Krzysztof Ciesielski,* Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA (e-mail: K_Cies@math.wvu.edu web page: `http://www.math.wvu.edu/~kcies`)
Hajrudin Fejzić, Department of Mathematics, California State University, San Bernardino, CA 92407, USA (e-mail: hfejzic@csusb.edu)
Chris Freiling, Department of Mathematics, California State University, San Bernardino, CA 92407, USA (e-mail: cfreilin@csusb.edu)

Measure zero sets with non-measurable sum

Abstract

For any $C \subseteq \mathbb{R}$ there is a subset $A \subseteq C$ such that $A + A$ has inner measure zero and outer measure the same as $C + C$. Also, there is a subset A of the Cantor middle third set such that $A + A$ is Bernstein in $[0, 2]$. On the other hand there is a perfect set C such that $C + C$ is an interval I and there is no subset $A \subseteq C$ with $A + A$ Bernstein in I.

1 Introduction.

It is not at all surprising that there should be measure zero sets, A, whose sum $A+A = \{x+y : x \in A, y \in A\}$ is non-measurable. Ask a typical mathematician why this should be so and you are likely to get the following response:

> The Cantor middle-third set, when added to itself gives an entire interval, $[0, 2]$. So certainly there exists a measure zero set that when added to itself gives a non-measurable set.

The intuition being that an interval has much more content than is needed for a non-measurable set.

Indeed such sets do exist (in ZFC). Sierpiński (1920) seems to be the first to address this issue. Actually, he shows the existence of measure zero sets

Key Words: sets, algebraic sum, measurability.
Mathematical Reviews subject classification: 26A21,28A05
*The work of the first author was partially supported by NATO Grant PST.CLG.977652.
Papers authored or co-authored by a Contributing Editor are managed by a Managing Editor or one of the other Contributing Editors.

X, Y such that $X+Y$ is non-measurable (see [7]). The paper by Rubel (see [6]) in 1963 contains the first proof that we could find for the case $X = Y$ (see also [5]). Ciesielski [3] extends these results to much greater generality, showing that A can be a measure zero Hamel basis, or it can be a (non-measurable) Bernstein set and that $A+A$ can also be Bernstein. He also establishes similar results for multiple sums, $A + A + A$ etc.

This paper is mainly about the statement above and the intuition behind it. Below we list four conjectures, each of which seems justified by extending this line of reasoning.

1. Not only does such a set exist, but it can be taken to be a subset of the Cantor middle-third set, $C_{\frac{1}{3}}$. (This does not seem to immediately follow from any of the above proofs. Thomson [9, p. 136] claims this to be true, but without proof.)

2. The intuition really has nothing to do with the precise structure of the Cantor set, which might lead one to conjecture the following. Suppose C is any set with the property that $C + C$ contains a set of positive measure. Then there must exist a subset $A \subseteq C$ such that $A + A$ is non-measurable.

3. The intuition relies on the fact that non-measurable sets can have far less content than an entire interval. Therefore, the claim should also hold when *non-measurable* is replaced by other similar qualities. Recall that if I is a set then a set S is called *Bernstein in* I if and only if both S and its complement intersect every non-empty perfect subset of I. Constructing a set that is Bernstein in an interval is one of the standard ways of establishing non-measurability. Certainly, any set that is Bernstein in an interval has far less content than the interval itself. Therefore, we might conjecture that there is a subset $A \subseteq C_{\frac{1}{3}}$ with $A+A$ Bernstein in [0,2].

4. Combining the reasoning behind the Conjectures 2 and 3, let C be any set with the property that $C + C$ contains an interval, I. We might conjecture that there must exist a subset $A \subseteq C$ such that $A + A$ is Bernstein in I.

