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MB-representations and topological algebras

Abstract

For an algebra $\mathcal{A}$ and an ideal $\mathcal{I}$ of subsets of a set X we consider pairs $\langle \mathcal{A}, \mathcal{I} \rangle$ which have the common inner Marczewski-Burstin representation. The main goal of the paper is to investigate which inner Marczewski-Burstin representable algebras and pairs are topological.

1 Introduction

Let X be a nonempty set and let $\mathcal{F}$ be a nonempty family of nonempty subsets of X. Following the idea of Burstin and Marczewski we define:

$$S(\mathcal{F}) = \{A \subset X\colon (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$$

and

$$S_0(\mathcal{F}) = \{A \subset X\colon (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus A)\}.$$

Then $S(\mathcal{F})$ is an algebra of subsets of X and $S_0(\mathcal{F})$ is an ideal on X. (See [BBRW]. In this paper family $S_0(\mathcal{F})$ is denoted by $S^0(\mathcal{F})$.) Burstin [Bu] showed that if we take as $\mathcal{F}$ the family of perfect subsets of $\mathbb{R}$ with a positive Lebesgue measure then $S(\mathcal{F})$ equals to the σ-algebra of measurable sets and $S_0(\mathcal{F})$ is the ideal of null sets. On the other hand, if $\mathcal{F}$ is the family of all perfect subsets of $\mathbb{R}$ then $S(\mathcal{F})$ and $S_0(\mathcal{F})$ become Marczewski's σ-algebra and Marczewski's σ-ideal, which are closely related to a class of Sierpiński functions [Ma].

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We say that an algebra $\mathcal{A}$ (an ideal $\mathcal{I}$) of subsets of X *has a Marczewski-Burstin representation* if there exists a nonempty family $\mathcal{F}$ of nonempty subsets of X such that $\mathcal{A} = S(\mathcal{F})$ ($\mathcal{I} = S_0(\mathcal{F})$, respectively). If in addition $\mathcal{F} \subset \mathcal{A}$ then we say that $\mathcal{A}$ *is inner MB-representable*. For $\mathcal{I} \subset \mathcal{A}$ we say that the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ *is MB-representable* provided $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ for some family $\mathcal{F}$. If in addition $\mathcal{F} \subset \mathcal{A}$ then we say that $\langle \mathcal{A}, \mathcal{I} \rangle$ *is inner MB-representable*. MB-representations of algebras and ideals were recently considered in the papers [Re, BR, BET, BBRW, BBC]. In the first three of these papers a family $\mathcal{F}$ was always chosen from "nice" sets: Borel or perfect with respect to some topology; papers [BBRW, BBC] contain the systematic studies of MB-representations for quite arbitrary families $\mathcal{F}$.

We say that the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ (an algebra $\mathcal{A}$, or an ideal $\mathcal{I}$) *is topological* provided there exists a topology τ on X such that $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ ($\mathcal{A} = S(\mathcal{F})$, or $\mathcal{I} = S_0(\mathcal{F})$, respectively), where $\mathcal{F} = \tau \setminus \{\emptyset\}$. It was noticed in [BBRW, prop. 1.3] that $\mathcal{I} = S_0(\tau \setminus \{\emptyset\})$ is equal to the ideal $NWD(\tau)$ of τ-nowhere dense sets, while $\mathcal{A} = S(\tau \setminus \{\emptyset\})$ is the algebra of subsets of X with nowhere dense boundary. Clearly every topological pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is inner MB-representable. The main question we investigate in this note is whether the converse is also true, that is, more precisely

Which inner MB-representable pairs $\langle \mathcal{A}, \mathcal{I} \rangle$ are topological?

We say that the families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subsets of X are *mutually coinitial* provided

$$(\forall U \in \mathcal{F}_1)(\exists V \in \mathcal{F}_2)(V \subset U) \quad \text{and} \quad (\forall U \in \mathcal{F}_2)(\exists V \in \mathcal{F}_1)(V \subset U).$$

We will need the following facts from [BBRW].

