{SCIVP} = \mathrm{CIVP} \begin{array}{l} \nearrow \ \mathrm{PR} \longrightarrow \mathrm{PC} \\ \searrow \ \mathrm{WCIVP} \end{array}$$

Chart 4: Darboux-like Baire class 2 functions from $\mathbb{R}$ into $\mathbb{R}$.

The most involved work in arguing for this chart is the nonreversability of the inclusions. (See [6, thm. 1.2].) Chart 4 remains unchanged if we restrict Darboux-like functions to *Borel functions* in place of Baire class 2. Within the class of additive functions Chart 3 remains almost unchanged: the only difference is that in this case we have PR = WCIVP and that the example of additive function $f\colon \mathbb{R} \to \mathbb{R}$ from Conn $\setminus$ AC (which is also CIVP) is known only under an extra set theoretical assumption that the union of less than continuum many meager subsets of $\mathbb{R}$ is meager in $\mathbb{R}$. (A subset of a topological space X is **meager**, or of the first category, if it is a countable union of *nowhere dense* subsets of X.)

The Darboux-like classes of functions are not closed under arithmetic operations. (See e.g. surveys [13, 3].) For example, if f_0 is the $\sin(1/x)$ function defined above and $f_1 = f_0 + \chi_{\{0\}}$, where χ_A is a *characteristic function* of A, then both f_0 and f_1 are Darboux Baire class 1, so they are also extendable. However, $f_1 - f_0 = \chi_{\{0\}}$ is clearly not even in PC. In fact, in 1927 A. Lindenbaum noticed that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as a sum of two Darboux functions, while H. Fast in 1959 proved that for every family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ of cardinality continuum there is just one Darboux function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g+F \stackrel{\text{def}}{=} \{g+f\colon f \in \mathcal{F}\}$ is a subset of D. The result of H. Fast is a generalization of that of A. Lindenbaum, since it is easy to see that $\mathbb{R}^{\mathbb{R}} = \mathcal{F} - \mathcal{F}$ if and only if for every $f, f' \in \mathbb{R}^{\mathbb{R}}$ there exists a $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + f, g + f' \in \mathcal{F}$. (See [3, prop. 4.9].) This led T. Natkaniec [15] to study the following cardinal operator defined for every $\mathcal{F} \subset \mathbb{R}^X$, where $|X|$ stands for the cardinality of X:

$$\mathrm{A}(\mathcal{F}) = \min \left\{|H| \colon H \subset \mathbb{R}^X \ \& \ \neg\exists g \in \mathbb{R}^X \ g + H \subset \mathcal{F}\right\} \cup \left\{\left|\mathbb{R}^X\right|^+\right\}.$$

The values of the operator A for Darboux-like classes of functions from $\mathbb{R}$ to $\mathbb{R}$ are as follows (see e.g. [3, thms. 4.7 & 4.10]):

$$\mathfrak{c}^+ = \mathrm{A}(\mathrm{PR}) = \mathrm{A}(\mathrm{Ext}) \leq \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{D}) \leq \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}},$$

where the value of $\mathrm{A}(\mathrm{D})$ between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$ can vary in different models of ZFC. Moreover, the monotonicity of the operator A implies that $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{SCIVP}) = \mathrm{A}(\mathrm{CIVP}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$ and $\mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathrm{D})$.

The above discussion shows that unlike the derivatives, classes of Darboux-like functions are not closed under addition. It is also not difficult to see that none of these classes (including Δ', see [2, p. 14]) is closed under multiplication. Closure under composition gives a completely different picture. First of all, the derivatives are not closed under composition: by a theorem of I. Maximoff (see e.g. [2, p. 26]), for every Darboux Baire class 1 function $g \colon \mathbb{R} \to \mathbb{R}$ (which does not need to be a derivative) there exists a homeomorphism h of $\mathbb{R}$ such that $f = g \circ h$ is a derivative; so, the composition $g = f \circ h^{-1}$ does not need to be a derivative. It is obvious from the definition that the class D of Darboux functions is closed under composition, and clearly so is C. The other classes from Chart 3, except for Ext, are not closed under composition. The problem of closure of Ext under composition remains open [14, Q. 9.1]. In fact, it is even not known whether the composition of two derivatives must be in Conn. A partial positive result in this direction was recently obtained by M. Csörnyei, T. C. O'Neil, D. Preiss [10] and, independently, by M. Elekes, T. Keleti, V. Prokaj [11], who proved that the composition of two derivatives from $[0, 1]$ into $[0, 1]$ must have a fixed point. (So, we cannot exclude the possibility that $\Delta' \circ \Delta' \subset \mathrm{AC}$.)