We will settle these four conjectures in the next four sections. In particular, in Section 2 we will give a proof of the first conjecture using transfinite induction. This provides what is possibly the simplest proof of the original assertion. However, the proof depends on a particular property of symmetric perfect sets. In Section 3 we give a slightly more complicated argument to prove the second conjecture. We show that any set C contains a subset, A, such that the inner measure of $A+A$ is zero and the outer measure is the same as $C+C$. In Section 4 we will settle the third conjecture, finding a subset A of the Cantor set such that $A+A$ is Bernstein in [0,2]. Finally, in Section 5 we will give a counterexample to the fourth conjecture, showing that the above lines of reasoning are indeed limited. In all of these proofs, we will be assuming the axioms of ZFC. No additional set theoretical assumptions will be used.

The existence of these sets is interesting historically. Suppose E is a measurable set and f is a measurable function. Several researchers in the area of generalized derivatives have taken for granted the measurability of sets such as:

$$E_j = \left\{ x \in E \colon \left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \right| < j \text{ for all } 0 < |h| < \frac{1}{j} \right\}.$$

It wasn't until 1960 that Stein and Zygmund [8] pointed out that the measurability of these sets is not automatic, and not until 1993 that Fejzić and Weil re-proved these results without this assumption. In their paper [4] they also show that this measurability assumption can be reduced to the (false) claim that $A+A$ is measurable whenever A is measurable (see also [5]).

2 Conjecture 1 is true.

It is well known that every real x in $[0,2]$ can be expressed as $x = a+b$ where a, b are in the Cantor set $C_{\frac{1}{3}}$. It is also well known (a proof occurs below in Lemma 4) that for almost every such x there are continuum many such representations. The following theorem therefore gives a positive answer to Conjecture 1.

Theorem 1 *Let C be any subset of $\mathbb{R}$ and let*

$$R = \{x \in \mathbb{R} \colon x \text{ has } \mathfrak{c} \text{ many representations } x = a+b \text{ with } a, b \in C\}.$$

Then there is a subset $A \subseteq C$ such that $A+A$ is Bernstein in R.

Proof. If R has no non-empty perfect subsets then we are done. Otherwise it has continuum many such sets. Let $\{P_\xi \colon \xi < \mathfrak{c}\}$ be the family of all non-empty perfect subsets of R. We will find an $A \subseteq C$ such that each P_ξ intersects both $A+A$ and its complement. Construct a sequence,

$$\langle \langle a_\xi, b_\xi, c_\xi, d_\xi \rangle \in C \times C \times P_\xi \times P_\xi \colon \xi < \mathfrak{c} \rangle$$

such that for each $\xi < \mathfrak{c}$,

$(*)$ $c_\xi = a_\xi + b_\xi$ and $\{d_\eta \colon \eta \leq \xi\} \cap (A_{\xi+1} + A_{\xi+1}) = \emptyset$, where $A_\xi = \bigcup_{\eta<\xi}\{a_\eta, b_\eta\}$. This will ensure that $A = A_\mathfrak{c}$ has the desired properties, since then $\{c_\xi\colon \xi<\mathfrak{c}\} \subseteq A+A \subseteq \mathbb{R}\setminus\{d_\xi\colon \xi<\mathfrak{c}\}$.

To make an inductive step, assume that for some $\alpha<\mathfrak{c}$ we have already constructed a partial sequence satisfying $(*)$ for all $\xi<\alpha$. First choose a $c_\alpha \in P_\alpha \setminus \{d_\xi\colon \xi<\alpha\}$. Then choose a_α, b_α in C such that $a_\alpha+b_\alpha=c_\alpha$ and neither a_α nor b_α is in $\{d_\xi\colon \xi<\alpha\}-A_\alpha$. Construction is finished by choosing $d_\alpha \in P_\alpha - (A_{\alpha+1}+A_{\alpha+1})$. ■

3 Conjecture 2 is true.

Conjecture 2 is settled by the following theorem. Note that it proves more than what is needed for Conjecture 2. However, as is the case with Theorem 1, it falls short of establishing that $A+A$ is Bernstein in $C+C$.