Fact 1 *If families $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually coinitial then $S(\mathcal{F}_1) = S(\mathcal{F}_2)$ and $S_0(\mathcal{F}_1) = S_0(\mathcal{F}_2)$.*

Fact 2 *If $\mathcal{F}_1 \subset S(\mathcal{F}_1)$, $\mathcal{F}_2 \subset S(\mathcal{F}_2)$ and $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1) \rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2) \rangle$ then $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually coinitial.*

Since topological algebras are always inner MB-representable, the problem: *is a given pair $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$, with $\mathcal{F} \subset S(\mathcal{F})$, topological?* is equivalent to: *is $\mathcal{F}$ mutually coinitial with some topology (or with a base of some topology) on X?* If we consider only an inner MB-representable algebra $S(\mathcal{F})$, the problem *is $S(\mathcal{F})$ topological* cannot be formulated in these terms: the ideals $S_0(\mathcal{F})$ and $S_0(\tau \setminus \{\emptyset\})$ can be quite different and so $\mathcal{F}$ and $\tau \setminus \{\emptyset\}$ need not be mutually coinitial. On the other hand, any ideal $\mathcal{I}$ is the ideal of nowhere dense sets in some topology (see [CJ]), so in our terms any ideal of sets is topological.

2 The results

We use the standard set theoretic notation as in [Ci].

Theorem 1 *Let $|X| = \kappa \geq \omega$ and $\mathcal{I}$ be a proper ideal of subsets of X such that $\mathcal{I} \subset [X]^{<\kappa}$. If $\bigcup \mathcal{I} = X$ then the pair $\langle \mathcal{P}(X), \mathcal{I} \rangle$ is inner MB-representable but is not topological.*

PROOF. To see that $\langle \mathcal{P}(X), \mathcal{I} \rangle$ is inner MB-representable put $\mathcal{F} = \mathcal{P}(X) \setminus \mathcal{I}$ and notice that $S_0(\mathcal{F}) = \mathcal{I}$ and $S(\mathcal{F}) = \mathcal{P}(X)$. (It is true for any proper ideal I.)

To see that $\langle \mathcal{P}(X), \mathcal{I} \rangle$ is not topological suppose, by way of contradiction, that for some topology τ we have $\mathcal{I} = S_0(\tau \setminus \{\emptyset\})$ and $\mathcal{P}(X) = S(\tau \setminus \{\emptyset\})$. Consider a family $\{A_\alpha : \alpha < \kappa\} \subset \mathcal{P}(X)$ of disjoint sets such that $|A_\alpha| = \kappa$ for each $\alpha < \kappa$. For every $\alpha < \kappa$ the interior $\mathrm{int}(A_\alpha)$ of A_α is nonempty since the boundary of A_α belongs to $\mathcal{I}$ and has the cardinality less then κ. Moreover $|\mathrm{int}(A_\alpha)| = \kappa$. For each $\alpha < \kappa$ choose an $x_\alpha \in \mathrm{int}(A_\alpha)$. Then $A = \{x_\alpha : \alpha < \kappa\}$ has cardinality κ, so $\mathrm{int}(A) \neq \emptyset$. Pick $x_{\alpha_0} \in \mathrm{int}(A)$. Then $\{x_{\alpha_0}\} = \mathrm{int}(A_{\alpha_0}) \cap \mathrm{int}(A) \in \tau$. But $\{x_{\alpha_0}\} \in \mathcal{I} = NWD(\tau)$, a contradiction. ■

Remark 1 The condition $\bigcup \mathcal{I} = X$ in Theorem 1 is essential. For example, if $x_0 \in X$ and $\mathcal{I} = \{\emptyset, \{x_0\}\}$ then the pair $\langle \mathcal{P}(X), \mathcal{I} \rangle$ is made topological by a topology $\tau = \{A \subset X : x_0 \notin A\} \cup \{X\}$.

For a family $\mathcal{G}$ of sets we let $i(\mathcal{G}) \stackrel{\text{def}}{=} \{\bigcap \mathcal{G}_0 : \mathcal{G}_0 \in [\mathcal{G}]^{<\omega}\}$.