The main reason for the studies of classes of functions related to derivatives comes from the fact that the class Δ' of all derivatives does not have any known nice characterization. (See [2].) One of the recent attempts of finding a characterization was to **topologize** it, that is to find two *topologies* τ_0 and τ_1 on $\mathbb{R}$ for which Δ' is equal to the class $\mathrm{C}(\tau_0, \tau_1)$ of all continuous functions from $\langle\mathbb{R}, \tau_0\rangle$ into $\langle\mathbb{R}, \tau_1\rangle$. Unfortunately, Δ' cannot be topologized, as shown by K. Ciesielski and, independently, by M. Tartaglia. (See [3, cor. 5.5].) However K. Ciesielski [4] proved that it can be characterized by preimages of sets in the sense that there exist families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{R}$ with the property that $\Delta' = \{f \in \mathbb{R}^{\mathbb{R}} \colon f^{-1}(B) \in \mathcal{A} \text{ for every } B \in \mathcal{B}\}$. It is interesting, that if the *generalized continuum hypothesis* holds then many classes of functions can be topologized [3, sec. 5]. In particular, this is the case for any class $\mathcal{F}$ of functions containing all constant functions such that $\mathcal{F}$ is contained either in the class of analytic functions from $\mathbb{R}$ to $\mathbb{R}$ or in the class of harmonic functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.

The class Δ' is also closely related to the class Appr of **approximately continuous functions**, which was introduced by A. Denjoy in 1915. (See e.g. [2, 7].) Recall that $f\colon \mathbb{R}\to\mathbb{R}$ is in Appr provided for every $x_0\in\mathbb{R}$ the **approximate limit** $\mathrm{applim}_{x\to x_0} f(x)$ equals to $f(x_0)$, where $\mathrm{applim}_{x\to x_0} f(x)=L$ if there exists a set $S\subset\mathbb{R}$ such that x_0 is a (Lebesgue) **density point** of S (that is, $\lim_{h\to 0^+}\frac{\lambda(S\cap[x_0-h,x_0+h])}{2h}=1$, with $\lambda(A)$ standing for the *inner Lebesgue measure* of A) and $\lim_{x\to x_0,\, x\in S} f(x)=L$. The interest in Appr comes from the fact that every bounded approximately continuous functions is a derivative. Also, each function in Appr is Darboux Baire class 1, so it belongs to every class of Darboux-like functions. It was not until 1952 that O. Haupt and C. Pauc defined the **density topology** $\tau_{\mathcal{N}}$ on $\mathbb{R}$ (which is the family of all $G\subset\mathbb{R}$ such that every $x\in G$ is a density point of G) and showed that Appr is equal to the class $\mathrm{C}(\tau_{\mathcal{N}},\tau_O)$ of all functions continuous with respect to the density topology $\tau_{\mathcal{N}}$ on the domain and the ordinary topology τ_O on the range. Their paper seemed to have had almost no impact and the same results were rediscovered in 1961 by C. Goffman and D. Waterman. (See [7, sec. 1.5].) This led to deep studies of the density topology, as well as to its category analog $\tau_{\mathcal{I}}$, known under the name of **$\mathcal{I}$-density topology**. Extensive research have been also conducted on the classes $\mathrm{C}(\tau_{\mathcal{I}},\tau_O)$ of **$\mathcal{I}$-approximately continuous functions**, $\mathrm{C}(\tau_{\mathcal{N}},\tau_{\mathcal{N}})$ of **density continuous functions**, and $\mathrm{C}(\tau_{\mathcal{I}},\tau_{\mathcal{I}})$ of **$\mathcal{I}$-density continuous functions**. (See [7].) Classes $\mathrm{C}(\tau_{\mathcal{N}},\tau_O)$ and $\mathrm{C}(\tau_{\mathcal{I}},\tau_O)$ are closed under addition, while $\mathrm{C}(\tau_{\mathcal{N}},\tau_{\mathcal{N}})$ and $\mathrm{C}(\tau_{\mathcal{I}},\tau_{\mathcal{I}})$ are not.