Theorem 2 *For every $C\subset\mathbb{R}$ there exists an $A\subset C$ such that $A+A$ has inner measure zero and outer measure the same as $C+C$.*

Proof. We can assume that the inner measure of $C+C$ is positive, since otherwise $A=C$ is as desired.

Let G be a G_δ-set containing $C+C$ such that $G\setminus(C+C)$ has inner measure zero. We will construct a set $A\subset C$ such that every perfect set $P\subset G$ of positive measure intersects both $G\setminus(A+A)$ and $A+A$. This implies that $A+A$ has inner measure zero and the same outer measure as $C+C$.

Let $\bar{\mathcal{E}}$ be the family of all perfect subsets P of G of positive measure such that $|P\cap(C+C)|<\mathfrak{c}$.[1] Let $\mathcal{E}$ be a maximal subfamily of $\bar{\mathcal{E}}$ of pairwise disjoint sets. Then $\mathcal{E}$ is at most countable, so $E=\bigcup\mathcal{E}$ is an F_σ-set and $E\cap(C+C)$ has cardinality less than $\mathfrak{c}$. For every $e\in E\cap(C+C)$ fix $c_e, d_e\in C$ such that $e=c_e+d_e$. Let $Z=\bigcup\{\{c_e,d_e\}\colon e\in E\cap(C+C)\}$. Then $Z\subset C$ also has cardinality less than $\mathfrak{c}$ and $E\cap(C+C)\subset Z+Z$. Notice that by the maximality of $\mathcal{E}$,

(a) $|P\cap(C+C)|=\mathfrak{c}$ for every perfect set $P\subset G\setminus E$ of positive measure.

Next, let $\bar{\mathcal{K}}$ be the family of all perfect subsets F of $G\setminus E$ of positive measure for which there exists an $X_F\subset C$ such that

[1]Since $P\cap(C+C)$ must have positive outer measure, this set must be uncountable. Also $\bar{\mathcal{E}}$ is empty if either $C+C$ is measurable or Martin's axiom holds. However, there are models of ZFC containing sets C of cardinality less than $\mathfrak{c}$ with full outer measure. In that case, $C+C$ also has these properties and so $\bar{\mathcal{E}}$ contains all perfect subsets of G of positive measure.

$$|X_F| < \mathfrak{c} \text{ and } F \cap (C+C) \subset X_F + C.$$[2]

Let $\mathcal{K}$ be a maximal subfamily of $\bar{\mathcal{K}}$ of pairwise disjoint sets. Then $\mathcal{K}$ is at most countable, so $K = \bigcup \mathcal{K}$ is an F_σ-set and $X = \bigcup_{F\in\mathcal{K}} X_F \subset C$ has cardinality less than $\mathfrak{c}$. Clearly $K \cap (C+C) \subset X + C$. Moreover, by the maximality of $\mathcal{K}$,

(b) for every $P \subset G \setminus (E \cup K)$ of positive measure and any set $X \subseteq C$ of cardinality less than $\mathfrak{c}$, $(P \cap (C+C)) \nsubseteq (X+C)$.

Let $\mathcal{P}$ be the family of all non-empty perfect subsets P of $G \setminus E$ of positive measure such that either $P \subset K$ or $P \cap K = \emptyset$. Let $\{P_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of $\mathcal{P}$. We will find an $A \subset C$ containing Z such that each P_ξ intersects both $A + A$ and $G \setminus (A+A)$. First notice that such a set will be as desired.

Indeed, take a perfect set $P \subset G$ of positive measure. We have to show that P intersects both $G \setminus (A+A)$ and $A + A$. If $P \cap E$ has positive measure then $\emptyset \neq P \cap E \cap (C+C) \subset Z + Z \subset A + A$, while $P \cap E \not\subset A + A$ since $E \cap (A+A) \subset E \cap (C+C)$ has cardinality less than $\mathfrak{c}$. On the other hand, if $P \cap E$ has measure zero then $P \setminus E$ contains one of sets P_ξ, since in this case either $(P \setminus E) \cap K$ or $(P \setminus E) \setminus K$ has to have positive measure.