Theorem 2 *Let κ be an infinite cardinal and $\mathcal{F}$ be a family of nonempty subsets of X such that $\mathcal{F} \subset S(\mathcal{F})$, $|\mathcal{F}| \leq \kappa$, and*

- *$S_0(\mathcal{F})$ contains all sets $\bigcup \mathcal{J}$ where $\mathcal{J} \in [i(\mathcal{F}) \cap S_0(\mathcal{F})]^{<\kappa}$.*

Then the pair $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ is topological.

PROOF. Recall that, by [BBRW, prop. 1.1(3)], we have $\mathcal{F} \cap S_0(\mathcal{F}) = \emptyset$. Let $\mathcal{F} = \{P_\alpha : \alpha < \kappa\}$. For every $\alpha < \kappa$ put

$$Z_\alpha = \bigcup \big(S_0(\mathcal{F}) \cap i(\{P_\xi : \xi \leq \alpha\})\big) \quad \text{and} \quad Q_\alpha = P_\alpha \setminus Z_\alpha.$$

Note that by our assumptions we have $Z_\alpha \in S_0(\mathcal{F})$, so $Q_\alpha \in S(\mathcal{F}) \setminus S_0(\mathcal{F})$.

Let τ be a topology on X generated by $\mathcal{B} = i(\{Q_\alpha : \alpha < \kappa\})$. By Fact 1 it is enough to show that families $\mathcal{F}$ and $\mathcal{B} \setminus \{\emptyset\}$ are mutually coinitial.

Clearly for every $P_\alpha \in \mathcal{F}$ we have $Q_\alpha \subset P_\alpha$ and $Q_\alpha \in \mathcal{B} \setminus \{\emptyset\}$. So, $\mathcal{B} \setminus \{\emptyset\}$ is coinitial with $\mathcal{F}$. To see that $\mathcal{F}$ is coinitial with $\mathcal{B} \setminus \{\emptyset\}$ take $Q \in \mathcal{B} \setminus \{\emptyset\}$. Since $Q \in S(\mathcal{F})$ it is enough to show that $Q \notin S_0(\mathcal{F})$ (as for every $A \in S(\mathcal{F}) \setminus S_0(\mathcal{F})$ there are $P, P' \in \mathcal{F}$ with $P \subset A \cap P'$). Let $\alpha_1 < \cdots < \alpha_n < \kappa$ be such that

$$Q = \bigcap_{i=1}^{n} Q_{\alpha_i} = \bigcap_{i=1}^{n} (P_{\alpha_i} \setminus Z_{\alpha_i}) = \bigcap_{i=1}^{n} P_{\alpha_i} \setminus \bigcup_{i=1}^{n} Z_{\alpha_i}$$

Since $\bigcap_{i=1}^n P_{\alpha_i} \in i(\{P_\xi : \xi \le \alpha_n\})$, it cannot belong to $S_0(\mathcal{F})$, as otherwise we would have $\bigcap_{i=1}^n P_{\alpha_i} \subset Z_{\alpha_n}$ contradicting our assumption that $Q \neq \emptyset$. Thus $\bigcap_{i=1}^n P_{\alpha_i} \in S(\mathcal{F}) \setminus S_0(\mathcal{F})$ and $\bigcup_{i=1}^n Z_{\alpha_i} \in S_0(\mathcal{F})$, leading to $Q \in S(\mathcal{F}) \setminus S_0(\mathcal{F})$. ■

Remark 2 It was pointed to us by the referee that a very similar result (with almost identical proof) was proved earlier by Schilling in [S, thm. 3]. More precisely, Schilling considers the σ-ideals $S_0^\sigma(\mathcal{F})$ generated by $S_0(\mathcal{F})$ (which he denotes by $\mathcal{M}(\mathcal{F})$), defines $S^\sigma(\mathcal{F})$ as

$$\{A \subset X : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F},\ Q \subset P)(Q \cap A \in S_0^\sigma(\mathcal{F}) \text{ or } Q \setminus A \in S_0^\sigma(\mathcal{F}))\}$$