A function $f\colon\mathbb{R}\to\mathbb{R}$ is **symmetrically continuous**, $f\in\mathrm{SC}$, provided $\lim_{h\to 0}[f(x+h)-f(x-h)]=0$ for every $x\in\mathbb{R}$; it is **approximately symmetrically continuous**, $f\in\mathrm{ApprSC}$, if $\mathrm{applim}_{h\to 0}[f(x+h)-f(x-h)]=0$ for each $x\in\mathbb{R}$. The theory of symmetrically continuous functions stems from the theory of trigonometric series and dates back to the beginning of the 20th century. Rresearch in this area has been very active in the last several years (see [16]) after C. Freiling and D. Rinne in 1988 solved a long standing problem proving that every measurable function $f\colon\mathbb{R}\to\mathbb{R}$ having **approximate symmetric derivative**, $D^s_{ap}f(x)\stackrel{\mathrm{def}}{=}\mathrm{applim}_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$, equal 0 for all $x\in\mathbb{R}$ must be constant almost everywhere. C. Freiling [12] also proved that it is consistent with ZFC that in the above theorem the assumption of a measurability of f can be dropped. However, this cannot be proved in ZFC, as under the *continuum hypothesis*, CH, W. Sierpiński constructed a nonempty proper subset X of $\mathbb{R}$ for which $f=\chi_X$ has approximate symmetric derivative equal 0 for all $x\in\mathbb{R}$. The above classes can be added to Chart 3 as follows.

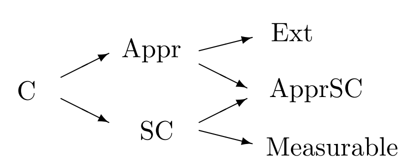

Chart 5: Approximately and symmetrically continuous functions.

The fact that every symmetrically continuous function is measurable follows from a theorem I.N. Pesin and D. Preiss [16, thm. 2.26] asserting that if an f is symmetrically continuous then its set of points of discontinuity is meager and of measure zero. On the other hand, an approximately symmetrically continuous function need not be measurable, as witnessed by the function $f = \chi_X$ mentioned above. It is unknown whether there exists a ZFC example of a non-measurable function in ApprSC.

It is easy to see that all inclusions in Chart 5 are proper. For example, clearly $\chi_{\{0\}} \in \mathrm{SC} \setminus \mathrm{Appr}$. To get $g \in \mathrm{Appr} \setminus \mathrm{SC}$ take $a_0 < b_0 < a_1 < b_1 < \cdots < 0$ such that 0 is an accumulation point of $E = \bigcup_{n<\omega}[a_n, b_n]$ and a density point of $\mathbb{R} \setminus E$. Put $g(x) = 0$ for $x \in \mathbb{R} \setminus E$ and $g(x) = (b_n - a_n)^{-1}\mathrm{dist}(x, \mathbb{R} \setminus [a_n, b_n])$ for $x \in [a_n, b_n]$. Finally $g + \chi_{\{0\}} \in \mathrm{ApprSC} \setminus (\mathrm{Appr} \cup \mathrm{SC})$.