To define A construct $\langle\langle a_\xi, b_\xi, c_\xi, d_\xi\rangle \in C \times C \times P_\xi \times P_\xi \colon \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$

- $c_\xi = a_\xi + b_\xi$ and $\{d_\eta \colon \eta \le \xi\} \cap (A_{\xi+1} + A_{\xi+1}) = \emptyset$,

where $A_\beta = Z \cup X \cup \bigcup_{\eta<\beta}\{a_\eta, b_\eta\}$. This will ensure that $A = A_{\mathfrak{c}}$ has the desired properties since then $\{c_\xi \colon \xi < \mathfrak{c}\} \subset A + A$ and $\{d_\xi \colon \xi < \mathfrak{c}\} \cap (A+A) = \emptyset$.

To make an inductive step assume that for some $\xi < \mathfrak{c}$ we have already constructed a partial sequence for which $\{d_\eta \colon \eta < \xi\} \cap (A_\xi + A_\xi) = \emptyset$. Let $Y = \mathrm{LIN}_{\mathbb{Q}}(A_\xi \cup \{d_\eta \colon \eta < \xi\})$, where $\mathrm{LIN}_{\mathbb{Q}}(S)$ denotes the set of finite linear combinations of elements in S with rational coefficients. Notice that $|Y| < \mathfrak{c}$ since $|S| < \mathfrak{c}$ implies $|\mathrm{LIN}_{\mathbb{Q}}(S)| < \mathfrak{c}$. Consider two cases.

Case 1: $P_\xi \subset K$. Choose $c_\xi \in P_\xi \cap (C+C) \setminus Y$. The choice can be made by (a), since $|P_\xi \cap (C+C)| = \mathfrak{c}$ as $P_\xi \subset G \setminus E$ has positive measure.

As $c_\xi \in P_\xi \cap (C+C) \subset K \cap (C+C) \subset X + C$ there are $a_\xi \in X \subset A_\xi$ and $b_\xi \in C$ such that $c_\xi = a_\xi + b_\xi$. We now show that $d_\eta \notin A_{\xi+1} + A_{\xi+1}$ for each $\eta < \xi$. Assume otherwise. Since $A_{\xi+1} = A_\xi \cup \{b_\xi\}$ and $d_\eta \notin A_\xi + A_\xi$, it must be that $d_\eta \in \{b_\xi\} + (A_\xi \cup \{b_\xi\})$. But then $b_\xi \in Y = \mathrm{LIN}_{\mathbb{Q}}(A_\xi \cup \{d_\eta \colon \eta < \xi\})$ and so also is $c_\xi = a_\xi + b_\xi$, contradicting the choice of c_ξ.

[2]The family $\bar{\mathcal{K}}$ may be non-empty even if C has measure zero. Although this cannot happen under Martin's axiom, it happens in any model of ZFC in which there is a non-meager measure zero set of cardinality less than $\mathfrak{c}$. (See e.g. [2, thms. 2.7.3 and 2.1.7].)

Case 2: $P_\xi \subset G \setminus (E \cup K)$. We will first show that

$(*)$ there exist $a, b \in C \setminus Y$ such that $a + b \in P_\xi \setminus Y$.

To see this, for each $p \in P_\xi \cap (C+C)$ choose $a_p, b_p \in C$ with $p = a_p + b_p$. If $(*)$ is false then for every $p \in P_\xi \cap (C+C) \setminus Y$ we would have $\{a_p, b_p\} \cap Y \neq \emptyset$. But then $P_\xi \cap (C+C) \setminus Y \subset (Y \cap C) + C$. So, for $X_0 = (Y \cap C) \cup \{a_p : p \in Y\}$ we have $P_\xi \cap (C+C) \subset X_0 + C$ contradicting (b).