(which he denotes by $\mathcal{B}(\mathcal{F})$[1]), and proves that if $\langle X, \mathcal{F} \rangle$ is a category base, $\kappa = |\mathcal{F}|$, and the condition • holds for $S_0^\sigma(\mathcal{F})$ in place of $S_0(\mathcal{F})$ then there exists a topology τ on X such that $S^\sigma(\mathcal{F}) = S^\sigma(\tau \setminus \{\emptyset\})$ and $S_0^\sigma(\mathcal{F})$ is equal to the σ-ideal $\mathcal{M}(\tau)$ of meager subsets of $\langle X, \tau \rangle$.

It is not difficult to see that our result implies Schilling's theorem since, by Fact 2, $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle = \langle S(\tau \setminus \{\emptyset\}), S_0(\tau \setminus \{\emptyset\}) \rangle$ implies that $\mathcal{F}$ and $\tau \setminus \{\emptyset\}$ are mutually coinitial so $S^\sigma(\mathcal{F}) = S^\sigma(\tau \setminus \{\emptyset\})$ and $S_0^\sigma(\mathcal{F})$ is equal to the σ-ideal generated by $S_0(\tau \setminus \{\emptyset\}) = NWD(\tau)$, that is, $S_0^\sigma(\mathcal{F}) = \mathcal{M}(\tau)$.

The relation between both results is the most straightforward when $S_0(\mathcal{F})$ is a σ-ideal, since then we have $S_0^\sigma(\mathcal{F}) = S_0(\mathcal{F}) = NWD(\tau) = \mathcal{M}(\tau)$ and $S^\sigma(\mathcal{F}) = S(\mathcal{F}) = S(\tau \setminus \{\emptyset\}) = S^\sigma(\tau \setminus \{\emptyset\})$.

We also should point here that our condition • implies that $\langle X, \mathcal{F} \cup \{X\} \rangle$ forms a category base.

Applying Theorem 2 to κ equal to continuum $\mathfrak{c}$ and the family $\mathcal{F}$ of perfect subsets of the real line we obtain immediately the following corollary.

Corollary 3 *The pair $\langle S, S_0 \rangle$ of the classical Marczewski sets is topological.*

[1] If $\langle X, \tau \rangle$ is a topological space then $\mathcal{B}(\tau \setminus \{\emptyset\}) = S^\sigma(\tau \setminus \{\emptyset\})$ is the family of all subsets of X with the Baire property.

The fact that $S = S^\sigma(\tau\setminus\{\emptyset\})$ (which is equal to $S(\tau\setminus\{\emptyset\})$) was first proved by Aniszczyk [A] under the additional set-theoretical assumption that the ideal S_0 is continuum additive. Schilling [S] noticed that there is a topology τ on the real line for which $\langle S, S_0\rangle = \langle S^\sigma(\tau\setminus\{\emptyset\}), S_0^\sigma(\tau\setminus\{\emptyset\})\rangle$ which, as we noticed in Remark 2, is equal to $\langle S(\tau\setminus\{\emptyset\}), S_0(\tau\setminus\{\emptyset\})\rangle$.

We also get

Corollary 4 *Assume the Continuum Hypothesis. If $\emptyset \notin \mathcal{F} \in [\mathcal{P}(X)]^{\leq\mathfrak{c}}$ is such that $S_0(\mathcal{F})$ is a σ-ideal and $\mathcal{F} \subset S(\mathcal{F})$ then the pair $\langle S(\mathcal{F}), S_0(\mathcal{F})\rangle$ is topological.*

For the rest of this note we will assume that X is a set of cardinality $\kappa \geq \omega$. We say that a family $\mathcal{F}_0 \subset [X]^\kappa$ is *almost disjoint* provided $|F_1 \cap F_2| < \kappa$ for every distinct $F_1, F_2 \in \mathcal{F}_0$. It is a basic fact that there exist an almost disjoint family $\mathcal{F}_0 \subset [X]^\kappa$ of cardinality greater than κ.

Notice the following simple fact.