Another interesting notion of generalized continuity is known as **countable continuity**. For topological spaces X, Y and $\mathcal{F} \subset Y^X$, we define **dec**$(\mathcal{F})$ as the smallest infinite cardinal κ such that for every $f \in \mathcal{F}$ there is a family of at most κ-many continuous functions g from a subset of X into Y such f is covered by all these functions g. A function $f\colon X \to Y$ is **κ-continuous** provided $\mathrm{dec}(\{f\}) \le \kappa$; it is **countably continuous** if f is ω-continuous. (See [3, sec. 4].) The study of these notions was initiated by a question of N. Luzin whether every Borel function from $\mathbb{R}$ into $\mathbb{R}$ is countably continuous. This question was answered negatively by P.S. Novikov and generalized by L. Keldyš. In fact we have already $\mathrm{dec}(\mathcal{B}_1) > \omega$, where $\mathcal{B}_1$ is the family of Baire class 1 functions from $\mathbb{R}$ to $\mathbb{R}$. The most general result in this direction was obtained by J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki [3, thm. 4.1] who proved that $\mathrm{cov}(\mathcal{M}) \le \mathrm{dec}(\mathcal{B}_1) \le d$, where $\mathrm{cov}(\mathcal{M})$ is the smallest cardinality of a *covering* of $\mathbb{R}$ by meager sets, and d is the *dominating number*. The consistency of $\mathrm{cov}(\mathcal{M}) < \mathrm{dec}(\mathcal{B}_1)$ and $\mathrm{dec}(\mathcal{B}_1) < d$ was proved by S. Steprāns [3, thm. 4.2], and S. Shelah, S. Steprāns [3, thm. 4.3], respectively. Number dec has been also studied by K. Ciesielski in [5], in which he proved that $\mathrm{cof}(\mathfrak{c}) \le \mathrm{dec}(\mathrm{SC}) = \mathrm{dec}(\mathrm{SZ}) = \mathrm{dec}(\mathbb{R}^{\mathbb{R}}) \le \mathfrak{c}$ and that each of the inequalities can be strict. $\mathrm{cof}(\mathfrak{c})$ stands for the *cofinality* of the continuum $\mathfrak{c}$ and SZ for the class of **Sierpiński-Zygmund functions** $f\colon \mathbb{R} \to \mathbb{R}$, that is, those whose restriction $f \restriction X$ is discontinuous for every $X \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

Finally, one can ask how much continuity an arbitrary function from a topological space X into Y must have. (See e.g. [3, pp. 148-149].) In 1922 H. Blumberg proved that for every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f \restriction D$ is continuous. This result was generalized by several authors to more general topological spaces. However, the most interesting discussion of Blumberg theorem remains in the case of functions from $\mathbb{R}$ to $\mathbb{R}$. Blumberg's set D is countable and in ZFC this is the best that can be proved, since under CH a restriction of a Sierpiński-Zygmund function to any uncountable set is discontinuous. A similar example can be also found in some models of ZFC (e.g. a *Cohen model*) in the absence of CH as noticed by several authors. (See [3, thm. 2.9].) At the same time S. Baldwin [3, thm. 2.8] showed that under *Martin's Axiom* for every function $f\colon \mathbb{R} \to \mathbb{R}$ and every cardinal number $\kappa < \mathfrak{c}$ there exists a set $D \subset \mathbb{R}$ such that $f \restriction D$ is continuous and D is **κ-dense**, that is, $D \cap I$ has cardinality at least κ for every nondegenerated interval I. In the same direction S. Shelah [3, thm. 2.10] showed that it is consistent with ZFC that for every function $f\colon \mathbb{R}\to\mathbb{R}$ there exists a set $D\subset\mathbb{R}$ such that $f\restriction D$ is continuous and D is **nowhere meager**, that is, $D\cap I$ is nonmeager for every nontrivial interval I. Most recently, A. Rosłanowski and S. Shelah (unpublished) also found a model of ZFC in which it is always possible to find the set D of positive *outer measure*, though in this case we cannot require that D is dense in $\mathbb{R}$. (See [3, thm. 2.11].)

Its is easy to find a function $f\colon \mathbb{R}\to\mathbb{R}$ which has no points of continuity—the characteristic function of the set of rational numbers has this property. But what if we ask for points of continuity in weaker sense? For example, a function $f\colon \mathbb{R}\to\mathbb{R}$ is **weakly continuous** at x if it has the Young property at x, that is, if there are sequences $a_n\nearrow 0$ and $b_n\searrow 0$ such that $\lim_{n\to\infty} f(x+a_n)=f(x)=\lim_{n\to\infty} f(x+b_n)$. This notion is so weak that it is impossible to find a function $f\colon \mathbb{R}\to\mathbb{R}$ which is nowhere weakly continuous. More precisely, every $f\colon \mathbb{R}\to\mathbb{R}$ is weakly continuous everywhere on the complement of a countable set. (See [3, thm. 2.16].) A natural symmetric counterpart of weak continuity is defined as follows: a function $f\colon \mathbb{R}\to\mathbb{R}$ is **weakly symmetrically continuous** at x if there is a sequence $h_n\to 0$ such that $\lim_{n\to\infty}(f(x+h_n)-f(x-h_n))=0$. The symmetric version of the theorem mentioned above badly fails: K. Ciesielski and L. Larson [3, thm. 2.17] constructed a nowhere weakly symmetrically continuous functions $f\colon \mathbb{R}\to\{0,1,2,3,\ldots\}$. It is unknown whether a nowhere weakly symmetrically continuous functions $f\colon \mathbb{R}\to\mathbb{R}$ can have finite range [3, prob. 2], though its range must have at least four elements. K. Ciesielski and S. Shelah [9] proved that such an f can have bounded countable range.