Now, take a and b as in $(*)$ and put $a_\xi = a$, $b_\xi = b$, and $c_\xi = a + b$. Then $a_\xi, b_\xi, c_\xi \notin Y$. In particular, $d_\eta \notin \{a_\xi, b_\xi, c_\xi\} + \mathrm{LIN}_{\mathbb{Q}}(A_\xi)$ for every $\eta < \xi$. So, $\{d_\eta : \eta < \xi\} \cap (A_{\xi+1} + A_{\xi+1}) = \emptyset$.

In both cases we finish the inductive step by choosing a $d_\xi \in P_\xi \setminus \mathrm{LIN}_{\mathbb{Q}}(A_{\xi+1})$.
■

4 Conjecture 3 is true.

In this section we embellish the argument in Section 1 to settle the third conjecture.

Definition 3 *Two real numbers x, y will be called equivalent and we write $x \sim y$ if and only if there is a ternary expansion of x and a ternary expansion of y such that the two expansions disagree on only finitely many digits.*

Note that if x is a ternary rational then it will have two possible ternary expansions. According to the definition, all such ternary rationals are equivalent. Every other real x has a unique ternary expansion. The following theorem fulfills the promise made in Section 2, showing that almost every $x \in [0,2]$ has $\mathfrak{c}$ many representations as the sum of elements in the Cantor set.

Lemma 4 *Let $x \in [0,2]$. If $x/2$ has infinitely many ones in its ternary expansion, then there are $\mathfrak{c}$ many representations of x as the sum of two Cantor-set elements. Otherwise, x has only finitely many such representations and all of the elements of $C_{\frac{1}{3}}$ used to represent x are equivalent to $x/2$.*

Proof. Every element of $C_{\frac{1}{3}}$ has a ternary expansion consisting of only even digits. Fix $x \in [0,2]$ and let $c = x/2$. If $x = a + b$ with $a, b \in C_{\frac{1}{3}}$, then c is the average of a and b. If c_i, a_i, b_i are the i^{th} digits of c, a, and b, respectively, then either

- $c_i = 0$ and $a_i = b_i = 0$,
- $c_i = 1$ and $a_i = 0, b_i = 2$,

- $c_i = 1$ and $a_i = 2, b_i = 0$, or
- $c_i = 2$ and $a_i = b_i = 2$.

Suppose first that c is a ternary rational. Then the digits of c must end in either a sequence of zeros or a sequence of two's. In either case, the digits of a and b must do likewise and so they are also ternary rationals. Therefore, $a \sim b \sim c$. Now consider the case when c is not a ternary rational, so there is a unique ternary expansion of c. Let us construct the numbers a and b using only even digits. For each c_i that is zero or two, we must have $a_i = b_i = c_i$. But for each c_i that is one, we have a choice, either $a_i = 0, b_i = 2$ or $a_i = 2, b_i = 0$. Thus if $k \in \{0, 1, 2, \ldots, \omega\}$ is the number of digits in c that have the value 1, then there are 2^k possible choices for the pair a, b. In particular, if c has infinitely many ones in its expansion, then there are $2^\omega = \mathfrak{c}$ many representations for x. If there are only finitely many ones, then the digits of a, b, c will all agree on a tail end. ■

Theorem 5 *There is a set $A \subseteq C_{\frac{1}{3}}$ such that $A+A$ is Bernstein in* $[0, 2]$.

PROOF. Let R_0 be the set of elements of $[0, 2]$ that can be expressed in $\mathfrak{c}$ many ways as the sum of elements of $C_{\frac{1}{3}}$, and let R_1 be the elements that can be expressed in only finitely many ways. Let $\{P_\xi : \xi < \mathfrak{c}\}$ be the family of all non-empty perfect subsets of $[0, 2]$. We will find an $A \subseteq C_{\frac{1}{3}}$ such that each P_ξ intersects both $A+A$ and its complement. Construct

$$\langle\langle a_\xi, b_\xi, c_\xi, d_\xi\rangle \in C \times C \times P_\xi \times P_\xi : \xi < \mathfrak{c}\rangle$$