Fact 3 *If $\mathcal{F}_0 \subset [X]^\kappa$ is almost disjoint and $\mathcal{F} = \{F\triangle A : F \in \mathcal{F}_0 \ \&\ A \in [X]^{<\kappa}\}$ then*

$$\mathcal{F} \subset S(\mathcal{F}) = \{A : (\forall F \in \mathcal{F})(|F\setminus A| < \kappa \ \text{ or } \ |F \cap A| < \kappa)\}$$

and $[X]^{<\kappa} \subset S_0(\mathcal{F}) = \{A : (\forall F \in \mathcal{F})(|F \cap A| < \kappa)\}$.

Moreover, $S_0(\mathcal{F}) = [X]^{<\kappa}$ if and only if $\mathcal{F}_0$ is maximal almost disjoint.

Theorem 5 *Let $\mathcal{F} = \{F\triangle A : F \in \mathcal{F}_0 \ \&\ A \in [X]^{<\kappa}\}$, where $\mathcal{F}_0 \subset [X]^\kappa$ is almost disjoint.*

(a) If κ is a regular cardinal and $|\mathcal{F}_0| \leq \kappa$ then the pair $\langle S(\mathcal{F}), S_0(\mathcal{F})\rangle$ is topological.

(b) If $|\mathcal{F}_0| > \kappa$ then the algebra $S(\mathcal{F})$ is not topological.

PROOF. (a) Let $X = \{x_\alpha : \alpha < \kappa\}$ and put

$$\mathcal{F}_1 = \{F \setminus \{x_\xi : \xi < \alpha\} : F \in \mathcal{F}_0 \ \&\ \alpha < \kappa\}.$$

Regularity of κ implies that families $\mathcal{F}$ and $\mathcal{F}_1$ are mutually coinitial. So, by Fact 1, we have $\langle S(\mathcal{F}), S_0(\mathcal{F})\rangle = \langle S(\mathcal{F}_1), S_0(\mathcal{F}_1)\rangle$. Clearly$|\mathcal{F}_1| \leq \kappa$ and $\mathcal{F}_1 \subset \mathcal{F} \subset S(\mathcal{F}) \subset S(\mathcal{F}_1)$.

Since regularity of κ implies also that $S_0(\mathcal{F})$ is κ-additive (i.e., union if less than κ-many sets from $S_0(\mathcal{F})$ belongs to $S_0(\mathcal{F})$), condition $\bullet$ from Theorem 2 holds and so $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1)\rangle$ is topological.

(b) By way of contradiction suppose that there exists a topology τ on X such that $S(\mathcal{F}) = S(\tau_0)$, where $\tau_0 = \tau \setminus \{\emptyset\}$.

Note that for every $F\in\mathcal{F}$ we have $F\in S(\mathcal{F})\setminus S_0(\mathcal{F})=S(\tau_0)\setminus NWD(\tau)$, so

$$\mathrm{int}_\tau(F)\neq\emptyset \ \text{ for every } F\in\mathcal{F}. \tag{1}$$

Also, if $F_0,F_1\in\mathcal{F}_0$ are different then

$$\mathrm{int}_\tau(F_0)\cap\mathrm{int}_\tau(F_1)\subset F_0\cap F_1\in[X]^{<\kappa}\subset S_0(\mathcal{F})=S_0(\tau_0)=NWD(\tau).$$

So, $\{\mathrm{int}_\tau(F)\colon F\in\mathcal{F}_0\}$ is the family of non-empty pairwise disjoint subsets of X of cardinality $|\mathcal{F}_0|>|X|$, which is impossible. ∎

Remark 3 Notice that if κ has uncountable cofinality, $\mathcal{F}_0\subset[X]^\kappa$ is maximal almost disjoint, and $\mathcal{F}$ is as in Fact 3 then the algebra $\mathcal{A}$ generated by the family $\mathcal{F}$ (i.e., the closure of $\mathcal{F}$ under finite unions, finite intersections and complements in X) is not inner MB-representable. This follows immediately from [BBC, thm. 13].

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