For functions from $\mathbb{R}$ to $\mathbb{R}$ many generalized continuities mentioned above can be viewed in the context of **path limit** $P\text{-}\lim_{x\to x_0} f(x)\stackrel{\text{def}}{=}\lim_{x\to x_0,\,x\in P} f(x)$ where x_0 is in the closure of $P\cap(x_0,\infty)$ and of $P\cap(-\infty,x_0)$. Thus, for a continuous function the path P at x_0 must be an interval; for $f\in\mathrm{PR}$ a path must be a perfect set; for $f\in\mathrm{PC}$ (i.e., weakly continuous) any path P works; for an approximately continuous function x_0 must be a density point of a path P; in any symmetric version of these notions the paths must be symmetric with respect to x_0. Luzin's theorem implies that every bounded measurable function $f\colon \mathbb{R}\to\mathbb{R}$ is approximately continuous almost everywhere. Blumberg's theorem and Sierpiński-Zygmund's example illustrate the extent to which arbitrary functions have sets of restricted continuity.

Certainly, the above discussion barely touches the tip of the iceberg of different notions of generalized continuities. From the notions not mentioned so far probably the most studied is that of quasi-continuity introduced in 1932 by Kempisty. (See [13, sec. 6].) Thus, a function f from a topological space X into $\mathbb{R}$ is **quasi-continuous**, $f\in\mathrm{QC}(X)$, if for every $x\in X$ and open sets $U\ni x$ and $V\ni f(x)$ there exists a nonempty open $W\subset U$ with $f[W]\subset V$. The other two closely related classes are defined as follows. A function $f\colon X\to\mathbb{R}$ is **cliquish**, $f\in\mathrm{CLIQ}(X)$, if for every $x\in X$, open $U\ni x$, and $\varepsilon>0$ there is a nonempty open $W\subset U$ such that $|f(y)-f(z)|<\varepsilon$ for all $y,z\in W$; f is **almost continuous in sense of Husain**, $f\in\mathrm{ACH}(X)$, if for every $x\in X$ and open $V\ni f(x)$ point x belongs to the interior of the closure of $f^{-1}(V)$. It is not difficult to see that $\mathrm{QC}(X) \subset \mathrm{CLIQ}(X)$ and that every $f \in \mathrm{CLIQ}(X)$ has the *Baire property*. Quasi-continuous functions need not to be in PC, as witnessed by $\chi_{(0,\infty)}$. Also, $\mathrm{Ext} \not\subset \mathrm{CLIQ}(\mathbb{R})$, since $\mathrm{Ext} + \mathrm{Ext} = \mathbb{R}^{\mathbb{R}} \neq \mathrm{CLIQ}(\mathbb{R}) = \mathrm{CLIQ}(\mathbb{R}) + \mathrm{CLIQ}(\mathbb{R})$. The relation $\mathrm{Ext} \not\subset \mathrm{ACH}(\mathbb{R}) \not\subset \mathrm{D}$ is justified by a $\sin(1/x)$-function and the characteristic function of the set of rational numbers, respectively. However, we have $\mathrm{ACH}(\mathbb{R}) \subset \mathrm{PC}$. Also, $\mathrm{QC}(\mathbb{R}) \not\subset \mathrm{ACH}(\mathbb{R})$ and $\mathrm{ACH}(\mathbb{R}) \not\subset \mathrm{CLIQ}(\mathbb{R})$, where the second relation is justified by the characteristic function of a *Bernstein set*.

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