such that for each $\xi < \mathfrak{c}$,

$(*)$ $c_\xi = a_\xi + b_\xi$ and $D_\xi \cap (A_{\xi+1} + A_{\xi+1}) = \emptyset$,

where $A_\xi = \bigcup_{\eta<\xi}\{a_\eta, b_\eta\}$ and $D_\xi = \{d_\eta : \eta \le \xi\}$. This will ensure that $A = A_{\mathfrak{c}}$ has the desired properties, since then $\{c_\xi : \xi < \mathfrak{c}\} \subseteq A + A \subseteq \mathbb{R} \setminus D_\xi$. To make the inductive step, assume that for some $\alpha < \mathfrak{c}$ we have already constructed a partial sequence satisfying $(*)$ for all $\xi < \alpha$. We first choose $a_\alpha, b_\alpha, c_\alpha$ such that $a_\alpha + b_\alpha = c_\alpha$ and such that neither a_α nor b_α is in $D_\alpha - A_\alpha$. We distinguish two cases.

Case 1: P_α intersects R_0 in a set of cardinality $\mathfrak{c}$. First choose $c_\alpha \in P_\alpha \cap R_0 \setminus D_\alpha$. Then choose a_α, b_α in $C_{\frac{1}{3}}$ such that $a_\alpha + b_\alpha = c_\alpha$ and neither a_α nor b_α is in $D_\alpha - A_\alpha$.

Case 2: P_α intersects R_1 in a set of cardinality $\mathfrak{c}$. First choose $c_\alpha \in P_\alpha \cap R_1 \setminus D_\alpha$ such that $c_\alpha/2$ is not equivalent to any element of $D_\alpha - A_\alpha$. Then choose $a_\alpha, b_\alpha \in C_{\frac{1}{3}}$ such that $a_\alpha + b_\alpha = c_\alpha$. Since, by Lemma 4, both a_α and b_α are equivalent to $c_\alpha/2$, neither of them is in $D_\alpha - A_\alpha$.

Construction is finished by choosing $d_\alpha \in P_\alpha \setminus (A_{\alpha+1} + A_{\alpha+1})$. ■

5 Conjecture 4 is false.

In this section we will construct a set A such that $A+A$ contains an interval, I, yet there is no subset $B \subseteq A$ with $B+B$ Bernstein in I. Let $C_{\frac{1}{2}}$ be the Cantor middle-half set, which will be the basis of our construction. That is, $C_{\frac{1}{2}}$ is the set of points, x, in the unit interval such that there is a base four expansion of x that uses only zeros and threes. Note that if the expansion of x ends in a sequence of zeros, then there will be an equivalent expansion ending in a sequence of threes. In this case we will say that x is a *quaternary rational*. In all other cases, the base 4 decimal expansion is unique.

When we dealt with sums from $C_{\frac{1}{3}}+C_{\frac{1}{3}}$, it was easier to think in terms of averages, so that the work could be carried out digit-wise. Similarly, when we work with $C_{\frac{1}{2}}$ it will be easier to think in terms of the following auxiliary sets. Let U be the set of elements of $[0,1]$ that use only zeros and twos in one of its base four expansions, and let V be the set of elements that use only zeros and ones. Since threes are not allowed in either set, each element of U and V has only one valid expansion. Furthermore, the sums in $U+V$ and the sums in $V+V$ can be carried out digit-wise since there will be no carries.

Our construction will be based on the following three lemmas. The proofs of the first two of them can be seen geometrically by examining Figure 1.

Lemma 6 *$C_{\frac{1}{2}}+C_{\frac{1}{2}}$ has measure zero.*

Proof. Let $x=a+b$ with $a,b\in C_{\frac{1}{2}}$. Let $c=\frac{1}{3}x=\frac{1}{3}a+\frac{1}{3}b$. Using a base 4 expansion of a and b where all digits are divisible by three, the computation of $\frac{1}{3}a+\frac{1}{3}b$ can be carried out digit-wise and results in an expansion where the digit 3 is not used. Therefore, unless c is a quaternary rational, its expansion will never use the digit three. Thus $\frac{1}{3}(C_{\frac{1}{2}}+C_{\frac{1}{2}})$ has measure zero and so does $C_{\frac{1}{2}}+C_{\frac{1}{2}}$. ■

Now consider a sum $c=u+v$ where $u\in U$ and $v\in V$. There are four possibilities for the i^{th} digits of this calculation.

- $c_i=0$, $u_i=v_i=0$
- $c_i=1$, $u_i=0$, $v_i=1$
- $c_i=2$, $u_i=2$, $v_i=0$
- $c_i=3$, $u_i=2$, $v_i=1$

Unlike the sums of $C_{\frac{1}{3}}+C_{\frac{1}{3}}$ this time there is no ambiguity. Every choice of the digits of c gives a unique choice for the digits of u and v. The next lemma relates this to the sum $C_{\frac{1}{2}}+\frac{1}{2}C_{\frac{1}{2}}$.

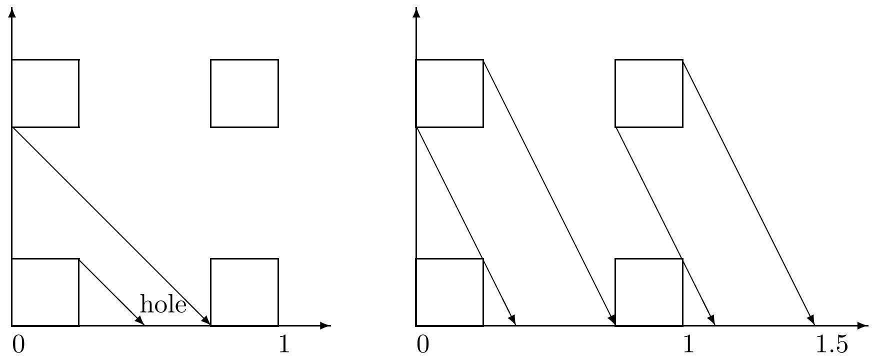

Figure 1: Consider the pairs of reals from the unit interval with the middle half removed. The projection along the direction with a slope -1 has holes, while the projection along the direction with a slope -2 fills the interval $[0, 1.5]$ without any overlaps.

Lemma 7 $C_{\frac{1}{2}} + \frac{1}{2}C_{\frac{1}{2}} = [0, 1.5]$. *Furthermore, each element in* $[0, 1.5]$ *can be expressed as such a sum in at most two ways, and except for a countable set, each element in* $[0, 1.5]$ *can be expressed in a unique way.*

PROOF. Fix an $x \in [0, 1.5]$, and suppose $x = a + b$ with $a \in C_{\frac{1}{2}}$ and $b \in \frac{1}{2}C_{\frac{1}{2}}$. Let $c = \frac{2}{3}x = \frac{2}{3}a + \frac{2}{3}b$. Using the fact that all of the digits of a are divisible by three, the computation of $\frac{2}{3}a$ can be carried out digit-wise and results in an element of U. Similarly, $\frac{2}{3}b \in V$. But each such c has at most two such representations and except when c is a quaternary rational, each such c has a unique representation.

The equality is justified by $\frac{2}{3}(C_{\frac{1}{2}} + \frac{1}{2}C_{\frac{1}{2}}) = U + V = [0, 1] = \frac{2}{3}[0, 1.5]$. ■

We are now ready to define the set A that will serve as our counterexample. Let $A = C_{\frac{1}{2}} \cup \frac{1}{2}C_{\frac{1}{2}}$.

Lemma 8 *There are two non-empty perfect subsets* P *and* Q *of* A *such that every element of* $P + Q$ *can be expressed uniquely as the sum of two elements in* A.

PROOF. $A + A$ is the union of three closed sets: $C_{\frac{1}{2}} + C_{\frac{1}{2}}$, $\frac{1}{2}C_{\frac{1}{2}} + \frac{1}{2}C_{\frac{1}{2}}$, and $C_{\frac{1}{2}} + \frac{1}{2}C_{\frac{1}{2}}$. By Lemma 6, the first and second sets are both measure zero. Choose an open interval $I \subseteq [0, 1.5]$ that is disjoint from the first two sets. By Lemma 7, the third set is the interval $[0, 1.5]$. Furthermore, this set can be partitioned into two sets X and Y such that X is countable and every element in Y has a unique representation as a sum of two elements, one in $C_{\frac{1}{2}}$ and the other in $\frac{1}{2}C_{\frac{1}{2}}$. Choose an x in $Y \cap I$ and let $x = a + b$ with $a \in C_{\frac{1}{2}}$ and $b \in \frac{1}{2}C_{\frac{1}{2}}$. Now choose a neighborhood J of a and a neighborhood K of b small enough that the closure $cl(J+K)$ of $J+K$ is a subset of I. Let R be the intersection of A with $cl(J)$ and let S be the intersection of A with $cl(K)$. Then both R and S are non-empty perfect subsets of A. Let D be the countable set consisting of all numbers used in the representations of elements of X, and let P and Q be non-empty perfect subsets of $R \setminus D$ and $S \setminus D$ respectively.

Fix $x \in P + Q$. We must show that x has a unique representation as a sum of elements of A. Since $x \in (R \setminus D) + (S \setminus D)$ then there exist $a \in R \setminus D$ and $b \in S \setminus D$ with $x = a + b$. Since $R \subseteq cl(J)$ and $S \subseteq cl(K)$, this gives us that $x \in I$. But then x is not in the first two pieces of $A + A$ so it must be that one of the elements a, b is in $C_{\frac{1}{2}}$ and the other is in $\frac{1}{2}C_{\frac{1}{2}}$. Since a and b are not in D, x cannot be in X. Therefore, $x \in Y$, and we are done. ∎

Theorem 9 *There is no subset $B \subseteq A$ such that $B+B$ is Bernstein in* $[0, 1.5]$.

Proof. Suppose that such a set B exists. Let P and Q be as in the previous lemma. B can't contain a non-empty perfect subset, since that would imply $B + B$ also contains a non-empty perfect subset of $[0, 1.5]$. Therefore, there is some element x in $P \setminus B$. Then $x + Q$ is a perfect subset of $P + Q$ and so each element of $x + Q$ has a unique representation as a sum of elements in A. But then since $x \notin B$ no element of $x + Q$ is in $B + B$. So, $B + B$ is not Bernstein in $[0, 1.5]$, which is a contradiction. ∎

References

[1] J.M. Ash, S. Catoiu, and R. Ríos-Collantes-de-Terán, *On the n-th quantum derivative*, preprint.

[2] T. Bartoszyński, H. Judah, *Set Theory*, A K Peters, 1995.

[3] K. Ciesielski, *Measure zero sets whose complex sum is non-measurable*, Real Anal. Exchange **26**(2) (2000-2001), 919–922.

[4] H. Fejzić and C.E. Weil, *Repairing the proof of a classical differentiation result*, Real Anal. Exchange **19**(2) (1993-94), 639–643.

[5] S.N. Mukhopadhyay and S. Mitra, *An Extension of a theorem of Ash on generalized differentiability*, Real Anal. Exchange **24**(1) (1998-99), 351–371.

[6] L.A. Rubel, *A pathological Lebesgue measurable function*, J. London Math. Soc. **38** (1963), 1–4.

[7] W. Sierpiński, *Sur la question de la measurabilité de la base de M. Hamel*, Fund. Math. **1** (1920), 105–111.

[8] E.M. Stein and A. Zygmund, *Smoothness and differentiability of functions*, Annals of the University of Sciences, Budapest, **III-IV** (1960-61), 295–307.

[9] B.S. Thomson, *Symmetric Properties of Real Functions*, Marcel Dekker, 1994.